

## CONSTRUCTION OF TWO SUMMATION FORMULAE ATTACHED WITH HYPERGEOMETRIC FUNCTION

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### ABSTRACT

*The main object of present paper is to develop two summation formulae involving the Contiguous relation [1] and derived formula [2]. The results are new and has general character.*

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### A. INTRODUCTION:

**Generalized Gaussian Hypergeometric function of one variable:**

$${}_A F_B(a_1, a_2, \dots, a_A; b_1, b_2, \dots, b_B; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \dots (a_A)_k}{(b_1)_k (b_2)_k \dots (b_B)_k k!} z^k \quad (1)$$

or

$${}_A F_B((a_A); (b_B); z) \equiv {}_A F_B((a_j)_{j=1}^A; (b_j)_{j=1}^B; z) = \sum_{k=0}^{\infty} \frac{((a_A))_k}{((b_B))_k k!} z^k \quad (2)$$

where the parameters  $b_1, b_2, \dots, b_B$  are neither zero nor negative integers and  $A, B$  are non negative integers.

### Contiguous Relations:

[Andrews p.363(9.16) , E.D. p.51(10), H.T.F.I. p.103(32)]

$$(a-b) {}_2 F_1(a, b; c; z) = a {}_2 F_1(a+1, b; c; z) - b {}_2 F_1(a, b+1; c; z) \quad (3)$$

[Abramowitz p.558 (15.2.19)]

$$(a-b)(1-z) {}_2 F_1(a, b; c; z) = (c-b) {}_2 F_1(a, b-1; c; z) + (a-c) {}_2 F_1(a-1, b; c; z) \quad (4)$$

### A New Summation Formula:

[Ref. [2] p.337 (10)]

$${}_2 F_1(a, b; \frac{a+b-1}{2}; \frac{1}{2}) = 2^{b-1} \frac{\Gamma(\frac{a+b-1}{2})}{\Gamma(b)} \left[ \frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-1}{2})} \left\{ \frac{b+a-1}{a-1} \right\} + 2 \frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a}{2})} \right] \quad (5)$$

