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AN ITERATIVE METHOD WITH QUADRATIC CONVERGENCE FOR NONLINEAR ILL-POSED PROBLEMS: FINITE-DIMENSIONAL REALIZATIONS

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ABSTRACT

An iterative regularization method which converges quadratically in the setting of a finite-dimensional subspace has been considered for obtaining stable approximate solution to nonlinear ill-posed operator equations $T(x) = y$. *The derived error estimate using an adaptive selection of the parameter in relation to the noise level are shown to be of optimal order with respect to certain natural assumptions on the ill-posed-ness of the equation. A stopping rule for the iteration index is provided. The results of computational experiments are provided which shows the reliability of our method.*

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1. INTRODUCTION:

We consider the problem of approximately solving the nonlinear ill-posed operator equation of the form

$$T(x) = y, \tag{1.1}$$

where $T : D(T) \subset X \to X$ is a nonlinear monotone operator and X is a real Hilbert space. We shall denote the inner product and the corresponding norm on X by $\langle \cdot,\cdot \rangle$ and $\| \cdot \|$ respectively.
Recall that T is monotone operator if it satisfies the relation

$$\langle T(x) - T(y), x - y \rangle \geq 0, \qquad \forall x, y \in D(T).$$

We assume that (1.1) has a solution, namely $x^{\dagger}$.

In application, usually only noisy data y^{δ} are available, such that

$$\| y - y^{\delta} \| \leq \delta. \tag{1.2}$$

Then the problem of recovery of $x^{\dagger}$ from noisy equation $T(x) = y^{\delta}$ is ill-posed, in the sense that a small perturbation in the data can cause large deviation in the solution.

Nonlinear ill-posed problems arise in a number of applications (see [4]). Since (1.1) is ill-posed, one has to replace the equation (1.1) by nearby equation whose solution is less sensitive to perturbation in the right side y. This replacement is known as regularization. A well known method for regularizing (1.1), when T is monotone is the method of Lavrentiev regularization (see [19]). In this method approximation x_{α}^{δ} is obtained by solving the singularly perturbed operator equation

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$$T(x)+\alpha(x-x_0)=y^\delta, \tag{1.3}$$

where $\alpha \geq 0$, called the regularization parameter, and x_0 is the initial guess for the solution $x^\dagger$.

In practice one has to deal with some sequence $(x_{n,\alpha}^\delta)$, converging to the solution x_α^δ of (1.3). Many authors considered such sequences (see [2, 3, 7, 8]). In [2] Bakushinsky and Smirnova considered an iteratively regularized Lavrentiev method:

$$x_{k+1}^\delta = x_k^\delta - (A_k^\delta + \alpha_\delta I)^{-1}(T(x_k^\delta) - y^\delta + \alpha_k(x_k^\delta - x_0)) \tag{1.4}$$

for $k = 0,1,2,\cdots$, where $A_k^\delta = T'(x_k^\delta)$ and (α_k) is a sequence of positive real numbers such that $\lim\limits_{k\to\infty}\alpha_k = 0$, as an approximate solution for (1.1). A general discrepancy principal, has been considered in [2] for choosing the stopping index k_δ and showed that $x_{k_\delta}^\delta \to x^\dagger$ as $\delta \to 0$. However no error estimate for $\left\|x_k^\delta - x^\dagger\right\|$ has been given in [2]. Later in [9], Mahale and Nair considered the method (1.4) and obtained an error estimate for $\left\|x_k^\delta - x^\dagger\right\|$, under weaker assumptions than the assumptions in [2] (see [9]).

In ([6]), George and Elmahdy considered an iterative regularization method;

$$x_{n+1,\alpha}^\delta = x_{n,\alpha}^\delta - (T'(x_{n,\alpha}^\delta)+\alpha I)^{-1}(T(x_{n,\alpha}^\delta) - y^\delta + \alpha(x_{n,\alpha}^\delta - x_0)), \tag{1.5}$$

where $x_{0,\alpha}^\delta = x_0$, as a modified iteratively regularized Lavrentiev method, and by using a majorizing sequence (see [1], page 28), proved that (1.5) converges quadratically to the unique solution x_α^δ of (1.3).

Recall that, a sequence (x_n) is said to be converges quadratically to x^* if there exist a positive number M, not necessarily less than 1, such that

$$\left\|x_{n+1} - x^*\right\| \leq M\left\|x_n - x^*\right\|^2,$$

for all n sufficiently large. And the convergence of (x_n) to x^* is said to be linear if there exist a positive number M_0, such that

$$\left\|x_{n+1} - x^*\right\| \leq M_0\left\|x_n - x^*\right\|.$$

Note that regardless of the value of M quadratic convergent sequence will always eventually converge faster than a linearly convergent sequence. For an extensive discussion of convergence rate see Ortega and Rheinboldt [11]. One of the advantage of the proposed method is that the analysis is based on a majorizing sequence and the stopping rule (which is different from the classical discrepancy principle (c.f., [2, 3, 8, 9]) is independent of the proposed method. More precisely, the proposed stopping rule, depends only on the choice of a real number $q \in (0,1)$ which depends on the starting point of the iteration. We provide an optimal order error estimate under a general source condition on $x_0 - x^\dagger$. Moreover we shall use the adaptive parameter selection procedure suggested by Pereverzev and Schock in [12], for choosing the regularization parameter α in $(x_{n,\alpha}^\delta)$.

In ([6]), George and Elmahdy proved that (1.5) converges quadratically to the unique solution x_α^δ of (1.3) under the following Assumptions:

Assumption: 1.1 There exists $r > 0$ such that $B_r(x_0) \cup B_r(x^\dagger) \subseteq D(T)$ and T is Frechet differentiable at all $x \in B_r(x_0) \cup B_r(x^\dagger)$.

Assumption: 1.2 There exists a constant $k_0 > 0$ such that for every $x, u \in B_r(x_0) \cup B_r(x^\dagger)$ and $v \in X$ there exists an element $\phi(x,u,v) \in X$ satisfying

$$[T'(x) - T'(u)]v = T'(u)\phi(x,u,v), \left\|\phi(x,u,v)\right\| \leq k_0\left\|v\right\|\left\|x-u\right\|.$$

Assumption: 1.3 There exists a continuous, strictly monotonically increasing function $\varphi:(0,a]\to(0,\infty)$ with $a \geq \|T'(x^{\dagger})\|$, satisfying $\lim_{\lambda\to 0}\varphi(\lambda)=0$ and there exist $v\in X$ with $\|v\|\leq 1$ such that $x_0 - x^{\dagger} = \varphi(T'(x^{\dagger}))v$ and $\sup_{\lambda\geq 0}\frac{\alpha\varphi(\lambda)}{\lambda+\alpha}\leq c_{\varphi}\varphi(\alpha), \qquad \forall\,\lambda\in(0,\mathrm{a}].$

The analysis in ([6]) as well as in this paper is based on a majorizing sequence.

