International Journal of Mathematical Archive-2(5), May - 2011, Page: 753-757
Available online through www.ijma.info **ISSN 2229 – 5046**

ON SYLOW NUMBERS OF THE ALTERNATING SIMPLE GROUPS[1]

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(Received on: 29-04-11; Accepted on: 11-05-11)

ABSTRACT

In this paper we give the formula of numbers of Sylow subgroups of alternating simple groups. This partially solves a problem posed by Jiping Zhang.

Keywords: *symmetric groups, alternating groups, Sylow numbers.*
MR (2000): *20D60, 20D06.*

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1. INTRODUCTION AND LEMMAS:

The structure of normalizer of Sylow 2-subgroup of symmetric groups S_n is studied by P. Hall (see Lemma 4 of [1], or [2]), he proved that Sylow 2-subgroup is self-normalized. The study in the numbers of Sylow subgroups (also called the Sylow numbers) is due to Professor Jiping Zhang [5]. He proved a conjecture of Huppert's. Then he put forward a problem in the end of his paper [5]. determine the number of Sylow p -subgroup of finite simple groups. In this paper we give the formula of Sylow numbers of the alternating simple group A_n, which partially solves above Zhang's problem. Let $\Omega = \{1, 2, 3, ..., n\}$, $S(\Omega)$ or S_n be symmetric group on Ω and P_n be a Sylow p -subgroup of symmetric group S_n. Then $|P_n| = (n!)_p = p^{s(n)}$, where $s(n) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + ...$.

Suppose that $n = a_r p^r + a_{r-1} p^{r-1} + ... + a_1 p + a_0$ is the p -adic expansion of the number n, in which $0 \le a_i \le p-1$, $i = 0,1,...,r$ and $a_r \ne 0$. In the following paper, the number p is always a prime. We will prove:

Theorem: The number of Sylow p -subgroup of the alternating group $A_n (n \ge 6)$ is

$$\frac{n!}{a_0!a_1!...a_r!\,p^{s(n)}(p-1)^{a_1+2a_2+...+ra_r}}.$$

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Next we will cite some lemmas.

Lemma: 1 Let $X=\{1,2,...,p\}$, $Y=\{1,2,...,t,i_1,i_2,...,i_{p-t}\}$ and $X\cap Y=\{1,2,...,t\}$ with $1\le t\le p-1$. Suppose that a,b are p-cycles on X and Y, respectively. Then the group $\langle a,b\rangle$ is not a p-group.

Proof: Suppose that $\langle a,b\rangle$ is a p-group. Clearly, it is a transitive permutation on the set $X\cap Y$, and its degree n is $|X\cap Y|=2p-t$. Since $1\le t\le p-1$, we have $1<n<2p$. On the other hand, the degree of $\langle a,b\rangle$ is a multiple of p since it is a p-group. In fact, we only need to choose a subgroup P of $\langle a,b\rangle$ whose kernel is 1 and order is maximal, then $n=|\langle a,b\rangle:P|$. So that $p\mid n$, which contradicts the facts that $p<n<2p$.

Lemma: 2 The normalizer of Sylow p-subgroup of the symmetric group S_p ($p\ge 5$) is the Frobenius group $Z_p:Z_{p-1}$.

Proof: Let $Z_p=\langle(123...p)\rangle$ is a Sylow p-subgroup of S_p. Choose an element $b=(12435687......)$. It is easy to check that $(123...p)^b\in Z_p$, and hence $Z_p:\langle a,b\rangle\le N_{S_p}(Z_p)$. On the other hand, S_p has $(p-1)!$ elements with order p exactly, so the number of subgroups of order p in S_p is $(p-2)!$. Obviously, the number $|S_p:N_{S_p}(Z_p)|$ of Sylow p-subgroups is also $(p-2)!$\$, thus $|N_{S_p}(Z_p)|=p(p-1)$. Therefore, $N_{S_p}(Z_p)$ is a Frobenius group with order $p(p-1)$.

2. Proof of Theorem: Denote by $A\wr B$ the wreath product of A and B, in which B regarded as a permutation group. Denote by $G^m:=\overbrace{G\times G\times\cdots\times G}^{m\ copies}$, $G^{\wr m}:=\overbrace{G\wr G\wr\cdots\wr G}^{m\ copies}$.

