**S_α– OPEN SETS IN TOPOLOGICAL SPACES**

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**ABSTRACT**

In this paper, we investigate a new class of semi open sets called $S_\alpha$-open sets in topological spaces and its properties are studied.

**Keywords:** Semi open sets, $\alpha$-closed sets, $S_\alpha$-open sets.

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1. **INTRODUCTION AND PRELIMINARIES**

Throughout this paper, a space means a topological space on which no separation axioms are assumed unless otherwise explicitly stated. In 1963 Levine [9] initiated semi open sets and gave their properties. Mathematicians gave in several papers interesting and different new types of sets. In 1965, O. Njastad [11] introduced $\alpha$-closed sets. We recall the following definitions and characterizations. The closure (resp., interior) of a subset $A$ of $X$ is denoted by $\text{cl} A$ (resp., $\text{int} A$). A subset $A$ of $X$ is said to be semi open [9] (resp., pre open [10], $\alpha$-open [11], regular open [13]) set if $A \subseteq \text{cl} \text{int} A$ (resp., $A \subseteq \text{int} \text{cl} A$, $A \subseteq \text{cl} \text{int} A$, $A = \text{int} \text{cl} \text{int} A$). The complement of semi open (resp., pre open, $\alpha$-open, regular open) set is said to be semi closed (resp., pre closed, $\alpha$-closed, regular closed). The intersection of all semi closed (resp., pre closed, $\alpha$-closed, regular closed) sets of $X$ containing $A$ is called semi closure (resp., pre closure, $\alpha$-closure, regular closure) and denoted by $\text{scl} A$ (resp., $\text{pcl} A$, $\alpha\text{cl} A$, $\text{rc} A$). The union of all semi open (resp., pre open, $\alpha$-open) sets of $X$ contained in $A$ is called the semi interior (resp., pre interior, $\alpha$-interior) and denoted by $\text{s int} A$ (resp., $\text{p int} A$, $\alpha\text{int} A$, $\alpha\text{int} A$). The family of all semi open (resp., pre open, $\alpha$-open, regular open, semi closed, pre closed, $\alpha$-closed, regular closed) subsets of a topological space $X$ is denoted by $\text{SO} (X)$ (resp., $\text{PO} (X)$, $\alpha\text{O} (X)$, $\text{RO} (X)$, $\text{SC} (X)$, $\text{PC} (X)$, $\alpha\text{C} (X)$, $\alpha\text{RC} (X)$).

**Definition:** 1.1 A topological space $(X, \tau)$ is said to be

1. Extremally disconnected if $\text{cl} V \in \tau$, for every $V \in \tau$.
2. Locally indiscrete if every open subset of $X$ is closed.
3. Hyperconnected if every nonempty open subset of $X$ is dense.

**Lemma:** 1.2

1. If $X$ is a locally indiscrete space, then each semi open subset of $X$ is closed and hence each semi closed subset of $X$ is open [2].
2. A topological space $X$ is hyperconnected if and only if $\text{R0} (X) = \{\emptyset, X\}$ [6]

**Theorem 1.3** Let $(X, \tau)$ be a topological space. Then $\text{SO} (X, \tau) = \text{SO} (X, \alpha\text{O} (X))$ [3].

**Theorem:** 1.4[9] Let $(X, \tau)$ be a topological space.

1. Let $A \subseteq X$. Then $A \in \text{SO} (X, \tau)$ if and only if $\text{cl} A = \text{cl} \text{int} A$.
2. If $\{A_\gamma : \gamma \in \Gamma\}$ is a collection of semi open sets in a topological space $(X, \tau)$, then $\bigcup \{A_\gamma : \gamma \in \Gamma\}$ is semi open.

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**Theorem: 1.5** If $Y$ is a semi open subspace of a space $X$, then a subset $A$ of $Y$, is a semi open set in $X$ if and only if $A$ is semi open set in $Y$ [12].

**Theorem: 1.6** [4] Let $(X, \tau)$ be a topological space.

If $A \in \tau$, and $B \in SO(X)$, then $A \cap B \in SO(x)\). 

