International Journal of Mathematical Archive-5(3), 2014, 290-294
Available online through www.ijma.info ISSN 2229 – 5046

S_{α}– OPEN SETS IN TOPOLOGICAL SPACES

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(Received on: 10-03-14; Revised & Accepted on: 29-03-14)

ABSTRACT

In this paper, we investigate a new class of semi open sets called S_{α}-open sets in topological spaces and its properties are studied.

Keywords: *Semi open sets, α-closed sets, S_{α}-open sets.*

2010 Mathematics subject classification: *54A05, 54A10, 54C05.*

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, a space means a topological space on which no separation axioms are assumed unless otherwise explicitly stated. In 1963 Levine [9] initiated semi open sets and gave their properties. Mathematicians gave in several papers interesting and different new types of sets. In 1965, O. Njastad [11] introduced α- closed sets. We recall the following definitions and characterizations. The closure (resp., interior) of a subset A of X is denoted by c l A (resp., int A),A subset A of X is said to be semi open [9] (resp, pre open [10], α- open [11], regular open [13]) set if A⊂ cl int A (resp., A⊂ int clA, A⊂ int cl int A, A=int cl (A) The complement of semi open (resp., pre open, α- open, regular open) set is said to be semi closed (resp., pre closed, α- closed, regular closed) The intersection of all semi closed (resp., pre closed, α- closed, regular closed) sets of X containing A is called semi closure (resp., pre closure, α–closure, regular closure) and denoted by scl A (resp., pcl A, αcl A, rcl A). The union of all semi open (resp., pre open, α- open) sets of X contained in A is called the semi interior (resp., pre interior, α-interior) and denoted by s int A (resp., p int A α int A). The family of all semi open (resp., pre open, α- open, regular open, semi closed, pre closed, α- closed, regular closed) subsets of a topological space X is denoted by SO (X) (resp., PO (X), αO(X) ,RO (X), SC(X), PC(X), αC(X),RC(X)).

Definition: 1.1 A topological space (X, τ) is said to be
1. Extremally disconnected of cl V∈τ, for every V∈τ.
2. Locally indiscrete if every open subset of X is closed.
3. Hyperconnected if every nonempty open subset of X is dense.

Lemma: 1.2
1. If X is a locally is indiscrete space, then each semi open subset of X is closed and hence each semi closed subset of X is open [2].
2. A topological space X is hyperconnected if and only if R0(X) = {∅, X} [6]

Theorem 1.3 .Let (X, τ)be a topological space. Then SO(X, τ) =SO(X, αO(X))[3].

Theorem: 1.4[9] Let (X, τ) be a topological space.
1. Let A⊂X. Then A∈SO (X, τ) if and only if cl A = cl int A.
2. If {Aγ: γ∈Γ} is a collection of semi open sets in a topological space (X, τ), then ∪{Aγ: γ∈Γ} is semi open.

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Theorem: 1. 5 If Y is a semi open subspace of a space X, then a subset A of Y, is a semi open set in X if and only if A is semi open set in Y [12].

Theorem: 1.6 [4] Let (X, τ) be a topological space.

If $A\in\tau$, and $B\in SO(X)$, then $A\cap B\in SO(x)$.

Theorem: 1.7 Let X and Y be spaces. If $A\subset X$ and $B\subset Y$ then s int $_{xxy}$ (AXB) =s int_x (A)X s int_y (B)[1].

Definition: 1.8 The subset A of a space X is said to be S_{p-} open [13] if for each $x\in A$, there exists a pre closed set F such that $x\in F\subset A$.

Theorem: 1.9 [4] Let A be any subset of a space X. Then $A\in SC(X)$ if and only if int cl $A\subset A$.

Theorem: 1.10 [12] A subset A of a space X is dense in X if and only if A is semi dense in X.

Theorem: 1.11 [7] A space X is extremely disconnected if and only if RO(x) =RC(X).

2. Sα-open sets

In this section, we introduce and study the concept of Sα- open sets in topological spaces and study some of its roperties.

