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ON T g'''-SPACES

[1]S. PIOUS MISSIER, [2]O. RAVI, [3]S. JEYASHRI AND [4]P. HERIN WISE BELL

[1]*Department of Mathematics, V. O. Chidambaram College, Thoothukudi, Tamil Nadu (India)*

[2, 3]*Department of Mathematics, P. M. Thevar College, Usilampatti, Madurai Dt., Tamil Nadu, (India)*

[4]*Department of Mathematics, Vivekananda College, Agasteswaram, Kanyakumari Dt. Tamil Nadu, (India)*

E-mail: spmissier@yahoo.com, siingam@yahoo.com, jeyabala1@yahoo.com, herinewisebell@yahoo.com

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ABSTRACT

The aim of this paper is to introduce Tg'''-spaces, ${}_gTg'''$-spaces and $\alpha Tg'''$-spaces. Moreover, we obtain certain new characterizations for the Tg'''-spaces, ${}_gTg'''$-spaces and $\alpha Tg'''$-spaces.

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1. INTRODUCTION:

Levine [10] introduced the notion of $T_{1/2}$-spaces which properly lie between T_1-spaces and T_0-spaces. Many authors studied properties of $T_{1/2}$-spaces: Dunham [8], Arenas et al. [3] etc. In this paper, we introduce the notions called Tg'''-spaces, ${}_gTg'''$-spaces and $\alpha Tg'''$-spaces and obtain their properties and characterizations.

2. PRELIMINARIES:

Throughout this paper (X, τ) (or X) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), cl(A), int(A) and A^c denote the closure of A, the interior of A and the complement of A respectively.

We recall the following definitions which are useful in the sequel.

Definition: 2.1
A subset A of a space (X, τ) is called:

(i) semi-open set [11] if $A \subseteq cl(int(A))$;

(ii) preopen set [13] if $A \subseteq int(cl(A))$;

(iii) α-open set [15] if $A \subseteq int(cl(int(A)))$;

(iv) β-open set [1] (= semi-preopen [2]) if $A \subseteq cl(int(cl(A)))$.

The complements of the above mentioned open sets are called their respective closed sets.

*Corresponding author: O. Ravi, *E-mail: siingam@yahoo.com
Department of Mathematics, P. M. Thevar College, Usilampatti, Madurai Dt., Tamil Nadu, (India)

The preclosure [16] (resp. semi-closure [5], α-closure [15], semi-pre-closure [2]) of a subset A of X, denoted by pcl(A) (resp. scl(A), αcl(A), spcl(A)), is defined to be the intersection of all preclosed (resp. semi-closed, α-closed, semi-preclosed) sets of (X, τ) containing A. It is known that pcl(A) (resp. scl(A), αcl(A), spcl(A)) is a preclosed (resp. semi-closed, α-closed, semi-preclosed) set. For any subset A of an arbitrarily chosen topological space, the semi-interior [5] (resp. α-interior [15], preinterior [16]) of A, denoted by sint(A) (resp. αint(A), pint(A)), is defined to be the union of all semi-open (resp. α-open, preopen) sets of (X, τ) contained in A.

Definition: 2.2
A subset A of a space (X, τ) is called:

(i) a generalized closed (briefly g-closed) set [10] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X, τ). The complement of g-closed set is called g-open set;

(ii) a generalized semi-closed (briefly gs-closed) set [4] if scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X, τ). The complement of gs-closed set is called gs-open set;

(iii) an α-generalized closed (briefly αg-closed) set [12] if αcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X, τ). The complement of αg-closed set is called αg-open set;

(iv) a generalized semi-preclosed (briefly gsp-closed) set [16] if spcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X, τ). The complement of gsp-closed set is called gsp-open set;

(v) a ĝ-closed set [19] (= ω-closed [18]) if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi-open in (X, τ). The complement of ĝ-closed set is called ĝ-open set;

(vi) a g'''-closed set [17] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is gs-open in (X, τ). The complement of g'''-closed set is called g'''-open set;

(vii) a g*-preclosed (briefly g*p-closed) set [20] if pcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is g-open in (X, τ). The complement of g*p-closed set is called g*p-open set.

The collection of all g'''-closed (resp. ω-closed, αg-closed, gsp-closed, gs-closed, α-closed, g*p-closed) sets is denoted by G'''C(X) (resp. ωC(X), αGC(X), GSPC(X), GSC(X), αC(X), G^*PC(X)).

The collection of all g'''-open (resp. ω-open, αg-open, gsp-open, gs-open, α-open, g*p-open) sets is denoted by G'''O(X) (resp. ωO(X), αGO(X), GSPO(X), GSO(X), αO(X), G^*PO(X)).

