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FUNCTIONS STRONGLY MCSHANE AND KURZWEIL- HENSTOCK INTEGRABLE IN BANACH SPACE

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ABSTRACT

*In this article, we characterize the S*M integrable functions within the S*HK integrable functions in Banach space, relativizing similar propositions in the real line case and generalizing classic results.*

Key Words and Phrases: *Banach space. McShane and Kurzweil-Henstock integrals.*

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1. INTRODUCTION AND PRELIMINARIES:

In this paper, we extend some relations between McShane integrals and Kurzweil- Henstock integrals, known for the real case [2]. We apply the extension for Banach space. All the material pertains to the propositions on mean value or so called "squeeze functions" that are characterized as the Vitali-Caratheodory propositions.

We consider functions $f : I \to X$ where $I \subset \mathrm{R}$ is a compact interval, and X is a Banach space with the norm $\|\cdot\|_X$. Based on the fact that respective definitions on Banach Space are well known if we consider e.g.: [1] , [3], [4]. By μ let the Lebesgue measure in R be denoted.

An interval I is a compact subinterval of R. A collection of intervals is called nonoverlapping if their interiors are disjoint. A partition P in I is a collection $\{(I_i, t_i) : i = 1, 2, ..., r\}$, where $I_1, ..., I_r$ are nonoverlapping subintervals of I and $t_1, ..., t_r \in I$. Let a compact interval $I \subset R$ be given, we say that P is

(i) a partition in I if $\bigcup_{i=1}^{r} I_i \subset I$

(ii) a partition of I if $\bigcup_{i=1}^{r} I_i = I$

(iii) a Perron partition (or K – partition) if $t_i \in I_i, \quad i = 1, ..., r.$

(iv) a M-partition of I if $t_i \in I, \quad i = 1, ..., r.$

Given $f : I \to X$ and partition $P = \{(I_i, t_i) : i = 1, ..., r\}$ in I , we set $\sigma(f, P) = \sum_{i=1}^{r} f(t_i)\mu(I_i)$ and call this number the Riemann sum, of f associated with P.

Given $\delta : I \to (0, +\infty)$, called a gauge, a partition $P = \{(I_i, t_i) : i = 1, ..., r\}$ in I is called $\delta - fine$ if $I_i \subset (t_i - \delta(t_i), t_i + \delta(t_i))$ $, i = 1, 2, ..., r$.

A function $f : I \to X$ is called - *strongly measurable* if there exists a sequence $(f_n)_n$ of simple functions such that $f_n(t) \overset{n}{\to} f(t)$ *a.e.*

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Definition: 1 A function $f : I \to X$ is said to be strongly measurable (or Bochner) integrable if there exists a sequence $(f_n)_n$ of simple functions such that

(i) $f_n(t) \to f(t), \quad a.e.$
(ii) $\int_A f_n$ converges in X for each measurable subset A of I.

In this case we put

$$(B)\int_A f = \lim_{n\to\infty}\int_A f_n$$

Definition: 2 A function $f : I \to X$ is said to be McShane integrable, respectively Kurzweil- Henstock integrable, (briefly *McS*- integrable, respectively *KH*-integrable) on I, if there exists $\omega \in X$ satisfying the following property: given $\varepsilon > 0$ there exists a gauge δ on I such that for each $\delta - fine$ partion, respectively M-partition (K-partion), $P = \{(I_i, t_i) : i = 1,\dots, r\}$ of I, we have

$$\| \sigma(f, P) - \omega \|_X < \varepsilon$$

Denote: $\omega = (M)\int_I f(t)d\mu$ ($\omega = (KH)\int_I f(t)d\mu$) and M (KH) denotes the set of all McShane (Kurzweil-Henstock) integrable functions.

Given a set $E \subset I$ we denote by χ_E its characteristic function ($\chi_E(t) = 1$ for $t \in E$, $\chi_E(t) = 0$ otherwise).
A function $f : I \to X$ is called *McShane* (*Kurzweil- Henstock*) *integral over the set* $E \subset I$ if the function $f \cdot \chi_E : I \to X$ is *McShane* (*Kurzweil- Henstock*) *intergable.*

In the case we write $\int_I f \cdot \chi_E = \int_E f$.

2. EXTENSION OF VITALI-CARATHEODORY THEOREM ON BANACH SPACE:

Recall some basic results of the integration on Banach spaces that used for our main proposition. We mainly refer to [1] Let Z denote the family of all compact subintervals $J \subset I$. A function $F : Z \to X$ is said to be *additive* if

$$F(J \cup L) = F(J) + F(L)$$

for any nonoverlapping $J, L \in Z$ such that $J \cup L \in Z$.