**Theorem: 1.7** Let $X$ and $Y$ be spaces. If $A \subset X$ and $B \subset Y$ then $s \text{ int}_x (AXB) = s \text{ int}_x (A)X s \text{ int}_y (B)[1].$

**Definition: 1.8** The subset $A$ of a space $X$ is said to be $S_p$ open [13] if for each $x \in A$, there exists a pre closed set $F$ such that $x \in F \subset A$.

**Theorem: 1.9** [4] Let $A$ be any subset of a space $X$. Then $A \in SC(X)$ if and only if $\text{int cl } A \subset A$.

**Theorem: 1.10** [12] A subset $A$ of a space $X$ is dense in $X$ if and only if $A$ is semi dense in $X$.

**Theorem: 1.11** [7] A space $X$ is extremely disconnected if and only if $RO(x) = RC(X)$.

2. $S\alpha$-open sets

In this section, we introduce and study the concept of $S\alpha$-open sets in topological spaces and study some of its properties.

**Definition: 2.1** A semi open set $A$ of a topological space $X$ is said to be $S\alpha$-open if for each $x \in A$, there exists a $\alpha$-closed set $F$ such that $x \in F \subset A$. A subset $B$ of a topological space $X$ is $S\alpha$-closed, if $X-B$ is $S\alpha$-open.

The family of $S\alpha$-open subsets of $X$ is denoted by $S\alpha O(X)$.

**Theorem: 2.2** A subset $A$ of a topological space $X$ is $S\alpha$-open if and only if $A$ is semi open and it is a union of $\alpha$-closed sets.

**Proof:** Let $A$ be $S\alpha$-open. Then $A$ is semi open implies, there exists $\alpha$-closed set $F_x$ such that $x \in F_x \subset A$ Hence $\bigcup_{x \in A} F_x \subset A$. But $x \in A, x \in F_x$ implies $A \subset \bigcup_{x \in A} F_x$. This completes one half of the proof.

Let $A$ be semi open and $A = \bigcup_{i \in I} F_i$, where each $F_i$ is $\alpha$-closed. Let $x \in A$. Then $x \in$ some $F_i \subset A$. Hence $A$ is $S\alpha$-open.

The following result shows that any union of $S\alpha$-open sets is $S\alpha$-open.

**Theorem: 2.3** Let $\{A_\alpha : \alpha \in \Delta\}$ be a family of $S\alpha$-open sets in a topological space $X$. Then $\bigcup_{\alpha \in \Delta} A_\alpha$ is an $S\alpha$-open set.

**Proof:** The union of an arbitrary semi open sets is semi open by theorem 1.4. Suppose that $x \in \bigcup_{\alpha \in \Delta} A_\alpha$. This implies that there exists $\alpha_0 \in \Delta$ such that $x \in A_{\alpha_0}$ and as $A_{\alpha_0}$ is an $S\alpha$-open set, there exists a $\alpha$-closed set $F$ in $X$ such that $x \in F \subset A_{\alpha_0} \subset \bigcup_{\alpha \in \Delta} A_\alpha$. Therefore $\bigcup_{\alpha \in \Delta} A_\alpha$ is a $S\alpha$-open set.

From theorem 2.3, it is clear that any intersection of $S\alpha$-closed sets of a topological space $X$ is $S\alpha$-closed. The following example shows that the intersection of two $S\alpha$-open sets need not be $S\alpha$-open.

**Example: 2.4** Let $X= \{a, b, c\}$

$\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$

$S\alpha$-open sets $= \{\emptyset, \{a\}, \{b\}, \{a, c\} \cap \{b, c\} = \{c\}$ is not an $S\alpha$-open set

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Theorem 2.5: A subset \( G \) of the topological space \( X \) is \( S_α \) -open if and only if for each \( x \in G \), there exists an \( S_α \) -open set \( H \) such that \( x \in H \subset G \).

**Proof:** Let \( G \) be a \( S_α \) -open set in \( X \). Then for each \( x \in G \), we have \( G \) is an \( S_α \) -open set such that \( x \in H \subset G \).