We denote the power set of X by P(X).

Definition: 2.3
A space (X, τ) is called:

(i) $T_{1/2}$-space [10] if every g-closed set is closed.

(ii) T_b-space [7] if every gs-closed set is closed.

(iii) αT_b-space [6] if every αg-closed set is closed.

(iv) T_ω-space [18] if every ω-closed set is closed.

(v) T_p*-space [20] if every g*p-closed set in it is closed.

(vi) $*_sT_p$-space [20] if every gsp-closed set in it is g*p-closed.

(vii) αT_d-space [6] if every αg-closed set is g-closed.

(viii) α-space [15] if every α-closed set is closed.

Definition: 2.4 [14]
Let (X, τ) be a topological space and A ⊆ X. We define the gs-closure of A (briefly gs-cl(A)) to be the intersection of all gs-closed sets containing A.

Result: 2.5 [17]
For a topological space X, the following hold:

(i) Every closed set is g'''-closed but not conversely.

(ii) Every g'''-closed set is ω-closed but not conversely.

(iii) Every g'''-closed set is g-closed but not conversely.

(iv) Every g'''-closed set is α g-closed but not conversely.

(v) Every g'''-closed set is gs-closed but not conversely.

(vi) Every g'''-closed set is gsp-closed but not conversely.

Theorem: 2.6[17]
A set A is g'''-closed if and only if cl(A) − A contains no nonempty gs-closed set.

3. PROPERTIES OF T g'''-SPACES:

We introduce the following definition:

Definition: 3.1
A space (X, τ) is called a T g'''-space if every g'''-closed set in it is closed.

Example: 3.2
Let X = {a, b, c} with τ = {ϕ, {b}, X}. Then G''' C(X) = {ϕ, {a, c}, X}. Thus (X, τ) is a T g'''-space.

Example: 3.3
Let X = {a, b, c} with τ = {ϕ, {a, c}, X}. Then G''' C(X) = {ϕ, {b}, {a, b}, {b, c}, X}. Thus (X, τ) is not a T g'''-space.

Proposition: 3.4
Every $T_{1/2}$-space is T g'''-space but not conversely.

Proof: Follows from Result 2.5 (iii).

The converse of Proposition 3.4 need not be true as seen from the following example.

Example: 3.5
Let X and τ as in the Example 3.2, G C(X) = {ϕ, {a}, {c}, {a, b}, {a, c}, {b, c}, X}. Thus (X,τ) is not a $T_{1/2}$-space.

Proposition: 3.6
Every T ω-space is T g'''-space but not conversely.

Proof: Follows from Result 2.5 (ii).
The converse of Proposition 3.6 need not be true as seen from the following example.

Example: 3.7
Let X = {a, b, c} with τ = {ϕ, {a}, {b, c}, X}. Then ωC(X) = P(X) and G''' C(X) = {ϕ, {a}, {b, c}, X}. Thus (X, τ) is T g'''-space but not a T ω-space.

Proposition: 3.8
Every α T_b-space is T g'''-space but not conversely.

Proof: Follows from Result 2.5 (iv).
The converse of Proposition 3.8 need not be true as seen from the following example.

Example: 3.9
Let X and τ as in the Example 3.2, αG C(X) = {ϕ, {a}, {c}, {a, b}, {a, c}, {b, c}, X}. Thus (X, τ) is not a α T_b-space.

Proposition: 3.10
Every $*_sT_p$-space and T_p*-space is T g'''-space but not conversely.

Proof: Follows from Result 2.5 (vi) and Definition 2.3 (vi) and (v).

The converse of Proposition 3.10 need not be true as seen from the following example.

Example: 3.11
Let X and τ as in the Example 3.2, GSP C(X) = {ϕ, {a}, {c}, {a, b}, {a, c}, {b, c}, X} and G^*P C(X) = {ϕ, {a}, {c}, {a, c}, {b, c}, X}. Thus (X, τ) is neither a $*_sT_p$-space nor a T_p*-space.

Proposition: 3.12
Every T_b-space is T g'''-space but not conversely.

Proof: Follows from Result 2.5 (v).

The converse of Proposition 3.12 need not be true as seen from the following example.

Example: 3.13
Let X and τ as in the Example 3.2, GS C(X) = {ϕ, {a}, {c}, {a, b}, {a, c}, {b, c}, X}. Thus (X, τ) is not a T_b-space.

Remark: 3.14
We conclude from the next two examples that T g'''-spaces and α -spaces are independent.

