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# ON AN UPPER BOUND FOR STRUCTURE GRACEFUL INDEX OF COMPLETE GRAPHS

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***ABSTRACT***

*A graph structure $G=(V,R_1,R_2,\ldots,R_k)$ consists of a non-empty set V together with relations $R_1,R_2,\ldots,R_k$ on V which are mutually disjoint such that each $R_i$, $1\leq i\leq k$, is symmetric and irreflexive. If $(u,v)\ \varepsilon\ R_i$ for some i, $1\leq i\leq k$, we call it a $R_i$-edge and write it as uv. The structure graceful index of a graph G is defined as the minimum k for which G is k-structure graceful. Let us denote it by SGI(G). In our previous paper, we prove that the $SGI(K_n)=2$, for $4 < n < 11$. In this paper we obtain the upper bound for the $SGI(K_n)$, for $n > 10$.*

## INTRODUCTION

In many real life situations, we are using complete graphs. Also, graceful labeling plays a vital role. But the complete graph $K_n$ is not graceful for $n > 4$.

A graph G = (V,E) is said to be k-structure graceful if E can be partitioned into k disjoint subsets $E_1,E_2,\ldots,E_k$ such that the graph structure $(V(G), E_1, E_2, \ldots, E_k)$ is graceful. The structure graceful index of a graph G is defined as the minimum k for which G is k-structure graceful. Let us denote it by SGI(G).

In our previous paper, we proved $SGI(K_n) = 2$, $4 < n < 11$. In the course of the proof, we found a graph $G_n$, which is graceful for $n > 4$. A $G_n$ graph has $V(G_n) = \{v_1,v_2,\ldots,v_n\}$ and $E(G_n) = \{v_1v_i/\ i > 1\} \cup \{ v_2v_i/\ i > 2\} \cup \{v_3v_i/\ i > 3\} \cup \{v_jv_n/\ 5\leq j < n\}$, for $n > 4$. Using this $G_n$ graph, we find the upper bound for the structure graceful index of $K_n$, $n > 10$.

**Definitions:**

1. A graph structure $G=(V,R_1,R_2,\ldots,R_k)$ consists of a non-empty set V together with relations $R_1,R_2,\ldots,R_k$ on V which are mutually disjoint such that each $R_i$, $1\leq i\leq k$, is symmetric and irreflexive.
2. A graph G = (V,E) is said to be k-structure graceful if E can be partitioned into k disjoint subsets $E_1,E_2,\ldots,E_k$ such that the graph structure $(V(G), E_1, E_2, \ldots, E_k)$ is graceful.
3. The structure graceful index of a graph G is defined as the minimum k for which G is k-structure graceful. Let us denote it by SGI(G).

**Theorem:**

$$SGI(K_n) \leq \begin{cases} \left\lceil \frac{n-5}{4} \right\rceil + 1, & \text{when } n\equiv 1 \pmod 4 \\ \left\lceil \frac{n-6}{4} \right\rceil + 1, & \text{when } n\equiv 2 \pmod 4 \\ \left\lceil \frac{n-7}{4} \right\rceil + 2, & \text{when } n\equiv 3 \pmod 4 \\ \left\lceil \frac{n-8}{4} \right\rceil + 2, & \text{when } n\equiv 0 \pmod 4 \end{cases}$$

where n >10.

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To prove this theorem we need the following lemma.

**Lemma:** One point union of $K_m$ and $K_{1,n}$ is graceful for $2 < m < 7$.

**Proof:** Let G be a one point union of $K_m$ and $K_{1,n}$, $n > 0$. Let $V(G) = \{v_1, v_2, \ldots, v_{m+n}\}$ and $E(G) = \{v_1v_i \,/\, 2 \le i \le m+n\} \cup \{v_jv_k \,/\, 2 \le j < k \le m\}$.

**Case - (i):** When m = 3

$$\text{Define } f(v_i) = \begin{cases} 0, & i = 1 \\ i, & i > 1 \end{cases}$$

<u>f is injective</u>:

$f(v_1) \ne f(v_i)$, since $f(v_1)$ is 0 and $f(v_i)$ is a positive integer for $i > 1$.

