

ON PRIME IDEAL CHARACTERIZATION
OF QUASI-COMPLEMENTED ALMOST DISTRIBUTIVE LATTICES

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ABSTRACT

Some necessary and sufficient conditions for an Almost Distributive Lattice (ADL) to become a quasi-complemented ADL in topological and algebraic terms are proved. Characterization of quasi-complemented Almost Distributive Lattice in terms of prime ideals and minimal prime ideals are established.

Key words: Almost Distributive Lattice, *-ADL, prime ideal, minimal prime ideal, quasi-complemented ADLs.

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1. INTRODUCTION

The concept of an Almost Distributive Lattice (ADL) was introduced by Swamy U M and Rao G C, as a common abstraction of existing lattice theoretic and ring theoretic generalizations of Boolean algebra. The class of ADLs with pseudo-complementation was introduced in [4], and it was observed that an ADL can have more than one pseudo-complementation unlike in the case distributive lattice. The concept of quasi-complemented Almost Distributive Lattices was introduced in [3] and proved that a uniquely quasi-complemented ADL is a pseudo-complemented ADL. Also, it was proved that an ADL L is quasi-complemented ADL if and only if every prime ideal in L is maximal.

In this paper, we characterize the quasi-complemented ADLs are both algebraically and topologically in terms of their prime ideals and minimal prime ideals with hull-kernel topology and dual hull-kernel topology.

2. PRELIMINARIES

In this section, we recall the definitions and certain properties Almost Distributive Lattice are taken from [4].

Definition: 2.1 An algebra of type $(L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (ADL), if it satisfies the following axioms:

- (1) $(a \vee b) \wedge c = (a \wedge b) \vee (b \wedge c)$
- (2) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
- (3) $(a \vee b) \wedge b = b$
- (4) $(a \vee b) \wedge a = a$
- (5) $a \vee (a \wedge b) = a$
- (6) $0 \wedge a = 0$ for all $a, b, c \in L$

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A non-empty subset I of an ADL L is called an ideal (filter) of L if $a \vee b \in I$ ($a \wedge b \in I$) and, $a \wedge x \in I$ ($x \wedge a \in I$) for any $a, b \in I$ and $x \in L$. If I is an ideal of L and $a, b \in L$, then $a \wedge b \in I \Leftrightarrow b \wedge a \in I$. The set $I(L)$ of all ideals of L is a complete distributive lattice under set inclusion with the least element (0) and the greatest element L in which, for any I, J , $I \cap J$ is the infimum of I, J and the supremum is given by $I \vee J = \{i \vee j / i \in I, j \in J\}$. Let S be a non-empty subset

of an ADL L . Then the set $\langle S \rangle = \left\{ \left(\bigvee_{i=1}^n \right) \wedge x / x \in L, s_i \in S \text{ and } n \in \mathbb{Z}^+ \right\}$ is the smallest ideal of L containing S .

For any $a \in L$, $\langle a \rangle = \{a \wedge x / x \in L\}$ is the principal ideal generated by a . Similarly, $a \in L$, $\langle a \rangle = \{a \wedge x / x \in L\}$ is the principal filter generated by a . The set $PI(L)$ of all principal ideals of L is a sub lattice of $I(L)$. A proper ideal P of L is said to be a prime if for any $x, y \in L$, $x \wedge y \in P$ implies either $x \in P$ or $y \in P$. A proper ideal M of L is said to be a maximal ideal if it is not contained in any proper ideal of L . For any subset S of an ADL L , we define $S^* = \{x \in L / x \wedge a = 0, \text{ for all } a \in S\}$. Then S^* is an ideal and is called the annihilator ideal of S . For $x \in L$, $\langle x \rangle^*$ is called an annulet of L .