Recall (see [1], Definition 1.3.11) that a nonnegative sequence (t_n) is said to be a majorizing sequence of a sequence (x_n) in X if

$$\|x_{n+1}-x_n\|\leq t_{n+1}-t_n \qquad \forall n\geq 0.$$

The main advantage of using the Assumption 1.2 is that, the majorizing sequence we are going to use in this paper is independent of the regularization parameter α. Further the majorizing sequence gives an a priori error estimate which can be used to determine the number of iterations needed to achieve a prescribed solution accuracy before actual computation take place.

In Section 2 we consider the sequence $(x_{n,\alpha}^{\delta})$ defined in (1.5), using Assumption 1.2 we proved that the sequence $(x_{n,\alpha}^{\delta})$ converges to x_{α}^{δ} and obtained an error estimate for $\|x_{n,\alpha}^{\delta}-x_{\alpha}^{\delta}\|$. In application, one looks for a sequence $(x_{n,\alpha}^{\delta})$ in a finite dimensional subspace X_h of X such that $x_{n,\alpha}^{h,\delta}\to x_{\alpha}^{\delta}$ as $h\to 0$ and $n\to\infty$. In Section 3 we considered an iteratively regularized projection method for obtaining a sequence $(x_{n,\alpha}^{h,\delta})$ in a finite dimensional subspace X_h of X and proved that $x_{n,\alpha}^{h,\delta}\to x_{\alpha}^{\delta}$. Also in Section 3 we obtained an estimate for $\|x_{n,\alpha}^{h,\delta}-x_{\alpha}^{\delta}\|$. Using an error estimate for $\|x_{\alpha}^{\delta}-x^{\dagger}\|$ (see [13]) we obtained an estimate for $\|x_{n,\alpha}^{h,\delta}-x^{\dagger}\|$ in Section 4. The error analysis for the order optimal result using an adaptive selection of the parameter α and a stopping rule using a majorizing sequence are also given in Section 4. Implementation of the adaptive choice of the parameter and the choice of the stopping rule are given in Section 5. Examples are given in Section 6. Finally the paper ends with some concluding remarks in Section 7.

2. CONVERGENCE ANALYSIS:

In [6] the following majorizing sequence (t_n) defined iteratively by, $t_0=0$, $t_1=\eta$, and

$$t_{n+1}=t_n+\frac{3k_0}{2}(t_n-t_{n-1})^2 \tag{2.6}$$

where k_0,η and $q\in[0,1)$ are nonnegative numbers such that

$$\frac{3k_0\eta}{2}q^n\leq q. \tag{2.7}$$

And

$$\omega:=\|T(x_0)-y^{\delta}\|\leq\eta\alpha, \tag{2.8}$$

where used for proving the quadratic convergence of the sequence $(x_{n,\alpha}^{\delta})$ to the unique solution x_{α}^{δ} of equation (1.3).

For proving the results in [6] as well as the results in this paper we use the following Lemma on majorization, which is a reformulation of Lemma 1.3.12 in [1].

Lemma: 2.1 Let (t_n) be a majorizing sequence for (x_n) in X. If $\lim_{n\to\infty}t_n=t^*$, then $x^*=\lim_{n\to\infty}x_n$ exists and

$$\|x^*-x_n\|\leq t^*-t_n \qquad \forall n\geq 0. \tag{2.9}$$

Throughout this paper we assume that the operator T satisfies the Assumptions 1.1, 1.2 and 1.3. The following Lemma is essentially a reformulation of a Lemma in [6].

Lemma: 2.2 Assume there exist nonnegative numbers $k_0, \eta,$ and $q \in [0,1)$ such that

$$\frac{3k_0\eta}{2} q^n \leq q. \tag{2.10}$$

Then the iteration (t_n), $n \geq 0$, given by $t_0 = 0, t_1 = \eta,$

$$t_{n+1} = t_n + \frac{3k_0}{2}(t_n - t_{n-1})^2 \tag{2.11}$$

is increasing, bounded above by $t^{**} := \frac{\eta}{1-q}$, and converges to some t^* such that $0 < t^* \leq \frac{\eta}{1-q}$. Moreover, for $n \geq 0$;

$$0 \leq t_{n+1} - t_n \leq q(t_n - t_{n-1}) \leq q^n \eta, \tag{2.12}$$

and

$$t^* - t_n \leq \frac{q^n}{1-q}\eta. \tag{2.13}$$

To prove the convergence of the sequence $(x_{n,\alpha}^\delta)$ defined in (1.5) we introduce the following notations:

$R_\alpha(x) = T'(x_0) + \alpha I$ and

$$G(x) := x - R_\alpha(x)^{-1}[T(x) - y^\delta + \alpha(x - x_0)]. \tag{2.14}$$

Note that with the above notation, $G(x_{n,\alpha}^\delta) = x_{n+1,\alpha}^\delta$ and

$$\left\| R_\alpha(x)^{-1} T'(x) \right\| \leq 1. \tag{2.15}$$

The following Lemma based on the Assumption 1.2 will be used in due course.

Lemma: 2.3 ([6] Lemma 2.3) For $u, v \in B_r(x_0)$

$$T(u) - T(v) - T'(u)(u - v) = T'(u)\int_0^1 \phi(v + t(u - v), u, u - v)dt.$$

Theorem: 2.4 Suppose $r \geq t^*$, and Assumption 1.2 and (2.8) hold. Also let the assumptions in Lemma 2.2 are satisfied. Then the sequence $(x_{n,\alpha}^\delta)$ defined in (1.5) is well defined and $x_{n,\alpha}^\delta \in B_{t^*}(x_0)$ for all $n \geq 0$. Further $(x_{n,\alpha}^\delta)$ is Cauchy sequence in $B_{t^*}(x_0)$ and hence converges to $x_\alpha^\delta \in \overline{B_{t^*}(x_0)} \subset B_{t^{**}}(x_0)$ and $T(x_\alpha^\delta) + \alpha(x_\alpha^\delta - x_0) = y^\delta$.