### Recurrence relation:

$$\Gamma(z+1) = z \Gamma(z) \quad (6)$$

### B. MAIN SUMMATION FORMULAE:

For both the formulae  $a \neq b$

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For  $a < 1$  and  $a > 19$

$$\begin{aligned}
 {}_2F_1(a, b ; \frac{a+b-19}{2}; \frac{1}{2}) = & 2^{b-1} \frac{\Gamma(\frac{a+b-19}{2})}{(a-b)\Gamma(b)} \frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-19}{2})} \\
 & \left\{ \frac{654729075a - 1396704420a^2 + 1094071221a^3 - 444647600a^4 + 107494190a^5 - 16486680a^6 + 1646778a^7}{(a-19)(a-17)(a-15)(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)} \right. \\
 & - 106800a^8 + 4335a^9 - 100a^{10} + a^{11} - 654729075b + 1919582505a^2b - 1779068820a^3b + 856680570a^4b \\
 & + \frac{(a-19)(a-17)(a-15)(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)}{-187881120a^5b + 33687066a^6b - 2688480a^7b + 216825a^8b - 5600a^9b + 189a^{10}b + 1396704420b^2} \\
 & + \frac{(a-19)(a-17)(a-15)(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)}{-1919582505ab^2 + 893172140a^5b^2 - 419849080a^6b^2 + 140368466a^7b^2 - 15938720a^8b^2 + 2201340a^9b^2} \\
 & + \frac{(a-19)(a-17)(a-15)(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)}{-71820a^8b^2 + 4655a^9b^2 - 1094071221b^3 + 17790688320ab^3 - 893172140a^5b^3 + 116147570a^6b^3} \\
 & + \frac{(a-19)(a-17)(a-15)(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)}{-27132000a^5b^4 + 6737780a^6b^4 - 310080a^7b^4 + 33915a^8b^4 + 444647600b^4 - 856680570ab^4 + 419849080a^2b^4} \\
 & + \frac{(a-19)(a-17)(a-15)(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)}{-116147570a^5b^4 + 4906370a^6b^4 - 452200a^7b^4 + 87210a^8b^4 - 107494190b^5 + 187881120ab^5 - 140368466a^2b^5} \\
 & + \frac{(a-19)(a-17)(a-15)(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)}{27132000a^5b^5 - 4906370a^4b^5 + 58786a^6b^5 + 164866803b^6 - 33687066ab^6 + 15938720a^2b^6 - 6737780a^3b^6} \\
 & + \frac{(a-19)(a-17)(a-15)(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)}{452200a^4b^6 - 53786a^5b^6 - 1646773b^7 + 2688480ab^7 - 2201340a^3b^7 + 810080a^5b^7 - 87210a^4b^7 + 106800b^8} \\
 & + \frac{(a-19)(a-17)(a-15)(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)}{-216825ab^8 + 71820a^2b^8 - 33915a^3b^8 - 4335b^9 + 5600ab^9 - 4655a^2b^9 + 100b^{10} - 189ab^{10} - b^{11}} \\
 & + \frac{(a-19)(a-17)(a-15)(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)}{\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-18}{2})} \left( \frac{-995790420a + 1550816724a^2 - 813092208a^3 + 299510128a^4 - 48599200a^5 + 7584996a^6 - 464112a^7 + 34032a^8}{(a-18)(a-16)(a-14)(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)} \right.} \\
 & + \frac{-660a^9 + 20a^{10} + 995790420b - 1144930128a^2b + 1026991136a^3b - 282607976a^4b + 74401824a^5b - 6350960a^6b}{(a-18)(a-16)(a-14)(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)} \\
 & + \frac{765792a^7b - 19404a^8b + 1120a^5b^2 - 1550816724b^2 + 1144830128ab^2 - 259801136a^3b^2 + 155725976a^4b^2}{(a-18)(a-16)(a-14)(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)} \\
 & + \frac{-21686448a^5b^2 + 4311328a^6b^2 - 150480a^7b^2 + 14364a^8b^2 + 813092208b^3 - 1026991136ab^3 + 259801136a^3b^3}{(a-18)(a-16)(a-14)(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)} \\
 & + \frac{-16460080a^4b^3 + 7162848a^5b^3 - 397936a^6b^3 + 62016a^7b^3 - 299510128b^4 + 282607976ab^4 - 155725976a^2b^4}{(a-16)(a-14)(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)} \\
 & + \frac{16460080a^5b^4 - 271320a^5b^4 + 90440a^6b^4 + 49598200b^5 - 74401824ab^5 + 21686448a^2b^5 - 7162848a^3b^5}{(a-16)(a-14)(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)} \\
 & + \frac{271320a^4b^5 - 7584696b^4 + 635096ab^6 - 4311328a^2b^6 + 897936a^3b^6 - 90440a^4b^6 + 464112b^7 - 765792ab^7}{(a-16)(a-14)(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)} \\
 & + \frac{1b0488a^2b^7 - 62016a^5b^7 - 34032b^8 + 19404ab^8 - 14364a^2b^8 + 660b^9 - 1120ab^9 - 400b^{10}}{(a-18)(a-16)(a-14)(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)} \} \\
 \end{aligned} \tag{7}$$