An element $a \in L$ is called a dense if $\langle a \rangle^* = (0)$ and set of all dense elements in L is denoted by D . Then D is a filter, whenever D is non-empty. An ADL L with 0 is called a $*$ -ADL if to each $x \in L$, there exists $y \in L$ such that $\langle x \rangle^{**} = \langle y \rangle^*$. An ADL L with 0 is a $*$ -ADL if and only if to each $x \in L$, there exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y \in D$. Every $*$ -ADL possesses a dense element. For any $x, y \in L$, define $x \leq y$ if and only if $x = x \wedge y$ or equivalently, $x \vee y = y$, then \leq is a partial ordering on L , in which 0 is the least element. An element m is maximal in (L, \leq) if and only if $m \wedge x = x$, for all $x \in L$. We don't know, so far, whether \vee is associative in an ADL or not. In this paper L denotes ADL with maximal element in which \vee is associative.

Lemma: 2.2 The set $I(L)$ of all ideals of L is complete pseudo-complemented distributive lattice with least element $\{0\}$, and greatest element L in which for any $I, J \in I(L)$ where $I \bar{\wedge} J = I \cap J$ is infimum of I and J and the supremum is given by $I \vee J = \{i \wedge j / i \in I \text{ and } j \in J\}$.

Lemma: 2.3 Let L be an ADL, Define a relation $\theta = \{(x, y) \in L \times L / \langle x \rangle^* = \langle y \rangle^*\}$. Then θ is a congruence relation on L .

If L is a bounded distributive lattice and X is the set of all prime ideals of L , then the hull-kernel topology τ_h^X on X is a topology on X for which $\{X_a / a \in L\}$ is a basis, where for any $a \in L$, $X_a = \{p \in X / a \notin p\}$. In this topology, for any

subset F of X the closure $\bar{F} = \left\{ Q \in X / \bigcap_{p \in F} P \subseteq Q \right\}$. The dual hull-kernel topology τ_d^X on X is the topology for which

$\{h_X(a) / a \in L\}$ is a base where $h_X(A) = \{p \in X / A \subseteq p\}$. For any $A \subseteq L$, write $h_X(A) = \{p \in X / A \subseteq p\}$ and it can be observe that $h_X(A) = \bigcap_{a \in A} h_X(a)$.

All these concepts can be analogously defined in case of ADLs also. Let L be an ADL and $S(M)$ denote the set of all prime ideals (minimal prime ideals) of L and also $m(I(L))$ denote the space of minimal prime ideals in $I(L)$. The hull-kernel topology on S is denoted by τ_h^S . In this topology for any $a \in L$, the corresponding basic open sets is denoted by S_a . We write τ_h^M for topology on M induced by τ_h^S . In this topology, the basic open sets are $\{M_a / a \in L\}$ where $M_a = M \cap S_a$. The dual hull-kernel topology on $S(M)$ is denoted by $\tau_d^S(\tau_d^M)$. In this topology, for any $a \in L$, the basic open sets are denoted by $h_s(a)(h_M(a))$. For a prime ideal P of $I(L)$, $C(P) = \bigcup \{J \in I(L) / J \in P\}$, Q is a prime ideal in L , $\tau(Q) = \{J \in I(L) / J \subseteq Q\}$. It is easy to see that $C(P)$, $\tau(Q)$ are prime ideals in L and $I(L)$ respectively.

Theorem: 2.4 Let P be a prime ideal of an ADL L . Then P is a minimal prime ideal if and only if for each $x \in P$, there exists $y \notin P$ such that $x \wedge y = 0$

Lemma: 2.5 A prime ideal P of an ADL L is minimal if and only if for each $x \in P$ implies $\langle x \rangle^* \not\subseteq P$.

Lemma: 2.6 For any $x, y, z \in L$, we have the following:

- [1] $M_x = h_M([x]^*)$
- [2] $h_M(x) = h_M([x]^{**})$
- [3] $[x]^* \subseteq [y]^* \Leftrightarrow h_M(x) \subseteq h_M(y)$
- [4] $[x]^* \subseteq [y]^* \Leftrightarrow M_y \subseteq M_x$
- [5] $[z]^* = [x]^* \cap [y]^* \Leftrightarrow h_M(z) = h_M(x) \cap h_M(y)$
- [6] $[x]^{**} = [y]^* \Leftrightarrow h_M(x) = h_M(y)$

Lemma: 2.7[2] Let L be an ADL such that $C(P) \in M$, for each $p \in m(I(L))$. Then the mapping $\phi: m(I(L)) \rightarrow M$ defined by $\phi(P) = C(P)$ for each $p \in m(I(L))$ is an onto continuous closed mapping.