Example: 3.15
Let X and τ as in the Example 3.2, α C(X) = {ϕ, {a}, {c}, {a, c}, X}. Thus (X, τ) is a T g'''-space but not a α -space.

Example: 3.16
Let X and τ as in the Example 3.3, α C(X) = {ϕ, {b}, X}. Thus (X, τ) is a α -space but not a T g'''-space.

Theorem: 3.17
For a space (X,τ) the following properties are equivalent:

(i) (X, τ) is a T g'''-space.

(ii) Every singleton subset of (X, τ) is either gs-closed or open.

Proof: (i) → (ii). Assume that for some x ∈ X, the set {x} is not a gs-closed in (X, τ). Then the only gs-open set containing $\{x\}^c$ is X and so $\{x\}^c$ is g'''-closed in (X, τ). By assumption $\{x\}^c$ is closed in (X, τ) or equivalently {x} is open.
(ii) → (i). Let A be a g'''-closed subset of (X, τ) and let x ∈ cl(A). By assumption {x} is either gs-closed or open.

Case (a) Suppose that $\{x\}$ is gs-closed. If $x \notin A$, then cl (A) −A contains a nonempty gs-closed set $\{x\}$, which is a contradiction to Theorem 2.6. Therefore $x \in A$.

Case (b) Suppose that $\{x\}$ is open. Since $x \in cl(A)$, $\{x\} \cap A \neq \phi$ and so $x \in A$. Thus in both case, $x \in A$ and therefore $cl(A) \subseteq A$ or equivalently A is a closed set of (X, τ).

Definition: 3.18
A topological space (X,τ) is called generalized semi-R_0 (briefly gs-R_0) if and only if for each gs-open set G and $x \in G$ implies gs-cl$(\{x\}) \subset G$.

Definition: 3.19
A topological space (X, τ) is called:

(i) generalized semi-T_0 (briefly gs-T_0) if and only if to each pair of distinct points x, y of X, there exists a gs-open set containing one but not the other.

(ii) generalized semi-T_1 (briefly gs-T_1) if and only if to each pair of distinct points x, y of X, there exists a pair of gs-open sets, one containing x but not y, and the other containing y but not x.

Theorem: 3.20
For a topological space X, each of the following statement is equivalent:

(i) X is a gs-T_1.

(ii) Each one point set is gs-closed set in X.

Proof: (i) ⇒ (ii) Let a space X be gs-T_1 and $x \in X$. Suppose gscl$(\{x\}) \neq \{x\}$. Then we can find an element $y \in$ gscl$(\{x\})$ with $y \neq x$. Since X is gs-T_1, there exist gs-open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. Now $x \in V^c$ and V^c is gs-closed. Therefore gscl$(\{x\}) \subseteq V^c$ which implies $y \in V^c$, contradiction. Hence gscl$(\{x\}) = \{x\}$ or $\{x\}$ is gs-closed.
(ii) ⇒ (i) Let $x, y \in X$ with $x \neq y$. Then $\{x\}$ and $\{y\}$ are gs-closed. Therefore $U = (\{x\})^c$ and $V = (\{y\})^c$ are gs-open and $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. Hence X is gs-T_1

Theorem: 3.21
For a space (X, τ) the following properties hold:

(i) If (X, τ) is gs-T_1, then it is T g'''.

(ii) If (X, τ) is T g''', then it is gs-T_0 .

Proof:
(i) The proof is obvious from Theorem 3.20.

(ii) Let x and y be two distinct elements of X. Since the space (X,τ) is T g''', we have that $\{x\}$ is gs-closed or open. Suppose that $\{x\}$ is open. Then the singleton $\{x\}$ is a gs-open set such that $x \in \{x\}$and $y \notin \{x\}$. Also, if $\{x\}$ is gs-closed, then $X \setminus \{x\}$ is gs-open such that $y \in X \setminus \{x\}$ and $x \notin X \setminus \{x\}$. Thus, in the above two cases, there exists a gs-open set U of X such that $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$. Thus, the space (X,τ) is gs-T_0 .

Theorem: 3.22
For a gs-R_0 topological space (X, τ) the following properties are equivalent:

(i) (X, τ) is gs-T_0.

(ii) (X, τ) is T g'''.

(iii) (X, τ) is gs-T_1.