It is known that the Sylow p-subgroup P_{p^r} of S_{p^r} is $Z_p^{\wr r}$ (see 1.6.19 in [4]). Of course, it is a transitive group with degree $P^{s(p^r)}$. If G acts on the set Δ and $X\subseteq\Delta$, then N_X stands for the global stabilizer of subset X in G. We will use the following steps to complete the proof of the Theorem.

Step: 1 If $n=p^r$, then $N_{S_{p^r}}(P_{p^r})\cong Z_p^{\wr r}:Z_{p-1}^r$.

Let $X=\{1,2,...,p^r\}$, $X_i=\{p(i-1)+1,p(i-1)+2,...,p(i-1)+p\}$, where $1\le i\le p^{r-1}$. Set $\mathrm{D}=\{X_1,X_2,...,X_{p^{r-1}}\}$. Then

$$S(X)_{\mathcal{D}}=(S(X_1)\times S(X_2)\times\cdots\times S(X_{p^{r-1}})):S_{p^{r-1}}\cong S_p\wr S_{p^{r-1}}.$$

Clearly, $S(X)_D$ must include a Sylow p-subgroup P_{p^r} of $S(X)$.

Let $K_i:=S(X_i)$, $1\le i\le p^{r-1}$, and $K:=K_1$. The class of subgroups conjugate to K in $S(X)$ referred to as the set of the fundamental subgroups in $S(X)$. Set

$$\Delta = Fun_{S(X)}(P_{p^r}) = \{K^x \mid x \in S(X), K^x \cap P_{p^r} \in Syl_p(K^x)\}.$$

By the definition of Δ, we have $\{K_i \mid 1 \le i \le p^{r-1}\} \subseteq \Delta$. Suppose that there exists an element $g \in S(X)$ such that $K^g \in \Delta$, and $K^g \ne K_i$ for each i. Now we let $K^g := S(g(X_1))$. of course $g(X_1) \ne K_i$ for each i. By the definition of Δ, we know that $K^g \cap P_{p^r} \in Syl_p(S(X))$. Then there exists a Sylow p-subgroup P of K^g such that $P \le P_{p^r}$. Without loss of generality, we can assume that $a = (123...p) \in K \cap P_n$, then $\langle a, P\rangle \le P_{p^r}$. Suppose that $X_i \cap g(X_1) \ne \varnothing$ for some i, and set $|X_i \cap g(X_1)| = t$, where $1 \le t \le p-1$. Then we can assume that $g(X_1) = \{1, 2, ..., t, i_1, i_2, ..., i_{p-t}\}$. By the Lemma 1, for any p-cycle b on $g(X_1)$, we have $\langle a, b\rangle$ is not a p-group. Then such P does not exist, contradicts. So $X_i \cap g(X_1) = \varnothing$ for any i, that is $g(X_1) \cap X = g(X_1) \cap (\cup_{i=1}^{p^{r-1}} X_i) = \varnothing$,which contradicts the fact that $K^g \le S(X)$. Thus $\{K_i \mid 1 \le i \le p^{r-1}\} = \Delta$. Moreover, by the definition of Δ, we know that $K^x \cap P_{p^r} \in Syl_p(K^x)$ for any $K^x \in \Delta$. If $g \in N_{S(X)}(P_{p^r})$, i.e., $P_{p^r}{}^g = P_{p^r}$, then $K^{xg} \cap P_{p^r} \in Syl_p(K^{xg})$, and hence $K^{xg} \in \Delta$. Thus $g \in N_{S(X)}(\Delta)$, then

$$N_{S(X)}(P_{p^r}) \le N_{S(X)}(\Delta) = S(X)_{\mathcal{D}} \cong S_p \wr S_{p^{r-1}}.$$

Without loss of generality, we suppose that P_{p^r} is a Sylow p-subgroup of $S_p \wr S_{p^{r-1}}$. of course, Now we $N_{S(X)}(P_{p^r}) = N_{S_p \wr S_{p^{r-1}}}(P_{p^r})$. denote by $G := S_p \wr S_{p^{r-1}}$, $A := S_p^{p^{r-1}}$,