Theorem: 2.8 For any ideal I in L , $M_I = \bigcap_{x \in I} M_x = \bigcap_{x \in I} M_{[x]^{**}}$

Theorem: 2.9 [2] Let L be an ADL. Then the following are equivalent:

- (1) M is compact, Hausdorff and extremally disconnected space.
- (2) The space M and $m(I(L))$ are homeomorphic.
- (3) $J^{**} \in A_0(L)$, for each $J \in I(L)$

3. ON PRIME IDEAL CHARACTERIZATION OF QUASI-COMPLEMENTED ALMOST DISTRIBUTIVE LATTICES

In this section, we characterize the quasi-complemented ADL in terms of hull - kernel topology and dual hull-kernel topology. Recall that an Almost Distributive Lattice L with 0 is called quasi-complemented if for each $x \in L$, there is an element $y \in L$ such that $x \wedge y = 0$ and $x \vee y$ is a maximal. Here y is called a quasi-complement of x .

Theorem: 3.1 Let L be an ADL with maximal element m in which every dense element is maximal. Then L is quasi-complemented ADL if and only if for each $x \in L$, there exists $y \in L$ such that $M_x = h_M(y)$.

Proof: Suppose L is quasi-complemented ADL. We have every quasi-complemented ADL is a $*$ ADL. Let $x \in L$. Then there exists $y \in L$ such that $[x]^* = [y]^{**}$. Therefore $h_M([x]^*) = h_M([y]^{**})$ and hence lemma 2.6, $h_M([x]^*) = h_M(y)$. Thus $M_x = h_M(y)$. Conversely, suppose that for each $x \in L$, there exists $y \in L$ such that $M_x = h_M(y)$. we shall prove that L is quasi complemented ADL. Since $M_x = h_M(y)$, $h_M([x]^*) = h_M([y]^{**})$. Hence by lemma 2.6, we get $[x]^* = [y]^{**}$. Therefore $x \wedge y = 0$ and $x \vee y$ is dense. It follows that $x \wedge y = 0$ and $x \vee y$ is a maximal. Thus L is quasi-complemented ADL.

Theorem: 3.2 Let L be an ADL with maximal element m in which every dense element is maximal. Then L is quasi-complemented ADL if and only if M is compact in the hull-kernel topology.

Proof: Suppose L is quasi-complemented ADL. Then for each $x \in L$ there exists $y \in L$ such that $M_x = h_M(y)$ and hence M_x is a basic closed set in M . Let $\{M_x \mid x \in \Delta\}$ be a family of closed sets in M with finite intersection property for some $\Delta \subseteq L$. Let F be a filter in L generated by Δ . Then for any $x_1, x_2, \dots, x_n \in \Delta$, $\bigcap_{i=1}^n M_{x_i} \neq \emptyset$ and hence $M_{\bigwedge_{i=1}^n x_i} \neq \emptyset$

It follows that $\bigwedge_{i=1}^n x_i \neq 0$. Therefore $0 \notin F$ and hence F is a proper filter of L . It follows that F is contained in a maximal filter say K of L . Therefore $L - M$ is minimal prime ideal of L . Let $x \in \Delta$. Then $x \notin L - K$. Therefore $L - K \in M_x$ for all $x \in \Delta$. Hence $L - K \in \bigcap_{x \in \Delta} M_x$, we get $\bigcap_{x \in \Delta} M_x \neq \emptyset$. Thus M is compact in hull-kernel topology.