Moreover, the following estimate hold for all $n \geq 0$,

$$\left\| x_{n+1,\alpha}^\delta - x_{n,\alpha}^\delta \right\| \leq t_{n+1} - t_n, \tag{2.16}$$

$$\left\| x_{n,\alpha}^\delta - x_\alpha^\delta \right\| \leq t^* - t_n \leq \frac{q^n \eta}{1-q}. \tag{2.17}$$

And

$$\left\| x_{n+1,\alpha}^\delta - x_\alpha^\delta \right\| \leq \tfrac{k_0}{2} \left\| x_{n,\alpha}^\delta - x_\alpha^\delta \right\|^2. \tag{2.18}$$

Proof: First we shall prove that

$$\left\|x^{\delta}_{n+1,\alpha} - x^{\delta}_{n,\alpha}\right\| \le \tfrac{3k_0}{2}\left\|x^{\delta}_{n,\alpha} - x^{\delta}_{n-1,\alpha}\right\|^2. \tag{2.19}$$

With G as in (2.14), we have for $u, v \in B_{t^*}(x_0)$,

$$\begin{aligned}
G(u)-G(v) &= u-v-R_\alpha(u)^{-1}[T(u)-y^\delta+\alpha(u-x_0)]+R_\alpha(v)^{-1}[T(v)-y^\delta+\alpha(v-x_0)]\\
&= u-v-[R_\alpha(u)^{-1}-R_\alpha(v)^{-1}](T(v)-y^\delta+\alpha(v-x_0))\text{ - }R_\alpha(u)^{-1}(T(u)-T(v)+\alpha(u-v)\\
&= R_\alpha(u)^{-1}[T'(u)(u-v)-(T(u)-T(v))]\text{ - }R_\alpha(u)^{-1}[T'(u)^{-1}-T'(v)^{-1}]R_\alpha(v)^{-1}(T(v)-y^\delta+\alpha(v-x_0))\\
&= R_\alpha(u)^{-1}[T'(u)(u-v)-(T(u)-T(v))]\text{ - }R_\alpha(u)^{-1}[T'(u)^{-1}-T'(v)^{-1}](v-G(v))\\
&= R_\alpha(u)^{-1}[T'(u)(u-v)+\int_0^1 T'(u+t(v-u))(v-u)dt]\text{ - }R_\alpha(u)^{-1}[T'(u)^{-1}-T'(v)^{-1}](v-G(v))\\
&= \int_0^1 R_\alpha(u)^{-1}[T'(u)+t(v-u))-T'(u))(v-u)dt]\text{ - }R_\alpha(u)^{-1}[T'(u)^{-1}-T'(v)^{-1}](v-G(v))
\end{aligned}$$

The last, but one step follows from the Fundamental Theorem of Integral Calculus. So by Assumption 1.2 and the estimate (2.15), we have

$$\|G(u)-G(v)\| \le \frac{k_0}{\alpha}\|u-v\|^2 + k_0\|u\text{-}v\|\|v\text{-}G(v)\|. \tag{2.20}$$

Now by taking $u = x^{\delta}_{n,\alpha}$ and $v = x^{\delta}_{n-1,\alpha}$ in (2.20) , we obtain (2.19).

Next we shall prove that the sequence (t_n), $n \ge 0$ defined in Lemma 2.2 is a mojorizing sequence of the sequence $(x^{\delta}_{n,\alpha})$.

Note that $\left\|x^{\delta}_{1,\alpha} - x_0\right\| = \left\|R_\alpha(x_0)^{-1}(T(x_0)-y^\delta)\right\| \le \dfrac{\omega}{\alpha} < \eta = t_1 - t_0$, assume that $\left\|x^{\delta}_{i+1,\alpha} - x^{\delta}_{i,\alpha}\right\| \le t_{i+1} - t_i, \forall\, i \le k$ for some k. Then by (2.19),

$$\left\|x^{\delta}_{k+2,\alpha} - x^{\delta}_{k+1,\alpha}\right\| \le \frac{3k_0}{2}\left\|x^{\delta}_{k+1,\alpha} - x^{\delta}_{k,\alpha}\right\|^2 \le \frac{3k_0}{2}(t_{k+1}-t_k)^2 = t_{k+2} - t_{k+1}.$$

Thus by induction $\left\|x^{\delta}_{n+1,\alpha} - x^{\delta}_{n,\alpha}\right\| \le t_{n+1} - t_n$ for all $n \ge 0$ all and hence (t_n), $n \ge 0$ is a majorizing sequence of the sequence $(x^{\delta}_{n,\alpha})$. So by Lemma 2.1 $(x^{\delta}_{n,\alpha})$ $n \ge 0.$ is a Cauchy sequence and converges to some $x^{\delta}_{\alpha} \in \overline{B_{t^*}(x_0)} \subset B_{t^{**}}(x_0)$ and

$$\left\|x^{\delta}_{n,\alpha} - x^{\delta}_{\alpha}\right\| \le t^* - t_n \le \frac{q^n\eta}{1-q}.$$

To prove (2.18), we observe that $G(x^{\delta}_{\alpha}) = x^{\delta}_{\alpha}$, so (2.18) follows from (2.20), by taking $u = x^{\delta}_{n,\alpha}$ and $v = x^{\delta}_{\alpha}$. Now by letting $n\to\infty$ in (1.5) we obtain $T(x^{\delta}_{\alpha}) + \alpha(x^{\delta}_{\alpha} - x_0) = y^\delta$.
This completes the proof of the Theorem.

Remark: 2.5 Note that (2.18) implies $(x^{\delta}_{n,\alpha})$ converges quadratically to x^{δ}_{α}.