Also $f(v_i) \ne f(v_j)$ if $i \ne j$. Therefore f is an injective function.

$$\begin{aligned} \text{Let } A_1 &= \{\ell(v_1v_i) \,/\, 2 \le i \le 3+n\} = \{|\ell(v_1) - \ell(v_i)| \,/\, 2 \le i \le 3+n\} \\ &= \{|\ell(v_1) - \ell(v_2)|\} \cup \{|\ell(v_1) - \ell(v_3)| \cup \ldots.. \cup \{|\ell(v_1) - \ell(v_{3+n})|\} \\ &= \{2, 3, \ldots, 3+n\} \end{aligned} \tag{1}$$

$$A_2 = \{\ell(v_2v_3)\} = \{|\ell(v_2) - \ell(v_3)|\} = \{|2-3|\} = \{1\} \tag{2}$$

From (1) & (2) $A_1 \cup A_2 = \{1, 2, \ldots 3+n\}$

Thus $f: V(G) \rightarrow Z_{3+n}$ is an injective function and the edges receive the labels from $\{1, 2, \ldots 3+n\}$. Hence one point union of $K_3$ and $K_{1,n}$ is graceful.

**Case - (ii):** When m = 4

$$\text{Define } f(v_i) = \begin{cases} 0, & i = 1 \\ 6, & i = 2 \\ 5, & i = 3 \\ 2, & i = 4 \\ i+2, & 5 \le i \le 4+n \end{cases}$$

<u>f is injective</u>:

$f(v_1) \ne f(v_i)$, for $i>1$, since $f(v_1)$ is 0 and $f(v_i)$ is not 0 for $i > 1$.

For $i \ge 5$, $f(v_i) \ge 7$. Therefore $f(v_i) \ne f(v_j)$ when $i \ge 5$ and $1 \le j \le 4$.

Clearly, $f(v_i) \ne f(v_j)$ for $5 \le i, j \le 4+n$. Therefore f is an injective function.

$$\begin{aligned} \text{Let } A_3 &= \{\ell(v_1v_i) \,/\, 2 \le i \le 4+n\} = \{|\ell(v_1) - \ell(v_i)| \,/\, 2 \le i \le 4+n\} \\ &= \{|\ell(v_1) - \ell(v_2)|, |\ell(v_1) - \ell(v_3)|, \ldots.., |\ell(v_1) - \ell(v_{4+n})|\} \\ &= \{|0-6|, |0-5|, |0-2|, |0-7|, |0-8|, \ldots, |0-(n+6)|\} \\ &= \{6, 5, 2, 7, 8, \ldots, n+6\} \end{aligned} \tag{3}$$

$$\begin{aligned} A_4 &= \{\ell(v_2v_i) \,/\, 3 \le i \le m\} = \{|\ell(v_2) - \ell(v_i)| \,/\, 3 \le i \le m\} \\ &= \{|\ell(v_2) - \ell(v_3)|, |\ell(v_2) - \ell(v_4)|\} \\ &= \{|6-5|, |6-2|\} = \{1, 4\} \end{aligned} \tag{4}$$

$$A_5 = \{\ell(v_3v_4)\} = \{|\ell(v_3) - \ell(v_4)|\} = \{|5-2|\} = \{3\} \tag{5}$$

From (3), (4) & (5) $A_3 \cup A_4 \cup A_5 = \{1, 2, \ldots, n+6\}$

Thus $f: V(G) \rightarrow Z_{6+n}$ is an injective function and the edges receive the labels from $\{1, 2, \ldots 6+n\}$. Hence one point union of $K_4$ and $K_{1,n}$ is graceful.

 

**Case - (iii):** When m = 5

$$\text{Define } f(v_i) = \begin{cases} 0, & i = 1 \\ 11, & i = 2 \\ 10, & i = 3 \\ 2, & i = 4 \\ 7, & i = 5 \\ 6, & i = 6 \\ i + 5, & 7 \le i \le 5+n \end{cases}$$

<u>f is injective</u>:

$f(v_1) \neq f(v_i)$, since $f(v_1)$ is 0 and $f(v_i)$ is a positive integer for $i > 1$.