Conversely suppose M is compact in the hull-kernel topology on M and $x \in L$. Then $h_M(x)$ being a closed subset of M , is compact. If $p \in h_M(x)$, then $x \in P$. Hence by lemma 2.5 $h_M(x) \cap h_M([x]^*) = \emptyset$. So that $h_M(x) \cap \bigcap_{t \in [x]^*} h_M(t) = \emptyset$.

Now $\{h_M(x) \cap h_M(t) / t \in [x]^*\}$ is a class of closed sets in $h_M(x)$ having empty intersection. There exist $t_1, t_2, \dots, t_n \in [x]^*$ such that $h_M(x) \cap h_M(t_1) \cap h_M(t_2) \cap h_M(t_3) \dots \cap h_M(t_n) = \phi$. Write $x' = \bigvee_{i=1}^n t_i$, then $h_M(x) \cap h_M(x') = \phi$. It follows that $M_x \cup M_{x'} = M$ and $M_x \cup M_{x'} = M_{x \wedge x'} = M_0 = \phi$. Therefore $M_x = h_M(x)$ and $M_{x'} = h_M(x')$. Hence $h_M([x]^{**}) = h_M(x) = M_x = h_M([x']^*)$. By lemma 2.6, we get $[x]^{**} = [x']^*$. It follows that $x \wedge y = 0$ and $x \vee y$ is dense. Thus $x \wedge y = 0$ and $x \vee y$ is a maximal and hence L is quasi-complemented ADL.

Hence, from the above theorem 3.1 and 3.2, we have the following

Corollary: 3.3 *Let L be an ADL with maximal element m in which every dense element is maximal. Then the following are equivalent:*

- (1) *L is quasi-complemented ADL*
- (2) $\tau_h^M = \tau_d^M$
- (3) *M is compact in hull-kernel topology.*

Next, we give an algebraic characterization of quasi-complemented ADL. First, we need the following

Lemma: 3.4 *Let L be an ADL with maximal element m. Then $M_m = M$ and $M_0 = \phi$.*

Proof: Suppose m is maximal element in L. Clearly $M_m \subseteq M$. Conversely suppose $P \in M$. Then $P \neq L$ and hence $m \notin P$. Therefore $P \in M_m$. Hence $M_m = M$ and clearly $M_0 = \phi$ since every ideal contains 0.

Theorem: 3.5 *Let L be an ADL with maximal element m in which every dense element is maximal. Then L is quasi-complemented ADL if and only if $B = \{M_x / x \in L\}$ is a Boolean algebra under the operations \cup and \cap .*

Proof: Suppose L is quasi-complemented ADL. It can be easily seen that B is a Boolean algebra under the operations \cup and \cap . Conversely, suppose that B is a Boolean algebra. Let $x \in L$. Then there exist $x' \in L$ such that $M_x \cup M_{x'} = M_m$ and hence $M_{x \wedge x'} = M_0$ and $M_{x \vee x'} = M_m$. It follows that $x \wedge x' = 0$ and $x \vee x'$ is dense. Hence $x \wedge x' = 0$ and $x \vee x'$ is maximal. Thus L is quasi-complemented ADL.

Recall that, the relation θ on an ADL L, defined by $(x, y) \in \theta \Leftrightarrow [x]^* = [y]^*$ is a congruence relation on L and hence L/θ of all congruence classes, is bounded distributive lattice under the induced operations on L. Now, we give another characterization of quasi-complemented ADL, if first we need the following lemma.

Lemma: 3.6 *Let L be an ADL with maximal element. Then we have the following:*

- (1) *If $m_1/\theta, m_2/\theta$ are two maximal elements in L/θ , then $m_1/\theta = m_2/\theta$*
- (2) *If d is dense element in L, then $d/\theta = m/\theta$, for all maximal element m in L*

Proof: Let $t \in m_1/\theta$. Then $(t, m_1) \in \theta$ implies $[t]^* = [m_1]^* = \{0\}$. Therefore $[t]^* = [m_2]^*$ and hence $(t, m_2) \in \theta$. Thus $m_1/\theta \subseteq m_2/\theta$. Similarly $m_2/\theta \subseteq m_1/\theta$. Hence $m_1/\theta = m_2/\theta$. Obviously $d/\theta = m/\theta$, for all maximal element m in L

It can be easily see that the elements $0/\theta$ and m/θ are bounds of L/θ and hence L/θ is a bounded distributive lattice.

