

Note on δ –semiopen sets

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ABSTRACT

We characterize preopen sets via δ – semiopen sets. Also, we characterize spaces which are s – closed, semiconnected, semi – T_2 , s – Urysohn, s – regular, semi – regular, s –normal and semi – normal by δ – semiopen sets.

Key words and Phrases: δ – semiopen, δ – open, semiopen, preopen, b – open and β – open sets, δ – semiclosure and δ – semiinterior, s – closed, semiconnected, semi – T_2 , s – Urysohn, s – regular, semi – regular, s – normal and semi – normal spaces.

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1. INTRODUCTION AND PRELIMINARIES:

In 1997, Park, Lee and Son [9] have introduced and studied δ – semiopen sets in topological spaces. Also, in 1997, Á.Császár [4] have introduced and studied generalized open subsets of a set X defined in terms of monotonic functions $\gamma: \wp(X) \rightarrow \wp(X)$. The family $\mu = \{A \subset X \mid A \subset \gamma(A)\}$ is called the family of γ – open sets which is closed under arbitrary union and $\emptyset \in \mu$. μ is called a *generalized topology*. The family of δ – semiopen sets in a space is a particular kind of generalized topology but is different from the well known family of generalized open sets, namely semiopen sets, preopen sets, b – open sets and β – open sets. In this paper, we characterize spaces which are s – closed, semiconnected, semi – T_2 , s – Urysohn, s – regular, semi – regular, s – normal and semi – normal by δ – semiopen sets.

By a space X , we will always mean the topological space (X, τ) . A subset A of a space X is said to be *regular open* if $A = \text{int}(\text{cl}(A))$ where *int* and *cl* are the *interior* and *closure* operators in the space X . The family of all regular open sets is a base for a topology τ_s coarser than τ , which is called the *semiregularization* of the topology τ . Elements of τ_s are called δ – open sets. δint and δcl are the *interior* and *closure* operators in (X, τ_s) . A space (X, τ) is said to be *semiregular* if $\tau = \tau_s$. A subset A of a space X is said to be α – open [15](resp. *semiopen* [10], *preopen* [14], b – open [1], β – open [2]) if $A \subset \text{int}(\text{cl}(\text{int}(A)))$ (resp. $A \subset \text{cl}(\text{int}(A))$, $A \subset \text{int}(\text{cl}(A))$, $A \subset \text{int}(\text{cl}(A)) \cup \text{cl}(\text{int}(A))$, $A \subset \text{cl}(\text{int}(\text{cl}(A)))$). A subset A of a space X is said to be α – closed (resp. *semiclosed*, *preclosed*, b – closed, β – closed) if $X - A$ is α – open (resp. semiopen, preopen, b – open, β – open). The family of all α – open (resp. semiopen, preopen, b – open, β – open) of a space will be denoted by $\tau^\alpha(X)$, $\sigma(X)$, $\pi(X)$, $b(X)$ and $\beta(X)$. The closure and interior operators of these families are denoted by c_α , i_α , c_σ , i_σ , c_π , i_π , c_b , i_b and c_β , i_β respectively. A subset A of a space X is said to be δ – semiopen [18] if $A \subset \text{cl}(\delta\text{int}(A))$. A is said to be δ – semiclosed if $X - A$ is δ – semiopen. We will denote the family of all δ – semiopen sets by $\xi(X)$. For any subset A of X , the δ – semiinterior of A , denoted by $i_\xi(A)$, is given by $i_\xi(A) = \bigcup \{U \in \xi \mid U \subset A\}$ and the δ – semiclosure of A , denoted by $c_\xi(A)$, is given by $c_\xi(A) = \bigcup \{X - U \mid U \in \xi, A \subset X - U\}$. Every δ – open set is a δ – semiopen set and δ – semiopen set is a semiopen set but the converses are not true. The following lemma will be useful in the sequel.

LEMMA: 1.1 Let X be a space and $A \subset X$. Then the following hold.

