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FIXED POINT THEOREMS ON CYCLIC GROUPS AND NORMAL SUB GROUPS

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ABSTRACT

In this paper, some properties of fixed points on the self maps on a group are derived. Some fixed point theorems on cyclic groups and normal subgroups are proved.

Key words: *Groups, sub groups, cyclic groups, normal subgroups, homomorphism.*

AMS subject classification: *47H10, 54H25.*

INTRODUCTION

An element x in a group G is called fixed point of a self map $f : G \to G$ if $f(x) = x$. The set of all fixed points of the map f is denoted by F_f .In 2006 J.Achari and Neeraj Anant Pande [1] established fixed point theorems for a family of self maps on groups using the following concept: Let (G, *) be a group and $f_i : G\to G$ be a self map on G given by $f_i (g) = g^i$ for every $g \in G$, then $x \in G$ is a fixed point of f_i iff $o(x) \mid i\text{-}1$.

Later in 2012, I.H. Naga Raja Rao *et.al* [2] established some results of fixed points on groups by using the above concept. In this paper we established some results of fixed points on cyclic groups of a group by using this concept. The following will be known from the previous observations. Let (G, *) be a group and $f_i : G\to G$ be a self map on G given by $f_i (g) = g^i$ for each $g \in G$.

The following will be known from the previous observations.

(i) $x \in G$ is a fixed point of f_i iff x^{-1} is a fixed point.
(ii) If x, y are fixed points of f_i implies that x*y is also a fixed point of f_i . $F_{f\,i}$ the set of all fixed points of f_i, is itself a group w. r. t to * and hence a sub group of G.
(iii) For an abelian group (G, *) $F_{f\,i}$ the set of all fixed points of f_i, is a normal subgroup of G.
(iv) For any group (G, *) , the self map f_i on G is a homomorphism and $F_{f\,i}$ and ker f_i are such that ker f_i is a sub group of $F_{f\,i}$ iff ker $f_i = \{e\}$.
(v) If x is a fixed point of f_i and f_j then x is also a fixed point of $f_i \circ f_j$.
(vi) x is a fixed point of f_i iff $o(x) \mid i\text{-}1$.

Throughout this paper, For any group G under multiplication, let $f_i : G\to G$ be a self map on G defined by $f_i (g) = g^i$ for each $g \in G$, and $F_{f\,i}$ be the set of all fixed points of f_i .The following results on cyclic groups are established .

Lemma 1: If G is a cyclic group of order n, then g is a fixed point of f_i where $i< n$ implies $i\text{-}1 \mid n$.

Proof: g is a fixed point of f_i $\Rightarrow f_i (g) = g$
$\Rightarrow g^i = g$
$\Rightarrow g^{i-1} = e$
$\Rightarrow i\text{-}1 \mid n$ (since $o(G) = n$, $o(g) \mid o(G)$).

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Lemma 2: If G is a cyclic group of order n and G= < g >, and if i-1 | n and $\frac{n}{i-1} = r$ an integer, then g^r is a fixed point of f_i .

Proof: Suppose G = < g > and 0(G) = n, then $g^n = e$.

Now, i-1 | n ⇒ n = (i-1) r ($\frac{n}{i-1} = r$ forsome integer)

⇒ n+r = ir
⇒ $g^{n+r} = g^{r\,i}$
⇒ $g^n.g^r = (g^r)^i$
⇒ $g^r = (g^r)^i = f_i(g^r)$ (since $g^n = e$)

Therefore g^r is a fixed point of f_i where $\frac{n}{i-1} = r$.

Theorem 3: If G is a cyclic group of order n and G = < g > and o(G) = n, for i < n, g is a fixed point of f_i iff i-1 = n.

Proof: If g is a fixed point of f_i, $f_i(g) = g$
⇒ $g^i = g$
⇒ $g^{i-1} = e$
⇒ i-1 = n. (since g is the generator of G, G = <g>, n is least positive integer such that g^n=e)
.
Conversly, i-1 = n ⇒ $g^{i-1} = g^n = e$ (since G = < g, o (G) = n)
⇒ $g^i = g$
⇒ $f_i(g) = g$.

Therefore g is a fixed point of f_i

Example 4: Let G = < i > ={ 1, -1, i, -i }.Then G is a cyclic group of order 4 and i^2 is the fixed point of f_3, and i is fixed point of f_5 .

For, 3-1 | 4 and $\frac{4}{2} = 2$, an integer, $f_3(i^2) = i^6 = i^2$.
5-1 | 4 and $\frac{4}{4} = 1$, an integer, $f_5(i) = i^5 = i$.

Lemma 5: If G is a cyclic group of order n , then every element of G is a fixed point of f_{n+1}.

Proof: $f_{n+1}(g) = g^{n+1} = g^n.g = e.\ g = g$ for each g in G.
Therefore $f_{n+1}(g) = g\ \ \forall g \in G$.

Lemma 6: Let G be a group. Then
(i) If G is abelian then f_i is a homomorphism on G,
(ii) If G is a cyclic group of order i then ker f_i =G iff G is cyclic group of order i.

Proof:
(i) If G is abelian
$f_i(ab) = (ab)^i = a^i b^i = f(a).f(b)$
Therefore f_i is a homomorphism.

(ii) Suppose G is a cyclic group of order i.
Let x ∈ G. Then $x^i = e$
⇒ $f_i(x) = e$
⇒ x ∈ ker f_i
∴ G ⊆ ker f_i .
Clearly ker f_i ⊆ G.
∴ G = ker f_i .