3. ITERATIVELY REGULARIZED PROJECTION METHOD:

Let H be a bounded subset of positive real such that zero is a limit point of H, and let $\{P_h\}, h\in H$ be a family of orthogonal projections from X into itself. We assume that

$$b_h := \|(I-P_h)x_0\| \to 0 \tag{3.21}$$

as $h \to 0.$ The above assumption is satisfied if $P_h \to I$ pointwise. Let

$$x_{n+1,\alpha}^{h,\delta} = x_{n,\alpha}^{h,\delta} - (P_h T'(x_{n,\alpha}^{h,\delta}) + \alpha I)^{-1} P_h (T(x_{n,\alpha}^{h,\delta}) - y^\delta + \alpha(x_{n,\alpha}^{h,\delta} - x_0)), \tag{3.22}$$

where $x_{0,\alpha}^{h,\delta} := P_h x_0$. Then

$$\|x_{1,\alpha}^{h,\delta} - P_h x_0\| = \|(P_h T'(x_{0,\alpha}^{h,\delta}) + \alpha I)^{-1} P_h (T(x_{0,\alpha}^{h,\delta}) - y^\delta)\| \le \frac{\|T(P_h x_0) - y^\delta\|}{\alpha} \le \eta_h.$$

Hereafter we assume that,

$$\frac{3k_0}{2} q^n \tilde{\eta} \le q. \tag{3.23}$$

and let $\tilde{\eta} = \max\{\eta, \eta_h\}$.

As in Section 2 one can prove that, the sequence $(t_n),\ n \ge 0$ is a majorizing sequence of the sequence $(x_{n,\alpha}^{h,\delta}).$ Hence $x_{n,\alpha}^{h,\delta} \in B_{t^*}(P_h x_0)$ where $t^* \le \dfrac{\tilde{\eta}}{1-q}$, so

$$\|x_{n,\alpha}^{h,\delta} - P_h x_0\| \le \frac{\tilde{\eta}}{1-q}. \tag{3.24}$$

To obtain an error estimate $\|x_{n,\alpha}^{h,\delta} - x_{n,\alpha}^{\delta}\|$, we observe that

$$\begin{aligned}
k_0\|x_{n-1,\alpha}^{h,\delta} - x_{n-1,\alpha}^{\delta}\| &= k_0\|x_{n-1,\alpha}^{h,\delta} - P_h x_0 + (P_h - I)x_0 + x_0 - x_{n-1,\alpha}^{\delta}\| \\
&\le k_0[\|x_{n-1,\alpha}^{h,\delta} - P_h x_0\| + \|(P_h - I)x_0\| + \|x_0 - x_{n-1,\alpha}^{\delta}\|] \\
&\le k_0[\frac{\tilde{\eta}}{1-q} + b_h + \frac{\eta}{1-q}] \\
&\le k_0[\frac{\tilde{\eta}}{1-q} + b_h + \frac{\tilde{\eta}}{1-q}] \\
&\le k_0[\frac{2\tilde{\eta}}{1-q} + b_h] \\
&\le [\frac{4q}{3(1-q)} + k_0 b_h] \\
&= Q.
\end{aligned} \tag{3.25}$$

Theorem: 3.1 Let $x_{n,\alpha}^{h,\delta}$ be as in (3.22) and $x_{n,\alpha}^{\delta}$ be as in (1.5). Let Assumptions in Theorem 2.4 and (3.23) hold. Then we have the following estimate,

$$\|x_{n,\alpha}^{h,\delta} - x_{n,\alpha}^{\delta}\| \le (\frac{Q}{2})^n b_h + (Q + b_h)\frac{(\frac{Q}{2})^n}{(\frac{Q}{2}) - q}\eta.$$

Proof:

$$x_{n,\alpha}^{h,\delta} - x_{n,\alpha}^{\delta} = x_{n-1,\alpha}^{h,\delta} - x_{n-1,\alpha}^{\delta} - (P_h T'(x_{n-1,\alpha}^{h,\delta}) + \alpha I)^{-1} P_h (T(x_{n-1,\alpha}^{h,\delta}) - y^\delta + \alpha(x_{n-1,\alpha}^{h,\delta} - x_0))$$

$$
\begin{aligned}
&+ (T'(x^{\delta}_{n-1,\alpha}) + \alpha I)^{-1}(T(x^{\delta}_{n-1,\alpha}) - y^{\delta} + \alpha(x^{\delta}_{n-1,\alpha} - x_0))\\
&= x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha} - [(P_hT'(x^{h,\delta}_{n-1,\alpha}) + \alpha I)^{-1}P_h - (T'(x^{\delta}_{n-1,\alpha}) + \alpha I)^{-1}\,(T(x^{h,\delta}_{n-1,\alpha}) - y^{\delta} + \alpha(x^{h,\delta}_{n-1,\alpha} - x_0))\\
&\quad - (P_hT'(x^{h,\delta}_{n-1,\alpha}) + \alpha I)^{-1}P_h(T(x^{h,\delta}_{n-1,\alpha}) - y^{\delta} + \alpha(x^{h,\delta}_{n-1,\alpha} - x_0)) - (T(x^{\delta}_{n-1,\alpha}) + \alpha(x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha}))\\
&= (P_hT'(x^{h,\delta}_{n-1,\alpha}) + \alpha I)^{-1}P_h[T'(x^{h,\delta}_{n-1,\alpha})(x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha}) - (T(x^{h,\delta}_{n-1,\alpha}) - T(x^{\delta}_{n-1,\alpha}))]\\
&\quad - [(P_hT'(x^{h,\delta}_{n-1,\alpha})P_h + \alpha P_h)^{-1} - (T'(x^{\delta}_{n-1,\alpha}) + \alpha I)^{-1}][T(x^{\delta}_{n-1,\alpha}) - y^{\delta} + \alpha(x^{\delta}_{n-1,\alpha} - x_0)]\\
&= \Gamma_1 - \Gamma_2
\end{aligned}
\tag{3.26}
$$