For  $a < 1$  and  $a > 20$

$$\begin{aligned}
 {}_2F_1(a, b ; \frac{a+b-20}{2}; \frac{1}{2}) = & 2^{b-1} \frac{\Gamma(\frac{a+b-20}{2})}{(a-b)\Gamma(b)} \frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-19}{2})} \\
 & \left\{ \frac{3715891200a - 7840972800a^2 + 6660433920a^3 - 2516910592a^4 + 721300608a^5 - 97228320a^6 + 12858384a^7}{(a-19)(a-17)(a-15)(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)} \right. \\
 & - 686928a^8 + 44352a^9 - 770a^{10} + 21a^{11} - 3715391200b + 2399109120ab + 5369825283a^2b - 6105745408a^3b \\
 & + \frac{(a-19)(a-17)(a-15)(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)}{3366978944a^4b - 713416256a^5b + 150993360a^6b - 10993664a^7b + 1139424a^8b - 25564a^9b + 1309a^{10}b} \\
 & + \frac{(a-19)(a-17)(a-15)(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)}{5441863680b^2 - 8765343744ab^2 + 2016699892a^2b^2 + 1689632000a^5b^2 - 1062548704a^4b^2 + 416085712a^5b^2} \\
 & + \frac{(a-19)(a-17)(a-15)(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)}{-46381888a^6b^2 + 7570816a^7b^2 - 229482a^8b^2 + 19019a^9b^2 - 3264915456b^3 + 5829540608ab^3 - 3406107904a^2b^3} \\
 & + \frac{(a-19)(a-17)(a-15)(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)}{363445376a^5b^3 + 168037520a^4b^3 - 54987520a^5b^3 + 16206848a^6b^3 - 739024a^7b^3 + 95931a^8b^3 + 1076416000b^4} \\
 & + \frac{(a-19)(a-17)(a-15)(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)}{} \\
 \end{aligned}$$

$$\begin{aligned}
& + \frac{-2153123128ab^4 + 1095107104a^2b^4 - 379926736a^5b^4 + 19173280a^6b^4 + b^788160a^5b^4 - 76874ea^6b^4}{(a-19)(a-17)(a-15)(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)} \\
& + \frac{177650a^7b^4 - 218683520b^5 + 385775040ab^5 - 305474960a^5b^5 + 61082112a^5b^5 - 14036288a^4b^5 + 271320a^5b^5}{(a-19)(a-17)(a-15)(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)} \\
& + \frac{58786a^6b^5 + 28865760b^6 - 60048912ab^6 + 28518336a^5b^6 - 12921216a^5b^6 + 895356a^4b^6 - 149226a^5b^6}{(a-19)(a-17)(a-15)(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)} \\
& + \frac{-2624369b^7 + 4131072ab^7 - 34890214a^2b^7 + 191568a^5b^7 - 149226a^4b^7 + 145200b^8 - 297792ab^8 + 98406a^5b^8}{(a-19)(a-17)(a-15)(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)} \\
& + \frac{-48279a^5b^8 - 5280a^5b^9 - 5280a^9b^8 + 6820ab^9 - 5775a^5b^9 + 110b^{10} - 209ab^{10} - b^{11}}{(a-17)(a-15)(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)} \\
& + \frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-20}{2})} \left\{ \frac{3715891200a - 5441863680a^2 + 3264915456a^3 - 1076416000a^4 + 218683520a^5 - 28865760a^6}{(a-20)(a-18)(a-16)(a-14)(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)} \right. \\
& + \frac{2524358a^7 - 146200a^8 + 5280a^9 - 110a^{10} + a^{11} - 3715891200b - 2399109120ab + 8765343744a^2b}{(a-20)(a-18)(a-16)(a-14)(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)} \\
& + \frac{-5529540608a^3b + 2153120128a^4b - 385775040a^5b + 60048912a^6b - 4131072a^7b + 297792a^8b}{(a-20)(a-18)(a-16)(a-14)(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)} \\
& + \frac{-6820a^9b + 209a^{10}b + 7840972800b^2 - 5369925280ab^2 - 2016699392a^2b^2 + 3406107904a^3b^2}{(a-20)(a-18)(a-16)(a-14)(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)} \\
& + \frac{-1095107104a^4b^3 + 305474960a^5b^4 - 28518336a^6b^5 + 3489024a^7b^6 - 98406a^8b^7 + 5775a^9b^8 - 6660433920b^9}{(a-20)(a-18)(a-16)(a-14)(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)} \\
& + \frac{6105745408ab^9 - 1689632000a^2b^9 - 363445376a^5b^9 + 379926736a^4b^9 - 61082112a^5b^9 + 12921216a^6b^9}{(a-20)(a-18)(a-16)(a-14)(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)} \\
& + \frac{-491568a^7b^9 + 48279a^8b^9 + 2516910592b^4 - 3866978944ab^4 + 1062548704a^2b^4 - 168037520a^5b^4}{(a-20)(a-18)(a-16)(a-14)(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)} \\
& + \frac{19173200a^4b^4 + 14030200a^5b^4 - 095355a^6b^4 + 149226a^7b^4 - 7213000c00b^5 + 713416256ab^5}{(a-20)(a-18)(a-16)(a-14)(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)} \\
& + \frac{-416085712a^2b^6 + 54987520a^5b^6 - 5788160a^4b^6 - 271320a^5b^6 + 149226a^6b^6 + 97228320b^6 - 150993360ab^6}{(a-20)(a-18)(a-16)(a-14)(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)} \\
& + \frac{46381388a^2b^6 - 16206843a^5b^6 + 768740a^4b^6 - 58786a^5b^6 - 12838384b^7 + 10993664ab^7 - 7570816a^2b^7}{(a-20)(a-18)(a-16)(a-14)(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)} \\
& + \frac{739024a^5h^7 - 177650a^4h^7 + 686928h^8 - 1139424ah^8 + 229482a^2h^8 - 959314h^8 - 44352h^9 + 25564ah^9}{(a-20)(a-18)(a-16)(a-14)(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)} \\
& + \frac{-19019a^2b^9 + 770b^{10} - 1309ab^{11} - 21b^{11}}{(a-20)(a-18)(a-16)(a-14)(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)} \} ] \quad (8)
\end{aligned}$$