Definition: 2.1 A semi open set A of a topological space X is said to be Sα-open if for each $x\in A$, there exists a α-closed set F such that $x\in F\subset A$. A subset B of a topological space X is $S_{\alpha-}$closed, if X-B is $S_{\alpha-}$open.

The family of $S_{\alpha-}$open subsets of X is denoted by $S_\alpha O(X)$.

Theorem: 2.2 A subset A of a topological space X is $S_{\alpha-}$open if and only if A is semi open and it is a union of α-closed sets.

Proof: Let A be $S_{\alpha-}$open. Then A is semi open $x\in A$ implies, there exists α-closed set F_x Such that $x\in F_x\subset A$ Hence $\bigcup_{x\in A} F_x\subset A$. But $x\in A$, $x\in F_x$ implies $A\subset\bigcup_{x\in A} F_x$. This completes one half of the proof.

Let A be semi open and $A=\bigcup_{i\in I} F_i$, where each F_i is α-closed. Let $x\in A$. Then $x\in$ some $F_i\subset A$. Hence A is S_α-open.

The following result shows that any union of S_α-open sets in S_α-open.

Theorem: 2.3 Let $\{A_\alpha : \alpha\in\Delta\}$ be a family of $S_{\alpha-}$open sets in a topological space X. Then $\bigcup_{\alpha\in\Delta} A_\alpha$ is an S_α -open set.

Proof: The union of an arbitrary semi open sets is semi open by theorem 1.4. Suppose that $x\in\bigcup_{\alpha\in\Delta} A_\alpha$. This implies that there exists $\alpha_0\in\Delta$ such that $x\in A_{\alpha 0}$ and as $A_{\alpha 0}$ is an S_α –open set, there exists a α-closed set F in X such that $x\in F\subset A_{\alpha 0}\subset\bigcup_{\alpha\in\Delta} A_\alpha$. Therefore $\bigcup_{\alpha\in\Delta} A_\alpha$ is a S_α-open set.

From theorem 2.3, it is clear that any intersection of S_α –closed sets of a topological space X is S_α –closed. The following example shows that the intersection of two S_α–open sets need not be S_α–open.

Example: 2.4 Let X= {a, b, c}

τ = {∅, {a}, {b}, {a, b}, X}

S_α–open sets = {∅, {a, c}, {b, c}, X}{a, c} ∩{b, c}={c} is not an S_α–open set

Theorem 2.5: A subset G of the topological space X is S_α -open if and only if for each $x\in G$, there exists an S_α–open set H such that $x\in H\subset G$.

Proof: Let G be a S_α–open set in X. Then for each $x\in G$, we have G is an S_α–open set such that $x\in G\subset G$.

Conversely, let for each $x\in G$, there exists an S_α–open set H such that $x\in H\subset G$. Than G is a union of S_α–open sets, hence by theorem 2.3, G is S_α–open.

Theorem: 2.6
1. Regular closed set is S_α–open set.
2. Regular open set is S_α–closed set.

Proof:
1. 1. Let A be regular closed in a topological space X. A=cl int A. A is semi open. A is α- closed. $x\in A$ implies $x\in A\subset A$. Hence A is S_α–open.
2. Obvious.

Theorem 2.7: If a space X is a T_1–space, then S_α (X) = SO(X).

Proof: S_α (X)□⊂ SO(X). Let A∈SO(X). Let x∈A. AS X is a T_1 –space, {x} is closed. Every closed set in X is α-closed. Hence $x\in\{x\} \subset A\in S\alpha O(X)$. This completes the proof.

Theorem: 2.8 If the family of all semi open subsets of a topological space is a topology on X, then the family of SαO (X) is also a topology on X.

Proof: Obvious.

Theorem: 2.9 If a space X is hyperconnected, then the only S_α–open sets of X are ∅ and X.