Also $f(v_i) \neq f(v_j)$, for $2 \le i < j \le 5+n$. Therefore f is an injective function.

$$\begin{aligned} \text{Let } A_6 &= \{ \ell(v_1v_i) \,/\, 2 \le i \le 5+n \} = \{|\ell(v_1) - \ell(v_i)| \,/\, 2 \le i \le 5+n\} \\ &= \{|\ell(v_1)-\ell(v_2)|, |\ell(v_1)-\ell(v_3)|, \ldots, |\ell(v_1)-\ell(v_{5+n})|\} \\ &= \{|0-11|, |0-10|, |0-2|, |0-7|, |0-6|, |0-12|, |0-13|, \ldots, |0-(n+10)|\} \\ &= \{11, 10, 2, 7, 6, 12, 13, \ldots, n+10\} \end{aligned} \tag{6}$$

$$\begin{aligned} A_7 &= \{ \ell(v_iv_j) \,/\, 2 \le i<j \le 5\} = \{|\ell(v_i) - \ell(v_j)| \,/\, 2 \le i<j \le 5\} \\ &= \{|\ell(v_2)-\ell(v_3)|, |\ell(v_2)-\ell(v_4)|, |\ell(v_2)-\ell(v_5)|\} \cup \{|\ell(v_3)-\ell(v_4)|, (v_3)-\ell(v_5)|\} \cup \{|\ell(v_4)-\ell(v_5)|\} \\ &= \{|11-10|,|11-2|,|11-7\} \cup \{|10-2|,|10-7|\} \cup \{|2-7|\} = \{1,9,4,8,3,5\} \end{aligned} \tag{7}$$

From (6) & (7) $A_6 \cup A_7 = \{1,2,\ldots,n+10\}$

Thus $f: V(G) \to Z_{10+n}$ is an injective function and the edges receive the labels from $\{1, 2, \ldots 10+n\}$. Hence one point union of $K_5$ and $K_{1,n}$ is graceful.

**Case - (iv):** When m = 6

$$\text{Define } f(v_i) = \begin{cases} 0, & i = 1 \\ 17, & i = 2 \\ 16, & i = 3 \\ 2, & i = 4 \\ 13, & i = 5 \\ 7, & i = 6 \\ 8, & i = 7 \\ 12, & i = 8 \\ i + 9, & 9 \le i \le 6+n \end{cases}$$

<u>f is injective:</u>

$f(v_1) \neq f(v_i)$, since $f(v_1)$ is 0 and $f(v_i)$ is a positive integer for $i > 1$.

Also $f(v_i) \neq f(v_j)$ if $i \neq j$, for $2 \le i,j \le 6+n$. Therefore f is an injective function.

$$\begin{aligned} \text{Let } A_8 &= \{ \ell(v_1v_i) \,/\, 2 \le i \le 6+n\} = \{|\ell(v_1) - \ell(v_i)| \,/\, 2 \le i \le 6+n\} \\ &= \{|\ell(v_1)-\ell(v_2)|, |\ell(v_1)-\ell(v_3)|, \ldots, |\ell(v_1)-\ell(v_{6+n})|\} \\ &= \{|0-17|, |0-16|, |0-2|, |0-13|, |0-7|, |0-8|, |0-12|, |0-18|, |0-19|, \ldots, |0-(n+15)|\} \\ &= \{17, 16, 2, 13,7, 8,12, 18, 19,\ldots, n+15\} \end{aligned} \tag{8}$$

$$\begin{aligned} A_9 &= \{ \ell(v_iv_j) \,/\, 2 \le i,j \le 6, i < j\} = \{|\ell(v_i) - \ell(v_j)| \,/\, 2 \le i,j \le 6, i < j\} \\ &= \{|\ell(v_2)-\ell(v_3)|,|\ell(v_2)-\ell(v_4)|,\ldots,|\ell(v_2)-\ell(v_6)|\} \cup \{|\ell(v_3)-\ell(v_4)|, |\ell(v_3)-\ell(v_5)|, \\ &\quad |\ell(v_3)-\ell(v_6)|\} \cup \{|\ell(v_4)-\ell(v_5)|,|\ell(v_4)-\ell(v_6)|\} \cup \{|\ell(v_5)-\ell(v_6)|\} \\ &= \{|17-16|,|17-2|,|17-13|,|17-7|\} \cup \{|16-2|,|16-13|,|16-7|\} \cup \{|2-13|,|2-7|\} \cup \{|13-7|\} \\ &= \{1, 15, 4, 10\} \cup \{14, 3, 9\} \cup \{11, 5\} \cup \{6\} \\ &= \{1, 3, 4, 5, 6, 9, 10, 11, 14, 15\} \end{aligned} \tag{9}$$

From (8) & (9) $A_8 \cup A_9 = \{1,2,\ldots,n+15\}$

Thus f: V (G)$\to Z_{15+n}$ is an injective function and the edges receive the labels from {1, 2,… 15+n}. Hence one point union of $K_6$ and $K_{1,n}$ is graceful.