Conversely, let for each \( x \in G \), there exists an \( S_α \) -open set \( H \) such that \( x \in H \subset G \). Then \( G \) is a union of \( S_α \) -open sets, hence by theorem 2.3, \( G \) is \( S_α \)-open.

**Theorem 2.6**
1. Regular closed set is \( S_α \) -open set.
2. Regular open set is \( S_α \) -closed set.

**Proof:**
1. Let \( A \) be regular closed in a topological space \( X \). \( A=\text{cl int } A \). \( A \) is semi open. \( A \) is \( S_α \) -closed. \( x \in A \) implies \( x \in A \subset A \). Hence \( A \) is \( S_α \) -open.

2. Obvious.

**Theorem 2.7:** If a space \( X \) is a \( T_1 \) -space, then \( S_α(X) = SO(X) \).

**Proof:** \( S_α(X) \subset SO(X) \). Let \( A \in SO(X) \). Let \( x \in A \). \( A \) is a \( T_1 \) -space, \( \{x\} \) is closed. Every closed set in \( X \) is \( S_α \) closed. Hence \( x \in \{x\} \subset A \in SO(X) \). This completes the proof.

**Theorem 2.8** If the family of all semi open subsets of a topological space is a topology on \( X \), then the family of \( S_α O \) (\( X \)) is also a topology on \( X \).

**Proof:** Obvious.

**Theorem 2.9** If a space \( X \) is hyperconnected, then the only \( S_α \) -open sets of \( X \) are \( \emptyset \) and \( X \).

**Proof:** Let \( A \subset X \) such that \( A \) is \( S_α \) -open in \( X \). If \( A=X \), there is nothing to prove. If \( A \neq X \) we have to prove \( A=\emptyset \). As \( A \) is \( S_α \) -open, for each \( x \in A \), there exists a \( \alpha \) -closed set \( F \) such that \( x \in F \subset A \). So \( X-A \subset X-F \). \( X-A \) is semi closed. Therefore \( \text{int cl}(X-A) \subset (X-A) \). Since \( X \) is hyper connected, then by definition 1.1 and theorem 1.10 \( \text{scl}(\text{int cl } (X-A))=X \subset X-A \). Hence \( X-A=X \). So \( A=\emptyset \).

**Theorem 2.10** If a topological space \( X \) is locally indiscrete, then every semi open set is \( S_α \) -open.

**Proof:** Let \( A \) be semi open in \( X \).

Then \( A \subset \text{cl int } A \). As \( X \) is locally indiscrete, \( \text{int } A \) is closed. Hence \( \text{int } A=\text{cl int } A \). So \( \text{cl int } A= \text{int } A \subset A \). So \( A \) is regular closed. By theorem 2.6(1)-A is \( S_α \) -open.

**Theorem 2.11** If a topological space \( (X,\tau) \) is \( T_1 \) or locally indiscrete, then \( \tau \subset S_0(X) \).

**Proof:** Let \( (X,\tau) \) be \( T_1 \). As every open set is semi open. \( \tau \subset S_0(X) \).

Let \( (X,\tau) \) be locally indiscrete then \( \tau \subset S_0(X) \).

**Theorem 2.12** If \( B \subset \text{clopen subset of a space } X \) and \( A \subset S_α(X) \) and \( B \) is open, then \( A \cap B \subset S_0(X) \).

**Proof:** Follows from theorem 2.11.

**Theorem 2.13** Let \( X \) be a locally indiscrete and \( A \subset X \), \( B \subset X \). If \( A \subset S_0(X) \) and \( B \) is open, then \( A \cap B \subset S_α \) -open in \( X \).

**Proof:** Follows from theorem 2.12.

**Theorem 2.14** Let \( X \) be extremally disconnected and \( A \subset X \), \( B \subset X \). If \( A \subset S_0(X) \) and \( B \subset R_0(X) \) then \( A \cap B \subset S_α \) -open in \( X \).