Where

$$\Gamma_1 = (P_hT'(x^{h,\delta}_{n-1,\alpha}) + \alpha I)^{-1}P_h[T'(x^{h,\delta}_{n-1,\alpha})(x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha}) - (T(x^{h,\delta}_{n-1,\alpha}) - T(x^{\delta}_{n-1,\alpha}))]$$

and

$$
\begin{aligned}
\Gamma_2 &= [(P_hT'(x^{h,\delta}_{n-1,\alpha})P_h + \alpha P_h)^{-1} - (T'(x^{\delta}_{n-1,\alpha}) + \alpha I)^{-1}][T(x^{\delta}_{n-1,\alpha}) - y^{\delta} + \alpha(x^{\delta}_{n-1,\alpha} - x_0)]\\
&= [(P_hT'(x^{h,\delta}_{n-1,\alpha})P_h + \alpha P_h)^{-1}[P_h(T'(x^{\delta}_{n-1,\alpha}) - P_hT'(x^{h,\delta}_{n-1,\alpha})P_h][P_h(T'(x^{\delta}_{n-1,\alpha}) + \alpha I)^{-1}]\\
&\qquad [T(x^{\delta}_{n-1,\alpha}) - y^{\delta} + \alpha(x^{\delta}_{n-1,\alpha} - x_0)]\\
&= (P_hT'(x^{h,\delta}_{n-1,\alpha})P_h + \alpha P_h)^{-1}P_h[(T'(x^{\delta}_{n-1,\alpha}) - T'(x^{h,\delta}_{n-1,\alpha})P_h](x^{\delta}_{n,\alpha} - x^{\delta}_{n-1,\alpha})\\
&= (P_hT'(x^{h,\delta}_{n-1,\alpha})P_h + \alpha P_h)^{-1}P_h[(T'(x^{\delta}_{n-1,\alpha}) - T'(x^{h,\delta}_{n-1,\alpha})](x^{\delta}_{n,\alpha} - x^{\delta}_{n-1,\alpha})\\
&\quad + (P_hT'(x^{h,\delta}_{n-1,\alpha})P_h + \alpha P_h)^{-1}P_h(T'(x^{h,\delta}_{n-1,\alpha})(I - P_h)(x^{\delta}_{n,\alpha} - x^{\delta}_{n-1,\alpha})\\
&= (P_hT'(x^{h,\delta}_{n-1,\alpha}) + \alpha I)^{-1}P_h(T'(x^{h,\delta}_{n-1,\alpha})\phi(x^{\delta}_{n-1,\alpha}, x^{h,\delta}_{n-1,\alpha}, x^{\delta}_{n,\alpha} - x^{\delta}_{n-1,\alpha})\\
&\quad + (P_hT'(x^{h,\delta}_{n-1,\alpha})P_h + \alpha P_h)^{-1}P_hT'(x^{h,\delta}_{n-1,\alpha})(I - P_h)(x^{\delta}_{n,\alpha} - x^{\delta}_{n-1,\alpha})
\end{aligned}
$$

Thus by Lemma 2.3 and Assumption 1.2 we have

$$
\begin{aligned}
\|\Gamma_1\| &\le \left\|(P_hT'(x^{h,\delta}_{n-1,\alpha}) + \alpha I)^{-1}P_hT'(x^{h,\delta}_{n-1,\alpha})\int_0^1 \phi(x^{\delta}_{n-1,\alpha} + t(x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha}), x^{h,\delta}_{n-1,\alpha}, x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha})dt\right\|\\
&\le k_0\int_0^1 \left\|x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha}\right\|\left\|x^{\delta}_{n-1,\alpha} + t(x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha}) - x^{h,\delta}_{n-1,\alpha}\right\|dt\\
&\le k_0\int_0^1 \left\|x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha}\right\|\left\|(t-1)(x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha})\right\|dt\\
&\le \frac{k_0}{2}\left\|x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha}\right\|^2.
\end{aligned}
\tag{3.27}
$$

and

$$
\begin{aligned}
\|\Gamma_2\| &\le \left\|(P_hT'(x^{h,\delta}_{n-1,\alpha}) + \alpha I)^{-1}P_h(T'(x^{h,\delta}_{n-1,\alpha})\phi(x^{\delta}_{n-1,\alpha}, x^{h,\delta}_{n-1,\alpha}, x^{\delta}_{n,\alpha} - x^{\delta}_{n-1,\alpha})\right\|\\
&\quad \left\| + (P_hT'(x^{h,\delta}_{n-1,\alpha})P_h + \alpha P_h)^{-1}P_hT'(x^{h,\delta}_{n-1,\alpha})(I - P_h)(x^{\delta}_{n,\alpha} - x^{\delta}_{n-1,\alpha})\right\|\\
&\le k_0\left\|x^{\delta}_{n,\alpha} - x^{\delta}_{n-1,\alpha}\right\|\left\|x^{\delta}_{n-1,\alpha} - x^{h,\delta}_{n-1,\alpha}\right\| + \left\|(I - P_h)\right\|\left\|(x^{\delta}_{n,\alpha} - x^{\delta}_{n-1,\alpha})\right\|\\
&\le k_0\left\|x^{\delta}_{n,\alpha} - x^{\delta}_{n-1,\alpha}\right\|\left\|x^{\delta}_{n-1,\alpha} - x^{h,\delta}_{n-1,\alpha}\right\| + b_h\left\|(x^{\delta}_{n,\alpha} - x^{\delta}_{n-1,\alpha})\right\|\\
&\le (Q + b_h)\left\|x^{\delta}_{n,\alpha} - x^{\delta}_{n-1,\alpha}\right\|.
\end{aligned}
\tag{3.28}
$$

Therefore by (3.26), (3.27) and (3.28) we have

$$\left\|x^{h,\delta}_{n,\alpha} - x^{\delta}_{n,\alpha}\right\| \le \frac{k_0}{2}\left\|x^{h,\delta}_{n-1,\alpha} - x^{\delta}_{n-1,\alpha}\right\|^2 + (Q + b_h)\left\|x^{\delta}_{n,\alpha} - x^{\delta}_{n-1,\alpha}\right\|$$

$$
\begin{aligned}
&\le \frac{Q}{2}\left\|x^{h,\delta}_{n-1,\alpha}-x^{\delta}_{n-1,\alpha}\right\|+(Q+b_h)\left\|x^{\delta}_{n,\alpha}-x^{\delta}_{n-1,\alpha}\right\| \\
&\le (\frac{Q}{2})^n\left\|x^{h,\delta}_{0,\alpha}-x^{\delta}_{0,\alpha}\right\|+(\frac{Q}{2})^{n-1}(Q+b_h)\eta+(\frac{Q}{2})^{n-2}(Q+b_h)\eta q+\cdots+(Q+b_h)\eta q^{n-1} \\
&\le (\frac{Q}{2})^n b_h+(\frac{Q}{2})^{n-1}(Q+b_h)\eta+(\frac{Q}{2})^{n-2}(Q+b_h)\eta q+\cdots+(Q+b_h)\eta q^{n-1} \\
&\le (\frac{Q}{2})^n b_h+(Q+b_h)[(\frac{Q}{2})^{n-1}\eta+(\frac{Q}{2})^{n-2}q+\cdots+q^{n-1}]\eta \\
&\le (\frac{Q}{2})^n b_h+(Q+b_h)\frac{(\frac{Q}{2})^n}{(\frac{Q}{2})-q}\eta.
\end{aligned}
$$

With $2q < Q < 2$

This completes the proof.