- (a) If A is semiopen, then $\text{cl}(A) = c_\alpha(A)$ [19, Lemma 1].
- (b) If A is semiopen, then $\text{cl}(A) = \delta\text{cl}(A) = c_\alpha(A)$ [8, Proposition 2.2].
- (c) A is preopen if and only if $c_\alpha(A) = \text{int}(\text{cl}(A))$ [8, Proposition .7(a)].

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2. δ – SEMIOPEN SETS:

We state the following Theorem 2.1 without proof, which gives the relation between the operators c_ξ and i_ξ with other operators which are essential to characterize the spaces already stated.

Theorem: 2.1 Let X be a space and $A \subset X$. Then the following hold.

- (a) $\delta \text{int}(c_\sigma(A)) = \delta \text{int}(cl(A))$.
- (b) $c_b(i_\xi(A)) = c_\sigma(i_\xi(A))$.
- (c) $c_\xi(i_\sigma(A)) = i_\sigma(A) \cup \text{int}(\delta cl(\text{int}(A)))$.
- (d) $c_\beta(i_\xi(A)) = c_\xi(i_\xi(A))$.
- (e) $cl(i_\xi(A)) = cl(\delta \text{int}(A))$.
- (f) $c_\xi(i_\pi(A)) = \text{int}(cl(A))$.
- (g) $c_\pi(i_\xi(A)) = cl(\delta \text{int}(A))$.
- (h) $c_\xi(i_\alpha(A)) = \text{int}(\delta cl(\text{int}(A)))$.

The following Theorem 2.2 gives characterizations of preopen and δ – open sets. Also, it gives properties of semiopen, δ – semiopen and δ – open sets.

Theorem: 2.2 Let X be a space and $A \subset X$. Then the following hold.

- (a) If A is δ – semiopen, then $c_b(A) = c_\sigma(A)$.
- (b) If A is semiopen, then $c_\xi(A) = c_\sigma(A)$ [16, Theorem 3.3].
- (c) If A is δ – semiopen, then $c_\xi(A) = c_\sigma(A) = c_b(A)$.
- (d) If A is δ – semiopen, then $c_\beta(A) = c_\xi(A)$ and so $c_\beta(A) = c_b(A) = c_\sigma(A) = c_\xi(A)$.
- (e) A is δ – semiopen if and only if $cl(A) = cl(\delta \text{int}(A))$ if and only if $cl(\delta \text{int}(A)) = \delta cl(A)$ if and only if $cl(\delta \text{int}(A)) = c_\pi(A)$ if and only if $cl(\delta \text{int}(A)) = c_\alpha(A)$.
- (f) A is preopen if and only if $c_\xi(A) = \text{int}(cl(A))$ if and only if $\text{int}(cl(A)) = c_\sigma(A)$ Lemma 1.1(c).
- (g) If A is α – open, then $c_\xi(A) = c_\beta(A)$ and so $c_\xi(A) = c_\beta(A) = c_b(A) = c_\sigma(A)$.

Proof: (a) The proof follows from Theorem 2.1(b).

(b) By Theorem 2.1(c), $c_\xi(A) = A \cup \text{int}(\delta cl(\text{int}(A))) = A \cup \text{int}(cl(\text{int}(A))) = A \cup \text{int}(cl(A)) = c_\sigma(A)$.

(c) The proof follows from (a) and (b), since every δ – semiopen set is a semiopen set.

(d) The proof follows from Theorem 2.1(d) and the fact that $\xi(X) \subset \sigma(X) \subset b(X) \subset \beta(X)$.

(e) By Theorem 2.1(e), it follows that $cl(A) = cl(\delta \text{int}(A))$. Since A is δ – semiopen, $\delta cl(A) = cl(\delta \text{int}(A))$. Again, by Theorem 2.1(g), $c_\pi(A) = cl(\delta \text{int}(A))$ and by Lemma 1.1(a), $c_\alpha(A) = cl(A)$. Hence the proof follows. The converses are clear.

(f) If A is preopen, then by Theorem 2.1(f), $c_\xi(A) = \text{int}(cl(A))$ and so $c_\xi(A) = A \cup \text{int}(cl(A)) = c_\sigma(A)$. The converses are clear.