Definition: 3 A function $f : I \to X$ is said to be *strongly McShane integrable* (*Kurzweil-Henstock integrable*) on I if there is an additive function $F : Z \to X$ such that for every $\varepsilon > 0$ there exists a gauge δ on I such that

$$\sum_{i=1}^{k} \| f(t_i)\mu(J_i) - F(J_i) \|_X < \varepsilon$$

for every $\delta - fine\ \ M - partion$ ($K - partion$) $P = \{(t_i, J_i) : i = 1, 2, \cdots, k\}$ of I.

Definition: 4 A function $f : I \to X$ has the *property* $S*M$ ($S*HK$) if for every $\varepsilon > 0$ there is a gauge δ on I such that

$$\sum_{i=1}^{k} \sum_{j=1}^{l} \| f(t_i) - f(s_j) \|_X \ \mu(J_i \cap L_j) < \varepsilon$$

for any $\delta - fine\ \ M - partions$ ($K - partions$) $P = \{(t_i, J_i) : i = 1, 2, \cdots, k\}$ and $Q = \{(s_j, L_j) : j = 1, 2, \cdots, l\}$ of I.

Theorem: 5 [1]. A function $f: I \to X$ is Bochner integrable if and only if f has the property $S*M$ or, equivalently, if and only if f is strongly McShane integrable.

Lemma: 6 [1]. Assume that $f: I \to X$ is Bochner integrable and let $\varepsilon > 0$ be given. Then there is a gauge $\delta: I \to (0, +\infty)$ and $\eta \in (0, \varepsilon)$ such that the following statement holds.

If $P = \{(H_m, t_m): m = 1, 2, \cdots, r\}$ is $\delta - fine\ \ M - system$ ($K - system$) for which

$$\sum_{m=1}^{r} \mu(H_m) < \eta$$

then

$$\sum_{m=1}^{r} \| f(t_m) \|_X \ \mu(H_m) < \varepsilon$$

MAIN RESULT:

Theorem: 7 Let $f: I \to X\ f$ be a strongly measurable function. The following two statements are equivalent

(a) function f is $S*M$ integrable on I
(b) there exists $S*HK$ integrable functions $g(x)$ and $h(x)$ such that for every $\eta > 0$

$$\| f(x) - [g(x) + \theta(h(x) - g(x))] \|_X < \eta \tag{1}$$

and for every $\varepsilon > 0$

$$(KH)\int_I \| h(x) - g(x) \|_X < \varepsilon \tag{2}$$

Proof: Assume that $\varepsilon > 0$ is given. Since function f is $S*M$ integrable then it is Bochner integrable [1, p.146], for which there is an T- Cauchy sequence (f_q), $q \in N$ of simple functions which converges to f almost everywhere in I. i.e.

$$\lim_{q \to \infty} \| f_q(t) - f(t) \|_X = 0 \tag{3}$$

for almost all $t \in I$.

Let $\alpha > 0$ be real number. Considering [3 p.9], function f is measurable if and only if function f has the separable range a.e. on I so we can find a set V on I such that $\mu(V) < \alpha$ and the range $f(I \setminus V)$ is separable. Construct the set $\{x_n \in f(I \setminus V): n \in N\}$ which is dense everywhere in $f(I \setminus V)$. Fix any $k \in N$ and denote

$$E_n^k = \left\{ s \in I \setminus V : \| f(s) - x_n \| < \frac{1}{k} \right\} = (I \setminus V) \cap f^{-1}[R_{\frac{1}{k}}(x_n)].$$

These sets are measurable. Since for every $s \in I \setminus V$ and for every k exists the natural number n such that

$$\| \mathrm{f(s)} - \mathrm{x}_n \|_X < 1/\mathrm{k},$$

then we get

$$\bigcup_{n=1}^{\infty} E_n^k = I \setminus V.$$

Construct the sequence of sets

$$B_n^k = E_n^k \setminus (E_1^k \cup E_2^k \cup \ldots \cup E_{n-1}^k).$$

We see that these sets are disjoint and

$$\bigcup_{n=1}^{\infty} B_n^k = I \setminus V.$$

Construct now the function with countable range

$$d^k(x) = \sum_{n=1}^{\infty} x_n 1_{B_n^k}(x) + 0 \cdot 1_{S \setminus V}(x) (k \in N).$$

We get that for every $s \in I \setminus V$ and every k

$$\| \mathrm{f}(\mathrm{s}) - d^k(x) \|_X < 1/\mathrm{k},$$

it follows that

$$\lim_{n\to\infty} \left\| f(s) - d^k(x) \right\|_X = 0$$

uniformly on $I \setminus V$. Since the set $I \setminus V$ is measurable and its measure is not greater than I, then exists a number $n_k \in N$ such that for $k > n_k$

$$\sum_{n=n_k+1}^{\infty} m(B_n^k) < \frac{1}{k}.$$

Considering the neighborhood $R_{\frac{1}{k}}(x_n)$ above mentioned, we construct $R_{\frac{1}{2k}}(x_n) \subset R_{\frac{1}{k}}(x_n)$.