4. ERROR BOUNDS UNDER SOURCE CONDITIONS:

To obtain an error estimate for $\left\|x^{h,\delta}_{n,\alpha}-x^{\dagger}\right\|$ it is enough to obtain an error estimate for $\left\|x^{\delta}_{\alpha}-x^{\dagger}\right\|$. To obtain an error estimate for $\left\|x^{\delta}_{\alpha}-x^{\dagger}\right\|$ we use the error estimate for $\left\|x^{\delta}_{\alpha}-x_{\alpha}\right\|$ and $\left\|x_{\alpha}-x^{\dagger}\right\|$ where x_α is the unique solution of the equation $T(x)+\alpha(x-x_0)=y$. It is known (cf. [13] Proposition 3.1) that

$$\left\|x^{\delta}_{\alpha}-x_{\alpha}\right\| \le \frac{\delta}{\alpha} \tag{4.29}$$

and (cf. [6] Theorem 3.1) that

$$\left\|x_{\alpha}-x^{\dagger}\right\| \le (k_0 r+1)c_{\varphi}\varphi(\alpha). \tag{4.30}$$

Combining the estimates in Theorem 2.4 and Theorem 3.1, (4.29) and (4.30) we obtaining the following,

Theorem: 4.1 Let $x^{h,\delta}_{n,\alpha}$ be as in (3.22) and let the assumptions in Theorem 2.4 and Theorem 3.1 be satisfied. Then we have the following:

$$
\begin{aligned}
\left\|x^{h,\delta}_{n,\alpha}-x^{\dagger}\right\| &\le \left\|x^{h,\delta}_{n,\alpha}-x^{\delta}_{n,\alpha}\right\|+\left\|x^{\delta}_{n,\alpha}-x^{\delta}_{\alpha}\right\|+\left\|x^{\delta}_{\alpha}-x_{\alpha}\right\|+\left\|x_{\alpha}-x^{\dagger}\right\| \\
&\le (\frac{Q}{2})^n b_h+(Q+b_h)\frac{(\frac{Q}{2})^n}{(\frac{Q}{2})-q}\eta+\frac{q^n\eta}{1-q}+\frac{\delta}{\alpha}+(k_0 r+1)c_{\varphi}\varphi(\alpha).
\end{aligned} \tag{4.31}
$$

Let

$$n_\delta := \min\{n : \max\{(\frac{Q}{2})^n b_h, Q\frac{(\frac{Q}{2})^n}{(\frac{Q}{2})-q}\eta, +b_h\frac{(\frac{Q}{2})^n}{(\frac{Q}{2})-q}\eta, \frac{q^n\eta}{1-q}\} \le \frac{\delta}{\alpha}\}. \tag{4.32}$$

Theorem: 4.2 Let $x^{h,\delta}_{n,\alpha}$ be as in (3.22) and let the assumptions in Theorem 2.4 and Theorem 3.1 be satisfied. Let n_δ be as in (4.32). Then we have the following:

$$\left\|x^{h,\delta}_{n,\alpha}-x^{\dagger}\right\| \le \max\{5,(k_0 r+1)c_{\varphi}\}(\varphi(\alpha)+\frac{\delta}{\alpha}). \tag{4.33}$$

4.1. A PRIORI CHOICE OF THE PARAMETER:

Note that the error estimate $\varphi(\alpha)+\dfrac{\delta}{\alpha}$ in (4.33) is of optimal order if $\alpha := \alpha_\delta$ satisfies, $\varphi(\alpha_\delta)\alpha_\delta = \delta$. Now using the function $\psi(\lambda) := \lambda\varphi^{-1}(\lambda)$, $0 < \lambda \le a$ we have $\delta = \alpha_\delta\varphi(\alpha_\delta) = \psi(\varphi(\alpha_\delta))$, so that $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$.

In view of the above observations we have the following.

Theorem: 4.4: Let $\psi(\lambda) := \lambda\varphi^{-1}(\lambda)$ for $0 < \lambda \le a$ and assumptions in Theorem 4.2 holds. For $\delta > 0$, let $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$ and let n_δ be as in (4.32) then

$$\left\| x^{\delta}_{n_\delta,\alpha} - x^{\dagger} \right\| = O(\psi^{-1}(\delta)).$$

4.2 AN ADAPTIVE CHOICE OF THE PARAMETER:

Now, we will present a parameter choice rule based on the adaptive method studied in [10, 12].
In practice, the regularization parameter α is often selected from some finite set

$$D_M(\alpha) := \{\alpha_i = \mu^i\alpha_0, i = 0,1,\cdots,M\} \tag{4.34}$$

Where $\mu > 1$ and M is big enough but not too large.

Let

$$n_M := \min\{n : \max\{(\frac{Q}{2})^n b_h, Q\frac{(\frac{Q}{2})^n}{(\frac{Q}{2})-q}\eta, + b_h\frac{(\frac{Q}{2})^n}{(\frac{Q}{2})-q}\eta, \frac{q^n\eta}{1-q}\} \le \frac{\delta}{\alpha_M}\}. \tag{4.35}$$

Then since $\dfrac{\delta}{\alpha_M} \le \dfrac{\delta}{\alpha_i}$ for $i = 0, 1, \cdots, M$, we have

$$\left\| x^{h,\delta}_{n_M,\alpha_i} - x^{\delta}_{\alpha_i} \right\| \le 4\frac{\delta}{\alpha_i}), \qquad \forall\ i = 0,\ 1,\ \cdots,\ M. \tag{4.36}$$

Let $x_i := x^{h,\delta}_{n_M,\alpha_i}$. The parameter choice strategy that we are going to consider in this paper, we select $\alpha = \alpha_i$ from $D_M(\alpha)$ and operates only with corresponding x_i, $i = 0,\ 1,\ \cdots,\ M.$