Theorem: 3.7 *Let L be an ADL with maximal element m in which every dense element is maximal. Then L is quasi-complemented ADL if and only if L/θ is a Boolean algebra*

Proof: Suppose L is a quasi-complemented ADL in which every dense element is maximal. Define $f: B \rightarrow L/\theta$ by $f(M_x) = x/\theta$, for each $x \in L$. Let $M_x, M_y \in B$. Then $M_x = M_y \Leftrightarrow [x]^* = [y]^* \Leftrightarrow (x, y) \in \theta \Leftrightarrow x/\theta = y/\theta \Leftrightarrow f(M_x) = f(M_y)$. Hence f is well-defined and one-one. Clearly f is an onto and hence f is a homomorphism. Also, $f(M_0) = 0/\theta$ and $f(M_m) = m/\theta$. Thus f is an isomorphism. Therefore L/θ is a Boolean algebra

Conversely, suppose that L/θ is a Boolean algebra. Let $x \in L$. Then there exists $x'/\theta \in L/\theta$ such that $x/\theta \wedge x'/\theta = 0/\theta$ and $x/\theta \vee x'/\theta = m/\theta$. It follows that $(x \wedge x')/\theta = 0/\theta$ and $(x \vee x')/\theta = m/\theta$. Therefore $[x \wedge x']^* = [0]^* = L$ and hence $x \wedge x' = 0$. Clearly $x \vee x'$ is dense. It follows that hypothesis $x \wedge x' = 0$ and $x \vee x'$ is a maximal. Hence L is quasi-complemented ADL.

Therefore from the above theorem 3.5 and 3.7, we have the following.

Corollary: 3.8 Let L be an ADL with maximal element m in which every dense element is maximal. Then the following are equivalent:

- (i) L is quasi-complemented ADL
- (ii) $B = \{M_x/x \in L\}$ is a Boolean algebra
- (iii) L/θ is a Boolean algebra

Recall that for a prime ideal P of $I(L)$, $C(P) = \bigcup \{J \in I(L)/J \in P\}$ is a prime ideal in L .

Theorem: 3.9 Let L be an ADL with maximal element m in which every dense element is maximal. Then L is quasi-complemented ADL if and only if $C(P) \in M$, for each $P \in m(I(L))$.

Proof: Suppose L is quasi-complemented ADL and $P \in m(I(L))$. Clearly $C(P)$ is a prime ideal in L . Let $x \in C(P)$. Then $x \in J$. Therefore $(x) \subseteq J$ and hence $(x) \in P$. Since L is quasi-complemented ADL, there exists $x' \in L$ such that $x \wedge x' = 0$ and $x \vee x'$ is maximal. But $(x) \vee (x') = (x \vee x') = L$ and $(x) \vee (x') \notin P$, since P is maximal prime ideal in $I(L)$, it follows that $(x') \notin P$. Hence $x' \notin P$. Thus for each $x \in C(P)$, Hence there exists $x' \notin C(P)$ such that $x \wedge x' = 0$. (by lemma 2.5), $C(P)$ is a minimal prime ideal of L . Hence $C(P) \in M$, for each $P \in m(I(L))$. Conversely assume the condition. We have the mapping $\phi: m(I(L)) \rightarrow M$ defined by $\phi(P) = C(P)$, for each $P \in m(I(L))$ is onto continues and closed since by lemma 2.7. Therefore M is a compact. It follows that, by theorem 3.2, L is quasi-complemented ADL.