Let y_n and z_n be elements of range $f(I \setminus V)$ such that $\left\| x_n - y_n \right\|_X < 1/2k$ and $\left\| x_n - z_n \right\|_X < 1/2k$ and $x_n = y_n + \theta(z_n - y_n)$ where θ is real number $0 \le \theta \le 1$. It easy to see that y_n and z_n are inner the $R_{\frac{1}{k}}(x_n)$

$$\left\| z_n - y_n \right\|_X < \left\| z_n - x_n \right\|_X + \left\| x_n - y_n \right\|_X < 1/k.$$

Construct two measurable functions

$$g(x) = \sum_{n=1}^{\infty} y_n 1_{B_n^k}(x) + 0 \cdot 1_v(x) \quad \text{and} \quad h(x) = \sum_{n=1}^{\infty} z_n 1_{B_n^k}(x) + 0 \cdot 1_v(x).$$

First, we prove the inequality (1). We obtain:

$\left\| f(s) - \left[g(s) + \theta(h(s) - g(s)) \right] \right\|_X \le \left\| f(s) - x_n \right\|_X + \left\| x_n - \left[g(s) + \theta(h(s) - g(s)) \right] \right\|_X < 1/k + 1/k = 2/k$ for every $B_n^k, n = 1, \ldots, n_k$ and inequality may not hold for $x \notin \bigcup_{n=1}^{n_k} B_n^k$. Since the function f is $S*M$ integrable, then it is Bochner integrable by [1, p.146]. To show that g and h are $S*M$ integrable we can prove that $d^k(x)$ is $S*M$ integrable.

Let p be a natural number $p \in N$. By (3) for every $x \in V$ and $q > p$ we get $\left\| f(x) - f_q(x) \right\|_X < 1/k$.

This inequality is satisfied for every B_n^k.. It follows that for $q > p$

$$\left\| d^k(x) - f_q(s) \right\|_X \le \left\| d^k(x) - f_q(s) \right\|_X + \left\| f(s) - f_q(s) \right\|_X < 2/k$$

Observing the inequalities

$$\sum_{n=1}^{\infty}\|x_n\|_X\ m(B_n^k)\le\sum_{n=1}^{\infty}\|x_n-f_q(x)\|_X\ m(B_n^k)+\sum_{n=1}^{\infty}\|f_q(x)\|_X\ m(B_n^k)<$$

$$<2/k\sum_{n=1}^{\infty}m(B_n^k)+\sum_{n=1}^{\infty}\|f_q(x)\|_X\ m(B_n^k)<(B)\int_S\|f\|_X<\infty$$

we obtain, by Lemma 1.4.1. [1, p.23], the function $d^k(x)$ is Bochner absolute integrable and satisfies the condition $S*M$ therefore also the $S*HK$ condition.

To prove the inequality (2), we can solve the inequality for the Bochner integral and $x\in I\setminus V$. We have that

$$(B)\int_{S/V}\|g(x)-h(x)\|_X\ dm=\sum_{n=1}^{\infty}\|z_n-y_n\|_X\ m(B_n^k)<\frac{1}{k}m(S).$$

The right side vanish to zero if for every $\varepsilon>0$ we choose the number k such that $\frac{1}{k}<\frac{\varepsilon}{2(b-a)}$.

For the second part of theorem, we suppose that (b) holds. Set $f_1(x)=g(x)+\theta(h(x)-g(x))$.

By the condition, for every $\varepsilon>0$

$$\|f(x)-f_1(x)\|<\varepsilon$$

It follows that

$$\|f(x)\|_X\le\|f_1(x)\|_X+\|f(x)-f_1(x)\|_X<\varepsilon+\|g(x)\|_X+\|h(x)-g(x)\|_X$$

By the [1, p.2], if f is measurable then $\|f\|_X$ is also measurable. By the [3], consequence 1.1.4., p.29], if the function $f(x)$ and $g(x)$ are KH absolute integrable then they are Mcshane integrable. This implies $\|f(x)\|_X$ is Bochner integrable.

According [1], proposition 5.1.2., p. 135] we obtain that $f(x)$ is $S*M$ integrable.