**Proof for the theorem:** Partition the edges of $K_n$ ie) $E(K_n)$ into two sets namely, $E(G_n)$ and $E(K_n \setminus G_n)$, then $E(K_n \setminus G_n)$ into $E(G_{n-4})$ and $E(\overline{K_{13} \setminus G_{13}} \setminus G_{n-4})$, then $E(\overline{K_n \setminus G_n} \setminus G_{n-4})$ into $E(G_{n-8})$ and $(E(\overline{K_n \setminus G_n \setminus G_{n-4}} \setminus G_{n-8})$, $(E(\overline{K_n \setminus G_n \setminus G_{n-4}} \setminus G_{n-8})$ into $E(G_{n-12})$ and $\overline{K_n \setminus G_n \setminus G_{n-4} \setminus G_{n-8}} \setminus G_{n-12}$ and so on.

From this partition, in the last step we arrive the following cases:

(i) $E(G_9)$ and edges in one point union of $K_5$ and $K_{1,\left\lfloor \frac{n}{4} \right\rfloor -1}$, when $n \equiv 1 \pmod 4$

(ii) $E(G_{10})$ and edges in one point union of $K_6$ and $K_{1,\left\lceil \frac{n}{4} \right\rceil -1}$, when $n \equiv 2 \pmod 4$

(iii) $E(G_7)$ and edges in one point union of $K_3$ and $K_{1,\left\lfloor \frac{n}{4} \right\rfloor}$, when $n \equiv 3 \pmod 4$

(iv) $E(G_8)$ and edges in one point union of $K_4$ and $K_{1,\left\lfloor \frac{n}{4} \right\rfloor}$, when $n \equiv 0 \pmod 4$

When $n \equiv 1 \pmod 4$:

We have subgraphs which contain the edges of $G_n$, $G_{n-4}$, $G_{n-8}$, …, $G_9$ and one point union of $K_5$ and $K_{1,\left\lfloor \frac{n}{4} \right\rfloor -1}$.

Let $G_n^{(j)} = \langle A_j \rangle$, j = 1, 2, …, m + 1, where m = $\left\lceil \frac{n-5}{4} \right\rceil$

where $\langle A_1 \rangle$ is the subgraph of $K_n$ induced by the edges in $G_n$.
$\langle A_2 \rangle$ is the subgraph of $K_n \setminus G_n$ induced by the edges in $G_{n-4}$.
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$\langle A_m \rangle$ is the subgraph of $K_n \setminus G_n \setminus G_{n-4} \setminus \ldots \setminus G_{13}$ induced by the edges in $G_9$ and
$\langle A_{m+1} \rangle$ is the subgraph of $K_n \setminus G_n \setminus G_{n-4} \setminus \ldots \setminus G_{13}$ induced by the edges in one point union of $K_5$ and $K_{1,\left\lfloor \frac{n}{4} \right\rfloor -1}$.

We have already proved that the graph $G_n$ is graceful for n > 4. Therefore $G_n^{(j)}$, j = 1, 2, …, m is graceful. By case (iii), $G_n^{(j)}$, j = m + 1 is graceful. Totally we have m + 1 = $\left\lceil \frac{n-5}{4} \right\rceil$+1 graceful subgraphs.

$$\therefore \text{ SHI } (K_n) \le \left\lceil \frac{n-5}{4} \right\rceil + 1.$$

When $n \equiv 2 \pmod 4$:

We have sub graphs which contain the edges of $G_n$, $G_{n-4}$, $G_{n-8}$, …, $G_{10}$ and one point union of $K_6$ and $K_{1,\left\lceil \frac{n}{4} \right\rceil -1}$.