On the other hand suppose ker f_i= G.

That is { x ∈ G | $f_i(x) = e$ } = G.

Then $f_i(x) = x^i = e\ \ \forall x \in G$.

∴ G is a cyclic group of order i.

Lemma 7: The set $\{f_i : G \to G \mid i \in Z_+\}$ is a commutative monoid under composition of mappings .

Proof:
(i) commutativity: For any i, j$\in Z_+$
$f_i \circ f_j(x) = f_i(x^j) = x^{j\,i}$
$= x^{i\,j} = f_j \circ f_j(x)$
$\therefore\ f_j \circ f_j = f_{j\,j} = f_{j\,i} = f_j \circ f_i\ \forall\ i,j \in Z_+$.
(ii) associativity : It is easy to observe for any i, j, k in Z_+
$(f_j \circ f_j) \circ f_k = f_i \circ (f_j \circ f_k) = f_{j\ j\,k} = f_k \circ (f_i \circ f_j)$
(iii) Identity: For 1 in Z_+ we have
$f_1 \circ f_i = f_{1\,i} = f_{i\,1} = f_i = f_i \circ f_1$.
$\therefore\ f_1$ is the identity element of $\{f_i \mid i \in Z_+\}$.

Lemma 8: If x is a fixed point of f_i or f_j then x is also a fixed point of $f_{lcm(i-1,j-1)+1.}$

Proof: $x \in F_{f\,i} \cup F_{f\,j} \Rightarrow x \in F_{f\,i}$ or $x \in F_{f\,j}$
$\Rightarrow f_i(x) = x$ or $f_j(x) = x$
$\Rightarrow x^i = x$ or $x^j = x$
$\Rightarrow o(x) \mid i-1$ or $o(x) \mid j-1$
$\Rightarrow o(x) \mid lcm(i-1, j-1)$
$\Rightarrow o(x) \mid lcm(i-1, j-1) +1-1$

$\therefore x \in F_{f\,lcm(i-1,j-1)+1}$, that is, x is a fixed point of $f_{lcm(i-1,j-1)+1}$. (From (vi))

Corollary 9: In general if x is a fixed point of $f_{i1}, f_{i2}, \ldots f_{i\,n}$ then x is a fixed point of $f_{lcm(i1-1,i2-1,\ldots,in-1)+1}$.

Theorem10: If G is a cyclic group of order n , then $F_{f\,i}$ is a cyclic subgroup of G.

Proof: Since $F_{f\,i} \subseteq G$, and a subgroup of G [1]
$F_{f\,i}$ is cyclic (subgroup of a cyclic group is cyclic)

Also $F_{f\,i}$ is abelian (Every cyclic group is abelian).

Now, we establish some results of fixed points on normal subgroups. We know that if N is a normal subgroup of a group G, then $G/N := \{xN \mid x \in G\}$ is a group under the operation on G.

Theorem 11: Let N be a normal subgroup of G, and x is a fixed point of $f_i : G \to G$ by $f_i(x) = x^i$, then xN is a fixed point of $g_i : G/N \to G/N$ defined by $g_i(xN) = x^iN$ iff $x^{i-1} \in N$.

Proof: xN is a fixed point of $g_i \Leftrightarrow x^iN = xN$
$\Leftrightarrow x^{i-1}N = N$
$\Leftrightarrow x^{i-1} \in N$.

In [3] if M, N are two normal subgroups of a group G, $M \cap N = \{e\}$ then MN=NM and hence MN is a subgroup of G.

We use this result in the following theorem.

Theorem 12: If M,N are two normal subgroups of G such that $M \cap N = \{e\}$ and x is a fixed point of $f_i \mid M$, y is a fixed point of of $f_i \mid N$, then xy is a fixed point of $f_i \mid MN$.

Proof: Let M, N be normal subgroups of G such that $M \cap N = \{e\}$. Then MN is a sub group of G and every element of M commutes with every element of N.

Now $(xy)^2 = (xy)(xy)$
$= xyyx$
$= xy^2x = xxy^2 = x^2y^2$,

Therefore $(xy)^i = x^i y^i$ for any positive integer i.

Let $h_i : MN \to MN$ defined by $h_i(xy) = (xy)^i$.

Then $h_i(xy) = (xy)^i = x^i y^i = xy$

Therefore xy is a fixed point of $f_i \mid MN$.

Now we observe that to prove the converse of the above it is needed that at least one of $o(x) \mid i-1$ or $o(y) \mid j-1$.

Corollary 13: If M, N are two normal subgroups of G such that $M \cap N = \{e\}$ if $o(x) \mid i-1$ or $o(y) \mid j-1$ then xy is a fixed point of $f_i \mid MN$, iff x is a fixed point of $f_i \mid M$, y is a fixed point of $f_i \mid N$.

Proof: If x is a fixed point of $f_i \mid M$, y is a fixed point of $f_i \mid N$,then xy is a fixed point of $f_i \mid MN$, was proved in the above theorem.

On the other hand suppose xy is a fixed point of $f_i \mid MN$.
Then $(xy)^i = xy$

$\Rightarrow x^i y^i = xy$

$\Rightarrow x^{i-1} y^{i-1} = e$

$\Rightarrow x^{i-1} = y^{i-1} \in M \cap N = \{e\}$

$\Rightarrow x^{i-1} = e,\ y^{i-1} = e$

$\Rightarrow x^i = x,\ y^i = y$

Therefore x is a fixed point of $f_i \mid M$ and y is a fixed point of $f_i \mid N$.

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