### C. DERIVATIONS OF SUMMATION FORMULAE (7) TO (8):

**Derivation of (7):** Replacing  $c = \frac{a+b-19}{2}$  and  $z = \frac{1}{2}$  in equation (4), we get

$$(a-b) {}_2F_1(a, b; \frac{a+b-19}{2}; \frac{1}{2}) = (a-b-19) {}_2F_1(a, b-1; \frac{a+b-19}{2}; \frac{1}{2}) + (a-b+19) {}_2F_1(a-1, b; \frac{a+b-19}{2}; \frac{1}{2})$$

Now with the help of the derived result from equation (5), we get

$$\begin{aligned}
L.H.S &= 2^{b-2} \frac{\Gamma(\frac{a+b-19}{2})}{(a-b+1)\Gamma(b-1)} \left[ \frac{(a-b-19)(b-1)}{(a-b+1)} \frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-19}{2})} \right. \\
&\quad \left. - 654729075 + 707515920a + 339870879a^2 - 531535680a^3 + 303314690a^4 - 75012000a^5 + 10891902a^6 - 966720a^7 \right. \\
&\quad \left. - (a-19)(a-17)(a-15)(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1) \right] \\
&\quad 51585a^8 - 1520a^9 + 19a^{10} + 1396704420b - 2143943946ab + 625999536a^2b + 321892072a^5b - 237910248a^4b \\
&\quad - (a-19)(a-17)(a-15)(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1) \\
&\quad + 66497796a^5b - 9387216a^6b + 826728a^7b - 36252a^8b + 950a^9b - 1094071221b^2 + 1888936416ab^2 \\
&\quad - (a-19)(a-17)(a-15)(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1) \\
&\quad - 915094340a^2b^2 + 94605408a^3b^2 + 59942986a^4b^2 - 19690080a^5b^2 + 3308812a^6b^2 - 217056a^7b^2 + 10659a^8b^2 \\
&\quad - (a-19)(a-17)(a-15)(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1) \\
&\quad + 444647600b^5 - 806337592ab^5 + 453354896a^2b^5 - 96308264a^3b^5 + 2971600a^4b^5 + 3113720a^5b^5 - 387600a^6b^5 \\
&\quad - (a-19)(a-17)(a-15)(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1) \\
&\quad + 38760a^7b^5 - 107494190b^6 + 196095360ab^6 - 118218190a^2b^6 + 29147520a^3b^6 - 2826250a^4b^6 + 41990a^5b^6 + \\
&\quad \left. (a-19)(a-17)(a-15)(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1) \right]
\end{aligned}$$



$$+\frac{-b^{19}b^{18}b^2+362b^{17}ab^2-10657a^2b^2+1520b^2-9b^2ab^2-19b^{10}}{(a-19)(a-17)(a-15)(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)})]$$

On simplification, we get the formula (7)

Similarly, we can prove the formula (8).

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