Let $G_n^{(j)} = \langle A_j \rangle$, j = 1, 2,…, m + 1, where m = $\left\lceil \frac{n-6}{4} \right\rceil$

where $\langle A_1 \rangle$ is the subgraph of $K_n$ induced by the edges in $G_n$.
$\langle A_2 \rangle$ is the subgraph of $K_n \setminus G_n$ induced by the edges in $G_{n-4}$.
……………………………………………………………………………
$\langle A_m \rangle$ is the subgraph of $K_n \setminus G_n \setminus G_{n-4} \setminus \ldots \setminus G_{14}$ induced by the edges in $G_{10}$ and
$\langle A_{m+1} \rangle$ is the subgraph of $K_n \setminus G_n \setminus G_{n-4} \setminus \ldots \setminus G_{14}$ induced by the edges in one point union of $K_6$ and $K_{1,\left\lceil \frac{n}{4} \right\rceil -1}$.

Again $G_n^{(j)}$, j = 1, 2, …, m are graceful and by case (iv), $G_n^{(j)}$, j = m + 1 is also graceful. we have

m + 1 = $\left\lceil \frac{n-6}{4} \right\rceil$+1 graceful subgraphs.

 

$\therefore$ SHI ($K_n$) $\leq \left[\dfrac{n-6}{4}\right] + 1$.

When $n \equiv 3 \pmod 4$:

We have sub graphs which contain the edges of $G_n, G_{n-4}, G_{n-8}, \ldots, G_7$ and one point union of $K_3$ and $K_{1,\left\lfloor \frac{n}{4} \right\rfloor}$.

Let $G_n^{(j)} = \langle A_j \rangle$, j = 1, 2, …, m + 2, where m = $\left[\dfrac{n-7}{4}\right]$

where $\langle A_1 \rangle$ is the subgraph of $K_n$ induced by the edges in $G_n$.
$\langle A_2 \rangle$ is the subgraph of $K_n \setminus G_n$ induced by the edges in $G_{n-4}$.
……………………………………………………………………………………
$\langle A_{m+1} \rangle$ is the subgraph of $K_n \setminus G_n \setminus G_{n-4} \setminus \ldots \setminus G_{11}$ induced by the edges in $G_7$ and
$\langle A_{m+2} \rangle$ is the subgraph of $K_n \setminus G_n \setminus G_{n-4} \setminus \ldots \setminus G_{11}$ induced by the edges in one point union of $K_3$ and $K_{1,\left\lfloor \frac{n}{4} \right\rfloor}$.

Here also $G_n^{(j)}$, j = 1, 2, …, m + 1 are graceful and by case (i), $G_n^{(j)}$, j = m + 2 is graceful. And we have m + 2 = $\left[\dfrac{n-6}{4}\right]$+2 graceful subgraphs.

$\therefore$ SHI ($K_n$) $\leq \left[\dfrac{n-7}{4}\right] + 2$

When $n \equiv 0 \pmod 4$:

In this form we have subgraphs which contain the edges of $G_n, G_{n-4}, G_{n-8}, \ldots, G_8$ and one point union of $K_4$ and $K_{1,\left\lfloor \frac{n}{4} \right\rfloor}$.

Again let $G_n^{(j)} = \langle A_j \rangle$, j = 1, 2, … , m + 2, where m = $\left[\dfrac{n-8}{4}\right]$

where $\langle A_1 \rangle$ is the subgraph of $K_n$ induced by the edges in $G_n$.
$\langle A_2 \rangle$ is the subgraph of $K_n \setminus G_n$ induced by the edges in $G_{n-4}$.
……………………………………………………………………………………
$\langle A_{m+1} \rangle$ is the subgraph of $K_n \setminus G_n \setminus G_{n-4} \setminus \ldots \setminus G_{12}$ induced by the edges in $G_8$ and
$\langle A_{m+2} \rangle$ is the subgraph of $K_n \setminus G_n \setminus G_{n-4} \setminus \ldots \setminus G_{12}$ induced by the edges in one point union of $K_4$ and $K_{1,\left\lfloor \frac{n}{4} \right\rfloor}$.

$G_n^{(j)}$, j = 1, 2, …, m + 1 are graceful and by case (ii), $G_n^{(j)}$, j = m + 2 is graceful. And we have m + 2 = $\left[\dfrac{n-6}{4}\right]$+2 graceful subgraphs.