$A := S_p^{p^{r-1}}, B := S_{p^{r-1}}$, $P_A = P_{p^r} \cap A, P_B = P_{p^r} \cap B$. Since $G = A : B$, we have $N_B(P_B) \cong N_{G/A}(P_{p^r}A/A) = N_G(P_{p^r})A/A \cong N_G(P_{p^r})/A \cap N_G(P_{p^r}) = N_G(P_{p^r})/N_A(P_{p^r})$, so that $N_G(P_{p^r}) = N_A(P_{p^r}) : N_B(P_B)$. In the sequel, we determine the structure of $N_A(P_{p^r})$. Obviously, $P_{p^r} = Z_p^{l_r} = Z_p \wr Z_p^{l_{r-1}} = Z_p^{p^{r-1}} : Z_p^{l_{r-1}}$. Choose an element $x = (x_1, x_2, ..., x_{p^{r-1}}; 1) \in N_A(P_{p^r})$, then we have $g^x \in P_{p^r}$ for any element $g = (g_1, g_2, ..., g_{p^{r-1}}; g_0) \in P_{p^r}$, where $g_0 \in Z_p^{l_{r-1}}$. That is

$$g^x = (x_1^{-1} g_1 x_{1g_0}, x_2^{-1} g_2 x_{2g_0}, \cdots, x_{p^{r-1}}^{-1} g_{p^{r-1}} x_{(p^{r-1})g_0}; g_0) \in Z_p^{p^{r-1}} : Z_p^{l_{r-1}}.$$

Since g_0 can be chosen randomly, we can get $x_i \in N_{S_p}(Z_p)$ if we choose $g_0 = 1$, and hence $x_i = x_{i^{g_0}}$. Furthermore, $Z_p^{l_{r-1}}$ is a transitive group with degree $p^{s(p^{r-1})}$, then for any pairs $i, j \in \{1, 2, ..., p^{r-1}\}$, there exists an element $g_0 \in Z_p^{l_{r-1}}$ such that $i^{g_0} = j$, and hence $x_1 = x_2 = ... = x_{p^{r-1}}$. Conversely, if $x = (x_1, x_2, ..., x_{p^{r-1}}; 1)$ and $x_1 \in N_{S_p}(Z_p)$, then $x \in N_A(P_{p^r})$. Thus $N_A(P_{p^r}) = Z_p^{p^{r-1}} : Z_{p-1}$. So we have $N_A(P_{p^r}) = N_A(P_{p^r}) : N_B(P_B) \cong (Z_p^{p^{r-1}} : Z_{p-1}) : N_B(P_B)$. Therefore, $N_{S_{p^r}}(P_{p^r}) \cong (Z_p^{p^{r-1}} : Z_{p-1}) : N_{S_{p^{r-1}}}(P_{p^{r-1}})$. By the induction for r, we can obtain $N_{S_{p^r}}(P_{p^r}) \cong Z_p^{l_r} : Z_{p-1}^r$.

Step: 2 If $n = a_r p^r + k$, then $N_{S_n}(P_n) \cong N_{S_k}(P_k) \times (N_{S_{p^r}}(P_{p^r}) \wr S_{a_r})$, here $0 \le k < p^r$.

Let $\Omega = \{1, 2, 3, ..., n\}$, $X_i = \{p^{i-1}, p^{i-1}+1, ..., p^{i-1}+p^r-1\}$ in which $1 \le i \le a_r$ and $X_0 = \Omega \setminus \cup_{i=1}^{a_r} X_i$. Set $D = \{X_{0,} X_1, X_2, ..., X_{p^{r-1}}\}$. Then

$$S(\Omega)_{\mathcal{D}} = S(X_0) \times ((S(X_1) \times S(X_2) \times \cdots \times S(X_{p^{r-1}})) : S_{a_r}) \cong S_k \times (S_{p^r} \wr S_{a^r}).$$

Clearly, $S(\Omega)_D$ includes a Sylow p-subgroup P_n of $S(\Omega)$. Denote by $K_i := S(X_i)$, $1 \le i \le a_r$, and $K := K_1$.