Proof: Let A⊂X such that A is S_α–open in X. If A=X, there is nothing to prove. If A ≠X we have to prove A=∅. As A is S_α –open, for each x∈A, there exists a α–closed set F such that x∈F⊂A. So X-A ⊂X-F. X-A is semi closed. Therefore int cl(X-A) ⊂ (X-A). Since X is hyper connected, then by definition 1.1 and theorem 1.10 scl(int cl (X-A))=X⊂X-A. Hence X-A=X. So A=∅.

Theorem: 2.10 If a topological space X is locally indiscrete, then every semi open set is S_α–open.

Proof: Let A be semi open in X.

Then A⊂ cl int A. As X is locally indiscrete, int A is closed. Hence int A=cl int A. So, cl int A= int A⊂A. So A is regular closed. By theorem 2.6(1)-A is S_α–open.

Theorem: 2.11 If a topological space (X,τ) is T_1 or locally indiscrete, then $\tau\subset S_\alpha 0$ (X).

Proof: Let (X, τ) be T_1 .As every open set is semi open, $\tau\subset S0(x) = S_\alpha 0$ (X).

Let (X, τ) be locally indiscrete then τ⊂SO(x) ⊂ SαO (X).

Theorem: 2.12 If B in clopen subset of a space X and A is S_α–open in X, then $A\cap B\in S_\alpha 0(X)$.

Proof: Let A be S_α–open .So A is semi open. B is open and closed in x. Then by theorem 1.6, A∩B is semi open in X. Let x ∈A∩B. Then x∈A and x∈B. Since A is S_α–open, there exists a α–closed set F such that x∈F⊂A. B is closed and hence α–closed. F∩B is α–closed. x∈F∩B⊂A∩B. So A∩B is S_α–open.

Theorem: 2.13 Let X be a locally indiscrete and A⊂X, B⊂X. It $A\in S_\alpha 0(X)$ and B is open, and then A∩B is S_α–open in X.

Proof: Follows from theorem 2.12.

Theorem: 2.14 Let X be extremally disconnected and A⊂X, B⊂X. If $A\in S_\alpha 0$ (X) and B∈R0(X) then A∩B is S_α–open in X.

Proof: Let $A\in S_\alpha 0$ (X) and B∈R0(X). Hence A is semi open. By Theorem 1.6, A∩B∈S0 (X).

Let $x \in A \cap B$. This implies $x \in A$ and $x \in B$. As A is S_{α} –open, there exists a α –closed set F such that $x \in F \subset A$. X is extremally disconnected. By Theorem 1.11. B is a regular closed set. This implies $F \cap B$ is α-closed. $x \in F \cap B \subset A \cap B$. So $A \cap B$ is S_{α} – open.

3. S_{α}- Operations

Definition: 3.1 A subset N of a topological space X is called S_{α} - neighborhood of a subset A of X, if there exists an S_{α}-open set U such that $A \subset U \subset N$. When A={x}, we say N is S_{α}- neighborhood of x.

Definition: 3.2 A point $x \in X$ is said to be an S_{α}- interior point of A, if there exists an S_{α}-open set U containing x such that $x \in U \subset A$. The set of all Sα-interior points of A is said to be S_{α}- interior of A and it is denoted by S_{α}- int A.

Theorem: 3.3 Let A be any subset of a topological space X. If x is a S_{α}-interior point of A, then there exists a semi closed set F of X containing x such that $F \subset A$.

Proof: Let $x \in S_{\alpha}$- int A. Then there exists a S_{α}-open set U containing x such that $U \subset A$. Since U is in S_{α}-open set, there exists a α-closed set F such that $x \in F \subset U \subset A$.

Theorem: 3.4 For any subset A of a topological space X, the following statements are true
1. The S_{α}-interior of A is the union of all S_{α}-open sets contained in A.
2. S_{α}- int A is the largest S_{α}-open set contained in A.
3. A is S_{α}- open set if and only of A=S_{α} int A.

Proof: obvious.

From 3, are see S_{α} int S_{α} int A= S_{α} int A.