Theorem: 4.4 Assume that there exists $i \in \{0,\ 1,\ \cdots,\ M\}$ such that $\varphi(\alpha_i) \le \dfrac{\delta}{\alpha_i}$. Let assumptions of Theorem 4.2 and Theorem 4.3 and (4.36) hold and let

$$l := \max\{i : \varphi(\alpha_i) \le \frac{\delta}{\alpha_i}\} < M\ ,$$

$$k := \max\{i : \left\|x_i - x_j\right\| \le 2(5 + (k_0 r + 1)c_\varphi)\frac{\delta}{\alpha_j}, \qquad j = 0,1,\cdots,i\}. \tag{4.37}$$

Then $l \le k$ and

$$\left\|x^{\dagger} - x_k\right\| \le c\,\psi^{-1}(\delta).$$

where $c = 3(5 + (k_0 r + 1)c_\varphi)\mu.$

Proof: To see that $l \le k$, it is enough to show that, for each $i \in \{1,\ \cdots,\ M\}$,

$$\varphi(\alpha_i) \le \frac{\delta}{\alpha_i} \Rightarrow \left\| x_i - x_j \right\| \le 2(5 + (k_0 r + 1)c_\varphi)\frac{\delta}{\alpha_j}, \qquad \forall j = 0,1,\cdots,i.$$

For $j \le i$, by (4.30) and (4.36) we have

$$\begin{aligned}
\left\| x_i - x_j \right\| &\le \left\| x_i - x^\dagger \right\| + \left\| x^\dagger - x_j \right\| \\
&\le (k_0 r + 1)c_\varphi \varphi(\alpha_i) + 5\frac{\delta}{\alpha_i} + (k_0 r + 1)c_\varphi \varphi(\alpha_j) + 5\frac{\delta}{\alpha_j} \\
&\le (5 + (k_0 r + 1)c_\varphi)\frac{\delta}{\alpha_i} + (5 + (k_0 r + 1)c_\varphi)\frac{\delta}{\alpha_i} \\
&\le 2(5 + (k_0 r + 1)c_\varphi)\frac{\delta}{\alpha_j}.
\end{aligned}$$

Thus the relation $l \le k$ is proved. Next we observe that

$$\begin{aligned}
\left\| x^\dagger - x_k \right\| &\le \left\| x^\dagger - x_l \right\| + \left\| x_l - x_k \right\| \\
&\le (k_0 r + 1)c_\varphi \varphi(\alpha_l) + 5\frac{\delta}{\alpha_l} + 2(5 + (k_0 r + 1)c_\varphi)\frac{\delta}{\alpha_l} \\
&\le 3(5 + (k_0 r + 1)c_\varphi)\frac{\delta}{\alpha_l}.
\end{aligned}$$

Now since $\alpha_\delta \le \alpha_{l+1} \le \mu\alpha_l$, it follows that

$$\frac{\delta}{\alpha_l} \le \mu\frac{\delta}{\alpha_\delta} = \mu\varphi(\alpha_\delta) = \mu\psi^{-1}(\delta).$$

Thus

$$\begin{aligned}
\left\| x^\dagger - x_k \right\| &\le 3(5 + (k_0 r + 1)c_\varphi)\mu\psi^{-1}(\delta) \\
&\le c\,\psi^{-1}(\delta).
\end{aligned}$$

This completes the proof of the theorem.

5. IMPLEMENTATION OF ADAPTIVE CHOICE RULE:

Here we provide an algorithm for the determination of a parameter fulfilling the balancing principle (4.37) and also provide a starting point for the iteration (3.22) approximating the unique solution x_α^δ of (1.3). The choice of the starting point involves the following five steps:

- Choose $\alpha_0 := \sqrt{\delta}$ and $\mu > 1$.
- Choose $\eta > 0$ and $q < 1$ such that $\frac{3k_0}{2}\eta \le q$.
- Choose $\eta_h > 0$ and $q < 1$ such that $\frac{3k_0}{2}\eta_h \le q$.
- Choose $\tilde{\eta} = \max\{\eta, \eta_h\}$ and $q < 1$ such that $\frac{3k_0}{2}\tilde{\eta} \le q$.
- Choose $x_0 \in D(T)$ such that $\left\| T(x_0) - y^\delta \right\| \le \tilde{\eta}\alpha_0$.

The choice of the stopping index n_M involves the following two steps:

- Choose the parameter $\alpha_M = \mu^M \alpha_0$ big enough with $\mu > 1$, not too large.

- Choose n_M such that $n_M := \min\{n : \max\{(\frac{Q}{2})^n b_h, Q\frac{(\frac{Q}{2})^n}{(\frac{Q}{2})-q}\eta, +b_h\frac{(\frac{Q}{2})^n}{(\frac{Q}{2})-q}\eta, \frac{q^n\eta}{1-q}\} \le \frac{\delta}{\alpha_M}\}$.

Finally the adaptive algorithm associated with the choice of the parameter specified in Theorem 4.4 involves the following steps:

5.1. ALGORITHM:

- Set $i \leftarrow 0$
- Solve $x_i := x^{h,\delta}_{n_M,\alpha_i}$ by using the iteration (3.22).
- If $\|x_i - x_j\| \le 2(5+(k_0r+1)c_\varphi)\frac{\sqrt{\delta}}{\mu^j}$, $\quad j \le i$, then take $k = i-1$.
- Set $i = i+1$ and return to step 2**.**

6. EXAMPLES:

In this section we consider examples satisfying the assumptions made in the paper.

We consider the operator $T : L^2[0,1] \to L^2[0,1]$ defined by

$$T(x)(s) = K^*K(s) + f(s), x, f \in L^2[0,1], s \in [0,1]. \tag{6.38}$$

where $K : L^2[0,1] \to L^2[0,1]$ is a compact linear operator such that the range of K denoted by $R(K)$ is not closed in $L^2[0,1]$. Then the equation $T(x) = y$ is ill-posed as K is compact with non-closed range, the Frechet derivative $T'(x)$ of T is given by

$$T'(x)z = K^*Kz, x, z \in L^2[0,1]. \tag{6.39}$$

So, T is monotone on $L^2[0,1]$. Further for $x, y, z \in L^2[0,1]$

$$[T'(x) - T'(y)]z = 0 \le k_0\|x-y\|, \forall k_0 \ge 0. \tag{6.40}$$

Hence Assumption 1.2 follows trivially. Again note that, since (6.40) holds for any $k_0 \ge 0$ we can choose any $\tilde{\eta} \ge 0$ in step 2 of the algorithm.