$\therefore$ SHI ($K_n$) $\leq \left[\dfrac{n-8}{4}\right] + 2$.

Hence the proof.

**Illustration:** The 3–structure graceful labeling of $K_{13}$ is shown below:

Here we have $13 \equiv 1 \pmod 4$.

Hence by the above result, SGI ($K_{13}$) = $\left[\dfrac{n-5}{4}\right]$ + 1 = 2 + 1 = 3.

 

For, partition the edges of $K_{13}$ into $E(G_{13})$ and $E(K_{13} \setminus G_{13})$. We have

**$G_{13}$** $K_n \setminus G_n = K_{13} \setminus G_{13}$

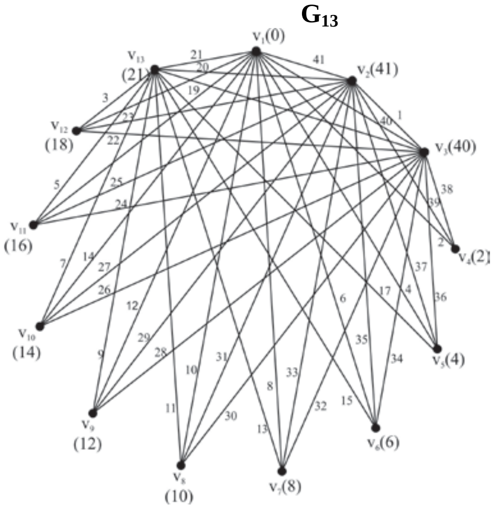


**Fig. 1**

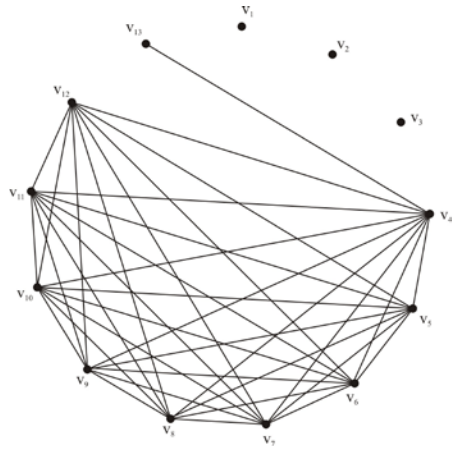


**Fig. 2**

Partition the edges of $K_{13} \setminus G_{13}$ into $E(G_{n-4} = G_9)$ and $E(\overline{K_{13} \setminus G_{13}} \setminus G_9)$

**$G_9$** $\overline{K_{13} \setminus G_{13}} \setminus G_9$ = One point union of $K_5$ and $K_{1,2}$

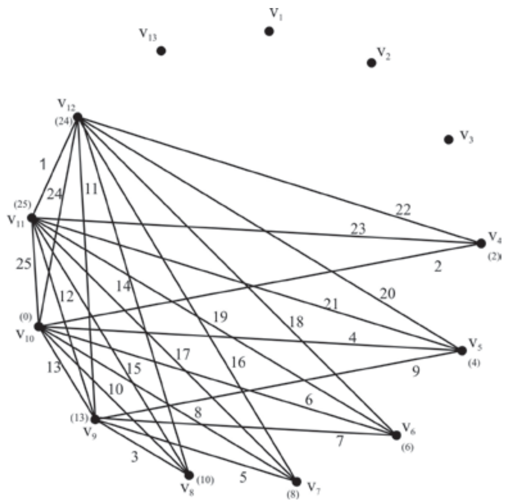


**Fig. 3**

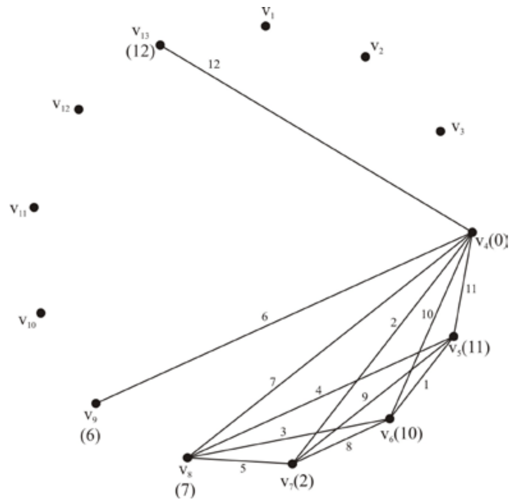


**Fig. 4**

Hence SGI $(K_{13}) = 3$.