(g) If A is α – open, then by Theorem 2.1(h), $c_\xi(A) = \text{int}(\delta cl(\text{int}(A))) = A \cup \text{int}(cl(\text{int}(A))) = c_\beta(A)$ and so $c_\xi(A) = c_\beta(A) = c_b(A) = c_\sigma(A)$.

Remark: 2.1 Theorem 2.2(e) is a generalization of Lemma 1.1(b) and characterizes δ – semiopens. Theorem 2.2(f) is a generalization of Lemma 1.1(c) which characterizes preopen sets in terms of δ – semiopen sets and also shows that

Theorem 3.3 of [18] is partially true for preopen sets. Theorem 2.2(g) shows that Theorem 2.2 (d) is also true for α – open sets.

A space X is said to be s – closed [5] if for every cover $\{V_\alpha \mid \alpha \in \Delta\}$ of X by semiopen sets of X , there exists a finite subset Δ_0 of Δ such that $X = \cup \{c_\alpha(V_\alpha) \mid \alpha \in \Delta_0\}$ or equivalently, for every cover $\{V_\alpha \mid \alpha \in \Delta\}$ of X by δ – semiopen sets of X , there exists a finite subset Δ_0 of Δ such that $X = \cup \{c_\alpha(V_\alpha) \mid \alpha \in \Delta_0\}$ [16, Theorem 6.1(2)]. The following Theorem 2.3 gives more characterizations of s – closed spaces, the proof of which follows from Theorem 2.2(d).

Theorem: 2.3 Let X be a space. Then the following are equivalent.

- (a) X is s – closed.
- (b) For every cover $\{V_\alpha \mid \alpha \in \Delta\}$ of X by δ – semiopen sets of X , there exists a finite subset Δ_0 of Δ such that $X = \cup \{c_b(V_\alpha) \mid \alpha \in \Delta_0\}$.
- (c) For every cover $\{V_\alpha \mid \alpha \in \Delta\}$ of X by δ – semiopen sets of X , there exists a finite subset Δ_0 of Δ such that $X = \cup \{c_\beta(V_\alpha) \mid \alpha \in \Delta_0\}$.

A space X is said to be *semiconnected* [17] if X cannot be expressed as the disjoint union of two nonempty semiopen sets. The following Theorem 2.4 gives more characterizations of semi -connected spaces.

Theorem: 2.4 Let X be a space. Then the following are equivalent.

- (a) X is *semiconnected*.
- (b) $c_\xi(A) = X$ for every nonempty δ – semiopen set A .
- (c) $c_\alpha(A) = X$ for every nonempty δ – semiopen set A .
- (d) $c_b(A) = X$ for every nonempty δ – semiopen set A .
- (e) $c_\beta(A) = X$ for every nonempty δ – semiopen set A .

Proof: (a) and (b) are equivalent by Theorem 6.3 of [16]. (b), (c), (d) and (e) are equivalent by Theorem 2.2(d). A space X is said to be *semi- T_2* [12] if for each pair of distinct points x and y , there exist semiopen sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Theorem 6.5 of [16] gives some characterizations of semi – T_2 spaces in terms of δ – semiopen sets. The following Theorem 2.5 gives some more characterizations of semi- T_2 spaces in terms of δ – semiopen sets.

Theorem: 2.5 Let X be a space. Then the following are equivalent.

- (a) X is *semi- T_2* .
- (b) For each pair of distinct points x and y , there exist δ – semiopen sets U and V such that $x \in U$, $y \in V$ and $c_\xi(U) \cap c_\xi(V) = \emptyset$.
- (c) For each pair of distinct points x and y , there exist δ – semiopen sets U and V such that $x \in U$, $y \in V$ and $c_\beta(U) \cap c_\beta(V) = \emptyset$.
- (d) For each pair of distinct points x and y , there exist δ – semiopen sets U and V such that $x \in U$, $y \in V$ and $c_b(U) \cap c_b(V) = \emptyset$.