Further we observe that, due to (6.39) the iteration $x^{h,\delta}_{m+1,\alpha}$ needs only one step to compute. This can be seen as follows:

$$x^{h,\delta}_{m+1,\alpha} = x^{h,\delta}_{m,\alpha} - (P_hT'(x^{h,\delta}_{m,\alpha}) + \alpha I)^{-1}P_h(T(x^{h,\delta}_{m,\alpha}) - y^\delta + \alpha(x^{h,\delta}_{m,\alpha} - x_0))$$

$$\begin{aligned}(P_hT'(x^{h,\delta}_{m,\alpha}) + \alpha I)^{-1}P_hx^{h,\delta}_{m+1,\alpha} &= (P_hT'(x^{h,\delta}_{m,\alpha}) + \alpha I)^{-1}P_hx^{h,\delta}_{m,\alpha} - P_h(T(x^{h,\delta}_{m,\alpha}) - y^\delta + \alpha(x^{h,\delta}_{m,\alpha} - x_0)) \\ &= (P_hK^*K + \alpha I)^{-1}P_hx^{h,\delta}_{m,\alpha} - P_h(K^*Kx^{h,\delta}_{m,\alpha} + f(s) - y^\delta + \alpha(x^{h,\delta}_{m,\alpha} - x_0)) \\ &= P_h(f(s) - y^\delta - \alpha x_0). \end{aligned} \tag{6.41}$$

Now we shall give the details for implementing the algorithm given in the above Section. Let (V_n) be a sequence of finite dimensional subspaces of X and let $P_h, h = 1/n$ denote the orthogonal projection on X with range $R(P_h) = V_n$. We assume that $\dim V_n = n+1$, and $\|P_hx - x\| \to 0$ as $h \to 0$ for all $x \in X$. Let $\{v_1, v_2, \cdots, v_{n+1}\}$ be a basis of $V_n, n = 1,2,\cdots$.

Note that $x_{m+1,\alpha}^{h,\delta} \in V_n$. Thus $x_{m+1,\alpha}^{h,\delta}$ is of the form $\sum_{i=1}^{n+1} \lambda_i v_i$ for some scalars $\lambda_1, \lambda_2, \cdots, \lambda_{n+1}$. It can be seen that $x_{m+1,\alpha}^{h,\delta}$ is a solution of (6.41) if and only if $\overline{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_{n+1})^T$ is the unique solution of

$$(M_n + \alpha B_n)\overline{\lambda} = \overline{a} \tag{6.42}$$

where

$$M_n = (\langle Kv_i, Kv_j \rangle), \qquad i, j = 1,2,3,\cdots,n+1$$
$$B_n = (\langle v_i, v_j \rangle), \qquad i, j = 1,2,3,\cdots,n+1$$

and

$$\overline{a} = (\langle P_h(y^\delta + \alpha x_0 - f(s)), vi \rangle)^T, \qquad i = 1,2,\cdots,n+1$$

Note that (6.42) is uniquely solvable because M_n is a positive definite matrix (i.e. $xM_n x^T > 0$ for all non-zero vector x) and B_n is an invertible matrix.

7. CONCLUDING REMARKS:

In this paper we consider the problem of approximately solving the nonlinear ill-posed operator equation $T(x) = y$, and an iterative method in the finite dimensional setting when the available noisy data y^δ in place of the exact data y.

We assumed that T is Frechet differentiable at all $x \in B_r(x_0) \cup B_r(x^\dagger)$.

The procedure involves finding the fixed point of the function

$$G(x) = x - (P_h T'(x_0) + \alpha I)^{-1} P_h (T(x) - y^\delta + \alpha(x - x_0)),$$

in an iterative manner in a finite dimensional subspace X_h of X. Here x_0 is an initial guess and P_h is the orthogonal projection onto X_h. The error analysis for the order optimal result using an adaptive selection of the parameter α and a stopping rule using a majorizing sequence we made use the adaptive method suggested by Pereversev and Schock in [12].

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REFERENCE:

[1] Argyros, I. K., Convergence and applications of Newton-type iterations, (2008) Springer.

[2] Bakushinsky, A. and Smirnova, A., On application of generalized discrepancy principle to iterative methods for nonlinear ill-posed problems, Numer. Funct. Anal. And Optimiz. 26(1), (2005), 35 - 48.

[3] Deuflhard, P., Engl, H. W. and Scherzer, O., A convergence analysis of iterative methods for the solution of nonlinear ill-posed problems under affinely invariant conditions, Inverse Problems, 14(5), (1998), 1081 - 1106.

[4] Engl, H. W., Hanke, M. and Neubauer, A., Regularization of inverse problems, Mathematics and its Applications, 375. Kluwer Academic Publishers Group, Dordrecht, viii+321pp. (1996). Dordrecht, Boston, London.

[5] George, S. and Elmahdy, A. I. ,An analysis of Lavrentiev regularization for nonlinear ill-posed problems using an iterative regularization method, Int.J.Comput.Appl.Math., 5(3), (2010) ,369 - 381.

[6] George, S. and Elmahdy, A. I., A quadratic convergence yielding iterative method for nonlinear ill-posed operator equations", CMAM, (2011) (To appear).

[7] Hanke, M., Neubauer, A. and Scherzer, O, A convergence analysis of the Landweber iteration for nonlinear ill-posed problems, Numer.Math. 72(1), (1995), 21 - 37.

[8] Jin.Qi-Nian, On the iteratively regularized Gauss-Newton method for solving nonlinear ill-posed problems, Mathematics of Computation, 69(232), (2000), 1603 - 1623.

[9] Mahale. P. and Nair. M. T., Iterated Lavrentiev regularization for nonlinear ill-posed problems, ANZIAM Journal, 51, (2009), 191 -217.

[10] Mathe, P. and Perverzev, S. V., Geometry of linear ill-posed problems in variable Hilbert scales, Inverse Problems, 19(3), (2003), 789 - 803.

[11] Ortega, J. M. and Rheinboldt, W. C., Iterative solution of nonlinear equations in general variables, (1970), Academic Press, New York.

[12] Perverzev, S. V. and Schock, E., On the adaptive selection of the parameter in regularization of ill-posed problems, SIAM J. Numer. Anal., 43(5), (2005), 2060 - 2076.

[13] Tautanhahn, U., On the method of Lavrentiev regularization for nonlinear ill-posed problems, Inverse Problems, 18(1), (2002), 191 - 207.

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