**Proof:** Let \( A \subset S_0(X) \) and \( B \subset R_0(X) \). Hence \( A \) is semi open. By theorem 1.6, \( A \cap B \subset S_0(X) \).
Let \( x \in A \cap B \). This implies \( x \in A \) and \( x \in B \). As \( A \) is \( S_{\alpha} \)-open, there exists a \( \alpha \)-closed set \( F \) such that \( x \in F \subset A \). \( X \) is extremally disconnected. By Theorem 1.11, \( B \) is a regular closed set. This implies \( F \cap B = \alpha \)-closed. \( x \in F \cap B \subset A \cap B \). So \( A \cap B \) is \( S_{\alpha} \)-open.

3. \( S_{\alpha} \)-Operations

**Definition:** 3.1 A subset \( N \) of a topological space \( X \) is called \( S_{\alpha} \)-neighborhood of a subset \( A \) of \( X \), if there exists an \( S_{\alpha} \)-open set \( U \) such that \( A \subset U \subset N \). When \( A = \{ x \} \), we say \( N \) is \( S_{\alpha} \)-neighborhood of \( x \).

**Definition:** 3.2 A point \( x \in X \) is said to be an \( S_{\alpha} \)-interior point of \( A \), if there exists an \( S_{\alpha} \)-open set \( U \) containing \( x \) such that \( x \in U \subset A \). The set of all \( S_{\alpha} \)-interior points of \( A \) is said to be \( S_{\alpha} \)-interior of \( A \) and it is denoted by \( S_{\alpha} \) int \( A \).

**Theorem:** 3.3 Let \( A \) be any subset of a topological space \( X \). If \( x \) is a \( S_{\alpha} \)-interior point of \( A \), then there exists a semi closed set \( F \) of \( X \) containing \( x \) such that \( F \subset A \).

**Proof:** Let \( x \in S_{\alpha} \) int \( A \). Then there exists a \( S_{\alpha} \)-open set \( U \) containing \( x \) such that \( U \subset A \). Since \( U \) is in \( S_{\alpha} \)-open set, there exists a \( \alpha \)-closed set \( F \) such that \( x \in F \subset U \subset A \).

**Theorem:** 3.4 For any subset \( A \) of a topological space \( X \), the following statements are true
1. The \( S_{\alpha} \)-interior of \( A \) is the union of all \( S_{\alpha} \)-open sets contained in \( A \).
2. \( S_{\alpha} \) int \( A \) is the largest \( S_{\alpha} \)-open set contained in \( A \).
3. \( A \) is \( S_{\alpha} \)-open set if and only if \( A = S_{\alpha} \) int \( A \).

**Proof:** obvious.

From 3, are see \( S_{\alpha} \) int \( A \) = \( S_{\alpha} \) int \( A \).

**Theorem:** 3.5 If \( A \) and \( B \) are any subsets of a topological space \( X \). Then,
1. \( S_{\alpha} \) int \( \emptyset = \emptyset \) and \( S_{\alpha} \) int \( X = X \)
2. \( S_{\alpha} \) int \( A \subset A \)
3. if \( A \subset B \), then \( S_{\alpha} \) int \( A \subset S_{\alpha} \) int \( B \)
4. \( S_{\alpha} \) int \( A \cup S_{\alpha} \) int \( B = S_{\alpha} \) int \( \emptyset \cup \emptyset \)
5. \( S_{\alpha} \) int \( (A \cap B) \subset S_{\alpha} \) int \( A \cap S_{\alpha} \) int \( B \)
6. \( S_{\alpha} \) int \( (A - B) \subset S_{\alpha} \) int \( A - S_{\alpha} \) int \( B \)

**Proof:** obvious.

6. Let \( x \in S_{\alpha} \) int \( (A - B) \). There exists an \( S_{\alpha} \)-open set \( U \) such that \( x \in U \subset A - B \). That is \( U \subset A \). \( U \cap B = \emptyset \) and \( x \notin B \). Hence \( x \in S_{\alpha} \) int \( A \), \( x \notin S_{\alpha} \) int \( B \). Hence \( x \in S_{\alpha} \) int \( A - S_{\alpha} \) int \( B \). This completes the proof.