Set $\Delta = Fun_{S(\Omega)}(P_n) = \{K^x \mid x \in S(\Omega), K^x \cap P_n \in Syl_p(K^x)\}$.

In this case, Δ is a maximal set of pair-wise commuting fundamental subgroups of K in S_{p^r}. It is clear that $\{K_i \mid 1 \le i < p^{r-1}\} \subseteq \Delta$. If there exists an element $g \in S(\Omega)$ such that $K^g \in \Delta$ and $K^g \ne K_i$ for each i. Denote by $K^g := S(g(X_1))$. Obviously, $g(X_1) \ne K_i$ for each i. Since $K^g \cap P_{p^r} \in Syl_p(S(\Omega))$, then there is a Sylow p-subgroup P of K^g such that $P \le P_n$. Now we can assume that $a = (123...p) \in K \cap P_n$, then $\langle a, P \rangle \le P_n$. Suppose that $X_i \cap g(X_1) \ne \varnothing$ for some i, and set $|X_i \cap g(X_1)| = t$, where $1 \le t \le p-1$. Without loss of generality, we can assume that $g(X_1) = \{1, 2, ..., t, \ \ i_1, i_2, ..., i_{p-t}\}$. Similarly, for any p-cycle b on $g(X_1)$, we have $\langle a, b \rangle$ is not a p-group by Lemma 1. Then such P does not exist, contradicts. So $X_i \cap g(X_1) = \varnothing$ for any i, then $g(X_1) \subseteq X_0$, which contradicts the fact that $|g(X_1)| > |X_0|$. Therefore, $\{K_i \mid 1 \le i \le p^{r-1}\} = \Delta$. Then

$$N_{S(\Omega)}(P_n) \le N_{S(\Omega)}(\Delta) = S(\Omega)_{\mathcal{D}} \cong S_k \times (S_{p^r} \wr S_{a_r}).$$

We can assume that P_n is a chosen Sylow p-subgroup of It is $S_k \times (S_{p^r} \wr S_{a_r})$. obvious that

$$N_{S(\Omega)}(P_n) = N_{S_k \times (S_{p^r} \wr S_{a_r})}(P_n).$$

It is easy to see that $P_n = P_k \times P_{p^r}^{a_r}$, so $N_{S(\Omega)}(P_n) = N_{S_k}(P_k) \times (N_{S_{p^r}}(P_{p^r}) \wr S_{a_r})$.

Step: 3 Now we continue to decompose k into the sum $a_{r-1}p^{r-1} + k_1$, in which $0 \le k_1 < p^{r-1}$ $. Next we repeat to decompose k_1, and so on. At last we can get the p-adic expansion $a_r p^r + a_{r-1}p^{r-1} + \ldots + a_1 p + a_0$ by r steps. Using the induction for r by the conclusions of step 1 and 2, we can obtain the result of the normalizer for general n:

$$N_{S_n}(P_n) \cong ((Z_p^{\wr r} : Z_{p-1}^r) \wr S_{a_r}) \times ((Z_p^{\wr r-1} : Z_{p-1}^{r-1}) \wr S_{a_{r-1}}) \times \cdots \times ((Z_p : Z_{p-1}) \wr S_{a_1}) \times S_{a_0}.$$

In the case of $p = 2$, it is not hard to see $|N_{S_n}(P_n)| = |P_n|$ since $a_i = 0$ or 1,

then S_n has a self-normalized Sylow 2-subgroup. Since $|S_n : A_n| = 2$, we have the Sylow p-number of S_n is same as the one of A_n for $p > 2$. If $p = 2$, we use the result of Kondrat'ev's [3]: A_n has a self-normalized Sylow 2-subgroup for $n \ge 6$.

So the number of Sylow p-subgroups of the alternating group $A_n (n \ge 6)$ is

$$\frac{n!}{a_0!a_1!\ldots a_r!\,p^{s(n)}(p-1)^{a_1+2a_2+\ldots+ra_r}}.$$

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[1]Project supported by the NNSF of China (No.11026195) and the foundation of Educational Department of Hubei Province in China (No.Q20111901).

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