Proof: (a) and (b) are equivalent by Theorem 6.5(3) of [16]. (b), (c) and (d) are equivalent by Theorem 2.2(d). A space X is said to be *s – Urysohn* [3] if for each pair of distinct points x and y , there exist semiopen sets U and V such that $x \in U$, $y \in V$ and $\text{cl}(U) \cap \text{cl}(V) = \emptyset$. In Theorem 6.6 of [16], it is established that a space X is *s – Urysohn* if and only if for each pair of distinct points x and y , there exist δ – semiopen sets U and V such that $x \in U$, $y \in V$ and $\text{cl}(U) \cap \text{cl}(V) = \emptyset$. The following Theorem 2.6 gives more characterizations.

Theorem: 2.6 Let X be a space. Then the following are equivalent.

- (a) X is *s – Urysohn*.
- (b) For each pair of distinct points x and y , there exist δ – semiopen sets U and V such that $x \in U$, $y \in V$ and $\text{cl}(U) \cap \text{cl}(V) = \emptyset$.

(c) For each pair of distinct points x and y , there exist δ -semiopen sets U and V such that $x \in U$, $y \in V$ and $\delta cl(U) \cap \delta cl(V) = \emptyset$.

(d) For each pair of distinct points x and y , there exist δ -semiopen sets U and V such that $x \in U$, $y \in V$ and $c_\alpha(U) \cap c_\alpha(V) = \emptyset$.

(e) For each pair of distinct points x and y , there exist δ -semiopen sets U and V such that $x \in U$, $y \in V$ and $c_\pi(U) \cap c_\pi(V) = \emptyset$.

Proof: (a) and (b) are equivalent by Theorem 6.6 of [16]. (b), (c), (d) and (e) are equivalent by Theorem 2.2(e). A space X is said to be s -regular [11] (resp. semi-regular [6]) if for each closed (resp. semiclosed) set F of X and a point $x \notin F$, there exist semiopen sets U and V such that $x \in U$, $F \subset V$ and $U \cap V = \emptyset$. The following Theorem 2.7 gives more characterizations of s -regular and semi-regular spaces in terms of δ -semiopen sets.

Theorem: 2.7 Let X be a space. Then the following are equivalent.

(a) X is s -regular (resp. semi-regular).

(b) For each point $x \in X$ and each open (resp. semiopen) set V containing x , there exists a δ -semiopen set U such that $x \in U \subset c_\alpha(U) \subset V$.

(c) For each point $x \in X$ and each open (resp. semiopen) set V containing x , there exists a δ -semiopen set U such that $x \in U \subset c_\beta(U) \subset V$.

(d) For each point $x \in X$ and each open (resp. semiopen) set V containing x , there exists a δ -semiopen set U such that $x \in U \subset c_\beta(U) \subset V$.

Proof: The proof follows from Theorem 6.7(3) of [16] and Theorem 2.2(d). A space X is said to be s -normal [15] (resp. semi-normal [7]) if for each disjoint closed (resp. semiclosed) sets F and K of X , there exist semiopen sets U and V such that $F \subset U$, $K \subset V$ and $U \cap V = \emptyset$. The following Theorem 2.8 gives more characterizations of s -normal and semi-normal spaces in terms of δ -semiopen sets.

Theorem: 2.8 Let X be a space. Then the following are equivalent.

(a) X is s -normal (resp. semi-normal).

(b) For each closed (resp. semiclosed) set F and each open (resp. semiopen) set V containing F , there exists a δ -semiopen set U such that $F \subset U \subset c_\alpha(U) \subset V$.

(c) For each closed (resp. semiclosed) set F and each open (resp. semiopen) set V containing F , there exists a δ -semiopen set U such that $F \subset U \subset c_\beta(U) \subset V$.

(d) For each closed (resp. semiclosed) set F and each open (resp. semiopen) set V containing F , there exists a δ -semiopen set U such that $F \subset U \subset c_\beta(U) \subset V$.

Proof: The proof follows from Theorem 6.8(3) of [16] and Theorem 2.2(d).

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