Theorem: 3.5 If A and B are any subsets of a topological space X. Then,
1. S_{α} int $\emptyset=\emptyset$ and S_{α} int X=X
2. S_{α} int $A \subset A$
3. if $A \subset B$, then S_{α} int $A \subset S_{\alpha}$ int B
4. S_{α} int $A \cup S\alpha$ int $B \subset S\alpha$int $(A \cup B)$
5. S_{α} int $(A \cap B)$□$\subset S_{\alpha}$ int $A \cap S_{\alpha}$ int B
6. S_{α} int (A-B)□$\subset S_{\alpha}$ int A-S_{α} int B

Proof: 1-5, obvious.
6. Let $x \in S_{\alpha}$ int (A-B). There exists an S_{α}-open set U such that $x \in U \subset A-B$. That is $U \subset A$. $U \cap B=\emptyset$ and $x \notin B$. Hence $x \in S_{\alpha}$ int A, $x \notin S_{\alpha}$ int B. Hence $x \in S_{\alpha}$ int A-S_{α} int B. This completes the proof.

Definition: 3.6 Intersection of S_{α}-closed sets containing F is called the S_{α}-closure of F and is denoted by S_{α} cl F.

Theorem: 3.7 Let A be a subset of the space X. $x \in X$ is in S_{α}-closed of A if and only if $A \cap U \neq \emptyset$, for every S_{α}-open set U containing x.

Proof: To prove the theorem, let us prove the contra positive.
$x \notin$ scl A $\Leftrightarrow$There exists an S_{α}-open set U containing x that does not intersect A. Let $x \notin S_{\alpha}$ cl A. X-S_{α} cl A is an S_{α}-open set containing x that does not intersect A. Let U be an S_{α}-open set set containing x that does not intersect A. X-U is a S_{α} –closed set containing A. S_{α} cl $A \subset$ (X-U)
$x \notin X-U \Rightarrow x \notin S_{\alpha}$cl A.

Theorem: 3.8 Let A be any subset of a space X .$A \cap F \neq \emptyset$ for every α closed set F of X containing x, then the point x is in the S_{α}- closure of A.

Proof: Let U be any S_{α}- open set containing x. So, there exists a α-closed set F such that $x \in F \subset U$. $A \cap F \neq \emptyset$ implies $A \cap U \neq \emptyset$ for every S_{α} -open set U containing x. Hence $x \in S_{\alpha}$ cl A, by theorem 3.7

Theorem: 3.9 For any subset F of a topological space X, the following statements are true.
1. S_{α} cl F is the intersection of all. S_{α}- closed sets in X containing F.
2. S_{α} cl F is the smallest . S_{α} –closed set containing F.
3. F is S_{α} closed if and only if F= S_{α} cl F.

Proof: Obvious.

Theorem: 3.10 If F and E are any subsets of a topological space X, then
1. S_α cl $\emptyset=\emptyset$ and. S_α cl X=X
2. For any subset F of X, $F\subset S_\alpha$ cl F.
3. If $F\subset E$, then S_α cl $F\subset S_\alpha$ cl E.
4. S_α cl $F\cup S_\alpha$ cl $E \subset S_\alpha$ cl $(F\cup E)$.
5. S_α cl $(F\cap E)\subset S_\alpha$ cl $F \cap S_\alpha$ cl E.

Proof: Obvious.

Theorem: 3.11 For any subset A of a topological space X, the following statements are true.
1. X- S_α cl A= S_α int(X-A).
2. X- S_α int A= S_α cl A.
3. S_α int A= X- S_α cl A.

Proof:
1. X- S_α cl A is a S_α-open set contained in (X-A). Hence X- S_α cl A $\subset S_\alpha$ int (X-A).

If X- S_α cl A $\neq S_\alpha$ int(X-A), then X- S_α int (X-A) is a S_α closed set properly contained in S_α cl A, a contradiction. Hence X- S_α cl A= S_α int(X-A). 2&3 follow from 1.

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Source of support: Nil, Conflict of interest: None Declared