Corollary: 8 (Theorem Vitaly- Carathedory) [2]

Let f be a function $f:I\to R$, the following statements are equivalent

(a) f is M-integrable on I.
(b) f is absolutely KH-integrable on I
(c) For every $\varepsilon>0$ there are absolutely KH-integrable functions g and h such that

$$g(x)\le f(x)\le h(x)\text{ on } I \text{ and } (KH)\int_I(h(x)-g(x)<\varepsilon$$

Proof: In the case where $X=R$, it is obvious that equality

$$f(x)=g(x)+\theta(h(x)-g(x)),\quad (0\le\theta\le 1)$$

implies

$$g(x)\le f(x)\le h(x).$$

For example, if $\theta=1/2$ we have

$$G(x)\le f(x)=\frac{g(x)+h(x)}{2}\le h(x).$$

Lemma: 9 [1, p.133]. Assume that $f: I \to X$ is Bochner integrable and let $\varepsilon > 0$. Then there is a gauge $\delta: I \to (0, \infty)$ and $\eta \in (0, \varepsilon)$ such that the following statement holds.

If is an $\{(H_m, t_m), m = 1, 2, 3 \cdots r\}$ $HK - system$ $(M - system)$ $\delta - fine$ for which

$$\sum_{m=1}^{r} \mu(H_m) < \eta$$

then

$$\sum_{m=1}^{r} \| f(t_m) \|_X \ \mu(H_m) < \varepsilon$$

Theorem: 10

Let f be a function $f: I \to R$, the following statements are equivalent:

(a) function f is $S * M$ -integrable

(b) For every $\varepsilon > 0$ there are absolutely $S * KH$ -integrable functions g and h such that

$$f(x) = g(x) + \phi(x) h(x) \text{ where } \phi(x): I \to \{0,1\} \tag{1}$$

and

$$(KH) \int_I \| H(x) - g(x) \|_X < \varepsilon . \tag{2}$$

Proof: Let us choose a gauge $\delta: I \to]0, \infty[$ as in Lemma 9 and $\eta \in]0, \varepsilon/2[$. Since function f is $S * M$ integrable then it is Bochner integrable and according to definition there exists a consequence of simple functions (f_q) with converge everywhere on $I \setminus Z_{a,}$ $\mu(Z_a) = 0.$. By the Egorov theorem, there exists a subsequence of this sequence which is uniformly convergent for every $x \in I \setminus V$, when $I \supset Z_a$ and $\alpha < \eta / 4$.This implies, that there exist the measurable disjoint sets $S_i \subset I$, such that $\bigcup_{i=1}^{\infty} S_i = V$ and

$$f(x) = \sum_{i=1}^{\infty} C_i . 1_{S_i}(x).$$

Since function f is Bochner integrable, it follows that below series is absolute convergent

$$\sum_{i=1}^{\infty} \| C_i \|_X \ \mu(S_i) = (B) \int_V \| f(x) \|_X < +\infty$$

We obtain

$$\sum_{i=N+1}^{\infty} \| C_i \|_X \ \mu(S_i) < \frac{\eta}{3}.$$

By the Lesbegue theorem there exists a closed set F_i and open set G_i such that

$$F_i \subset S_i \subset G_i$$

and $\mu(G_i \setminus F_i) < \varepsilon / 2^{i+1}.$

We observe that for every i the equality holds

$1_{S_i}(x) = 1_{F_i}(x) + \phi(x) \cdot 1_{G_i}(x)$ where $\phi: I \to \{0,1\}$. According this equality, we construct the functions

$$g(x) = \sum_{i=1}^{\infty} C_i . 1_{F_i}(x) + 0 \cdot 1_{I \setminus U}(x)$$

with $U = \bigcup_{i=1}^{\infty} F_i \subset V$ and

$$h(x) = \sum_{i=1}^{\infty} C_i \cdot 1_{G_i}(x) + 0 \cdot 1_{I \setminus T}(x)$$

with $T = \bigcup_{i=1}^{\infty} G_i \supset V$ .Reviewing the proof arguments of theorem 7, we conclude that these functions are Bochner integrable and it follows that they are absolute KH-integrable.

In order to prove (2) we consider inclusion $I \setminus U \subset I \setminus T \cup T \setminus I$.

Since

$$T \setminus U = \bigcup_{i=1}^{\infty} G_i \setminus \bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} (G_i \setminus F_i)$$

We get

$$m(T \setminus U) = m(\bigcup_{i=1}^{\infty} (G_i \setminus F_i)) \le \sum_{i=1}^{\infty} m(G_i \setminus F_i) < \frac{\eta}{2}$$

If above K-system of the set, $I \setminus U$ has been taken δ -fine and satisfy

$$\sum_{m=1}^{r} m(H_m) \le \sum_{i=1}^{\infty} m(G_i \setminus F_i) < \eta$$

then we have

$$\sum_{m=1}^{r} \| h(t_m) - g(t_m) \|_X \; m(H_m) < \varepsilon$$

This proves (2).

Second part of proof is the same as in Theorem 7.

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