Proof: It suffices to prove only (i) $\Rightarrow$ (iii). Let $x \neq y$ and since (X, τ) is gs-T_0, we may assume that $x \in U \subseteq X \setminus \{y\}$ for some gs-open set U. Then $x \in X \setminus$ gs-cl($\{y\}$) and $X \setminus$ gs-cl($\{y\}$) is gs-open. Since (X, τ) is gs-R_0, we have gs-cl($\{x\}$) $\subseteq X \setminus$ gs-cl($\{y\}$) $\subseteq X \setminus \{y\}$ and hence $y \notin$ gs-cl($\{x\}$). There exists gs-open set V such that $y \in V \subseteq X \setminus \{x\}$ and (X, τ) is gs-T_1.

4. ${}_gT g'''$-SPACES:

Definition: 4.1
A space (X, τ) is called a ${}_gT g'''$-space if every g-closed set in it is g'''-closed.

Example: 4.2
Let X and τ as in the Example 3.3, is a ${}_gT g'''$-space and the space (X, τ) in the Example 3.2, is not a ${}_gT g'''$-space.

Proposition: 4.3
Every $T_{1/2}$-space is ${}_gT g'''$-space but not conversely.

Proof: Follows from Result 2.5 (i).

The converse of Proposition 4.3 need not be true as seen from the following example.

Example: 4.4
Let X and τ as in the Example 3.3, is a ${}_gT g'''$-space but not a $T_{1/2}$-space.

Remark: 4.5
$T g'''$-space and ${}_gT g'''$-space are independent.

Example: 4.6
The space (X, τ) in the Example 3.3, is a ${}_gT g'''$-space but not a $T g'''$-space and the space (X, τ) in the Example 3.2, is a $T g'''$-space but not a ${}_gT g'''$-space.

Theorem: 4.7
If (X, τ) is a ${}_gT g'''$-space, then every singleton subset of (X, τ) is either g-closed or g'''-open.

Proof: Assume that for some $x \in X$, the set $\{x\}$ is not a g-closed in (X, τ). Then $\{x\}$ is not a closed set, since every closed set is a g-closed set. So $\{x\}^c$ is not open and the only open set containing $\{x\}^c$ is X itself. Therefore $\{x\}^c$ is trivially a g-closed set and by assumption, $\{x\}^c$ is an g'''-closed set or equivalently $\{x\}$ is g'''-open.

The converse of Theorem 4.7 need not be true as seen from the following example.

Example: 4.8
Let X and τ as in the Example 3.2. The sets $\{a\}$ and $\{c\}$ are g-closed in (X,τ) and the set $\{b\}$ is g'''-open. But the space (X,τ) is not a ${}_gT g'''$-space.

Theorem: 4.9
A space (X,τ) is $T_{1/2}$ if and only if it is both $T g'''$ and ${}_gT g'''$.

Proof: Necessity. Follows from Propositions 3.4 and 4.3.

Sufficiency. Assume that (X, τ) is both $T g'''$ and ${}_gT g'''$. Let A be a g-closed set of (X, τ). Then A is g'''-closed, since (X, τ) is a ${}_gT g'''$. Again since (X, τ) is a $T g'''$, A is a closed set in (X, τ) and so (X, τ) is a $T_{1/2}$.

5. $\alpha T g'''$-SPACES:

Definition: 5.1
A space (X, τ) is called a $\alpha T g'''$-space if every α g-closed set in it is g'''-closed.

Example: 5.2
Let X and τ as in the Example 3.3, is a α T g'''-space and the space (X, τ) in the Example 3.2, is not a α T g'''-space.

Proposition: 5.3
Every α T_b-space is α T g'''-space but not conversely.

Proof: Follows from Result 2.5 (i).

The converse of Proposition 5.3 need not be true as seen from the following example.

Example: 5.4
Let X and τ in the Example 3.3, is a α T g'''-space but not a α T_b-space.

Proposition: 5.5
Every α T g'''-space is a α T_d-space but not conversely.

Proof: Let (X, τ) be an α T g'''-space and let A be an α g-closed set of (X, τ). Then A is a g'''-closed subset of (X, τ) and by Result 2.5 (iii), A is g-closed. Therefore (X, τ) is an α T_d-space.

The converse of Proposition 5.5 need not be true as seen from the following example.

Example: 5.6
Let X and τ in the Example 3.3, is a α T_d-space but not a α T g'''-space.

Theorem: 5.7
If (X,τ) is a α T g'''-space, then every singleton subset of (X,τ) is either α g-closed or g'''-open.

Proof: Similar to Theorem 4.7.

The converse of Theorem 5.7 need not be true as seen from the following example.

Example: 5.8
Let X and τ as in the Example 3.2. The sets {a} and {c} are α g-closed in (X,τ) and the set {b} is g'''-open. But the space (X,τ) is not a α T g'''-space.

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