**3–structure graceful labeling of $K_{13}$**

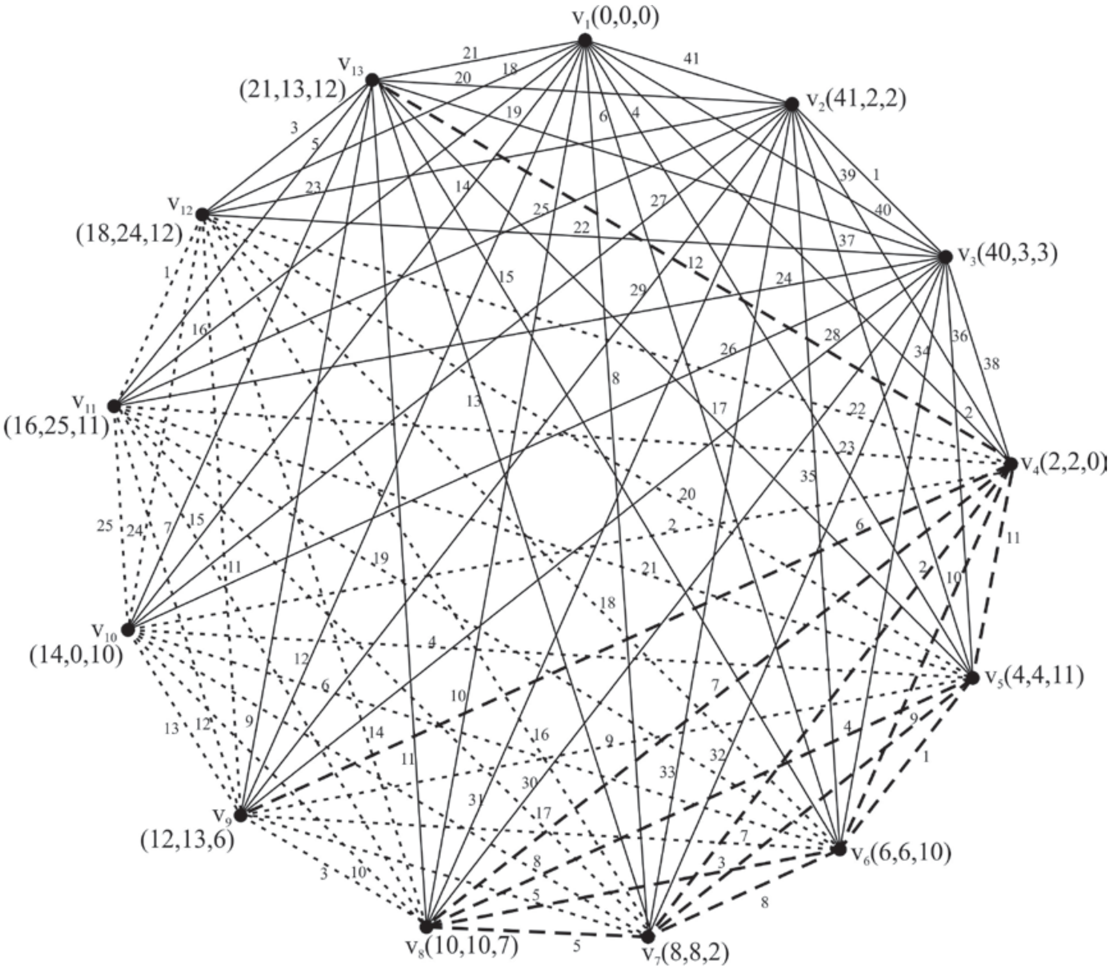


**Fig. 5**

## CONCLUSION

Decomposition of complete graphs $K_n$ into graceful subgraphs has been got for n > 10. This work may contribute much on application side. The sharpness of upper bounds for SGI ($K_n$) is yet to be tested. The extension of this sort of work to other important families of graphs such as Petersen graphs, etc. is our next target.

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