**Definition:** 3.6 Intersection of \( S_{\alpha} \)-closed sets containing \( F \) is called the \( S_{\alpha} \)-closure of \( F \) and is denoted by \( S_{\alpha} \) cl \( F \).

**Theorem:** 3.7 Let \( A \) be a subset of the space \( X \). \( x \in X \) is in \( S_{\alpha} \)-closed of \( A \) if and only if \( A \cap U \neq \emptyset \), for every \( S_{\alpha} \)-open set \( U \) containing \( x \).

**Proof:** To prove the theorem, let us prove the contra positive.\( x \notin S_{\alpha} \) cl \( A \) \( \Leftrightarrow \) There exists an \( S_{\alpha} \)-open set \( U \) containing \( x \) that does not intersect \( A \). Let \( x \notin S_{\alpha} \) cl \( A \). \( X - S_{\alpha} \) cl \( A \) is an \( S_{\alpha} \)-open set containing \( x \) that does not intersect \( A \). Let \( U \) be an \( S_{\alpha} \)-open set set containing \( x \) that does not intersect \( A \). \( X - U \) is a \( S_{\alpha} \)-closed set containing \( A \). \( S_{\alpha} \) cl \( A = (X - U) \)

\( x \notin X - U \Rightarrow x \notin S_{\alpha} \) cl \( A \).

**Theorem:** 3.8 Let \( A \) be any subset of a space \( X \). \( A \cap F \neq \emptyset \) for every \( \alpha \)-closed set \( F \) of \( X \) containing \( x \), then the point \( x \) is in the \( S_{\alpha} \)-closure of \( A \).

**Proof:** Let \( U \) be any \( S_{\alpha} \)-open set containing \( x \). So, there exists a \( \alpha \)-closed set \( F \) such that \( x \in F \subset U \). \( A \cap F \neq \emptyset \) implies \( A \cap U \neq \emptyset \) for every \( S_{\alpha} \)-closed set \( U \) containing \( x \). Hence \( x \in S_{\alpha} \) cl \( A \), by theorem 3.7

**Theorem:** 3.9 For any subset \( F \) of a topological space \( X \), the following statements are true.
1. \( S_{\alpha} \) cl \( F \) is the intersection of all \( S_{\alpha} \)-closed sets in \( X \) containing \( F \).
2. \( S_{\alpha} \) cl \( F \) is the smallest \( S_{\alpha} \)-closed set containing \( F \).
3. \( F \) is \( S_{\alpha} \)-closed if and only if \( F = S_{\alpha} \) cl \( F \).

**Proof:** Obvious.
Theorem: 3.10 If $F$ and $E$ are any subsets of a topological space $X$, then

1. $S_a\text{cl}\emptyset = \emptyset$ and $S_a\text{cl}\ X = X$
2. For any subset $F$ of $X$, $F \subset S_a\text{cl}\ F$.
3. If $F \subset E$, then $S_a\text{cl}\ F \subset S_a\text{cl}\ E$.
4. $S_a\text{cl}\ (F \cup E) \subset S_a\text{cl}\ F \cap S_a\text{cl}\ E$.
5. $S_a\text{cl}\ (F \cap E) \subset S_a\text{cl}\ F \cap S_a\text{cl}\ E$.

Proof: Obvious.

Theorem: 3.11 For any subset $A$ of a topological space $X$, the following statements are true.

1. $X - S_a\text{cl}\ A = S_a\text{int}(X - A)$.
2. $X - S_a\text{int}\ A = S_a\text{cl}\ A$.
3. $S_a\text{int}\ A = X - S_a\text{cl}\ A$.

Proof:

1. $X - S_a\text{cl}\ A$ is a $S_a$-open set contained in $(X - A)$. Hence $X - S_a\text{cl}\ A \subset S_a\text{int}(X - A)$.

If $X - S_a\text{cl}\ A \neq S_a\text{int}(X - A)$, then $X - S_a\text{int}(X - A)$ is a $S_a$ closed set properly contained in $S_a\text{cl}\ A$, a contradiction. Hence $X - S_a\text{cl}\ A = S_a\text{int}(X - A)$. 2 & 3 follow from 1.

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