Recall that the set $A_0(L)$ of all annulets of an ADL L forms a distributive lattice under the binary operations \vee and \wedge defined by $[x]^* \vee [y]^* = [x \vee y]^*$ and $[x]^* \wedge [y]^* = [x \wedge y]^*$, for any $[x]^*, [y]^* \in A_0(L)$.

Theorem: 3.10 Let L be an ADL with maximal element m in which every dense element is maximal. Then $J^{**} \in A_0(L)$, for each $J \in I(L)$ if and only if L is quasi-complemented ADL and for each $Q \in M$, there exists unique $P \in m(I(L))$ such that $C(P)=Q$.

Proof: Suppose $J^{**} \in A_0(L)$, for each $J \in I(L)$. Let $x \in L$. Then we have $[x]^{**} \in A_0(L)$. It follows that, there exist $x' \in L$ such that $[x]^{**} = [x']^*$. Hence $x \wedge x' = 0$ and $x \vee x'$ is maximal. Thus L is quasi-complemented ADL. Now, let $Q \in M$ and $P_1, P_2 \in m(I(L))$ such that $C(P_1) = C(P_2) = Q$. Let $J \in P_1$. Then $J^* \in I(L)$ and $J \cap J^* = \{0\}$. Since $P_1 \in m(I(L))$, $J \in P_1$, $J^* \notin P_1$, (by lemma 2.5). Again, since $J^* \in I(L)$, $J^* \in A_0(L)$. Hence there exist $y \in L$ such that $J^* = (y)^*$. Again, since $P_1 \in m(I(L))$, $J \in P_1$, $J^{**} \in P_1$. Therefore $(y)^{**} \in P_1$, and hence $(y) \in P_1$. It follows that $y \in C(P_1) = C(P_2)$ and hence $(y) \in P_2$. Therefore $[y]^* \notin P_2$, since P_2 is a minimal prime ideal. Hence $J^* \notin P_2$. Therefore $J \in P_2$ and hence $P_1 \subseteq P_2$. Hence $P_1 = P_2$, since P_1, P_2 are minimal prime ideals. Therefore for each $Q \in M$, there exist unique $P \in m(I(L))$ such that $C(P)=Q$.

Conversely, suppose L is Quasi-complemented ADL and for each $Q \in M$, there exist unique $P \in m(I(L))$ such that $C(P)=Q$. By lemma 2.7, we have there exists a mapping $\phi: m(I(L)) \rightarrow M$ defined $\phi(P) = C(P)$ is a homeomorphism.

Let $J \in I(L)$. Then $h_M(J^*)$ is a both open and closed set in M . Hence $M - h_M(J^*)$ is compact (being closed subset of compact space M is compact). Therefore $M - h_M(J^*) = \bigcup_{i=1}^n M_{a_i} = M_{\bigvee_{i=1}^n a_i}$. Now, put $y = \bigvee_{i=1}^n a_i$.

Then $M - h_M(J^*) = M_y = h_M([y]^*) = M_y - h_M([y]^{**})$. Hence $h_M[J]^* = h_M([y]^{**})$. It follows that $J^{**} \in A_0(L)$.

Corollary: 3.11 Let L be an ADL with maximal element m in which dense element is maximal. Then the following are

- (1) M is compact, Hausdorff and extremally disconnected space.
- (2) The space M and $m(I(L))$ are homeomorphic.
- (3) $J^{**} \in A_0(L)$, for each $J \in I(L)$

Further any of the above conditions implies that L is quasi-complemented ADL

Theorem: 3.12 Let L be an ADL with maximal element m in which every dense element is maximal. Then the following are equivalent:

- (1) L is quasi-complemented ADL
- (2) $\tau_h^M = \tau_d^M$
- (3) M is compact in hull-kernel topology
- (4) $(B = \{M_x / x \in L\}, \cap, \cup)$ is a Boolean algebra.
- (5) L/θ is Boolean Lattice, where θ is the congruence relation on L and defined by
 $\theta = \{(x, y) \in L \times L / [x]^* = [y]^*\}$
- (6) $C(P) \in M$, for each $P \in m(I(L))$.

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