FIXED POINT THEOREMS ON CYCLIC GROUPS AND NORMAL SUBGROUPS

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ABSTRACT

In this paper, some properties of fixed points on the self maps on a group are derived. Some fixed point theorems on cyclic groups and normal subgroups are proved.

Key words: Groups, sub groups, cyclic groups, normal subgroups, homomorphism.

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INTRODUCTION

An element $x$ in a group $G$ is called fixed point of a self map $f : G \rightarrow G$ if $f(x) = x$. The set of all fixed points of the map $f$ is denoted by $F_f$. In 2006 J. Acharya and Neeraj Anant Pande [1] established fixed point theorems for a family of self maps on groups using the following concept: Let $(G, \ast)$ be a group and $f_i : G \rightarrow G$ be a self map on $G$ given by $f_i(g) = g^i$ for every $g \in G$, then $x \in G$ is a fixed point of $f_i$ iff $o(x) \mid i - 1$.

Later in 2012, I.H. Naga Raja Rao et.al [2] established some results of fixed points on groups by using the above concept. In this paper we established some results of fixed points on cyclic groups of a group by using this concept. The following will be known from the previous observations. Let $(G, \ast)$ be a group and $f_i : G \rightarrow G$ be a self map on $G$ given by $f_i(g) = g^i$ for each $g \in G$.

The following will be known from the previous observations.

(i) $x \in G$ is a fixed point of $f_i$ iff $x^{-1}$ is a fixed point.
(ii) If $x, y$ are fixed points of $f_i$ implies that $x \ast y$ is also a fixed point of $f_i$. $F_{f_i}$ the set of all fixed points of $f_i$, is itself a group w. r. t $\ast$ and hence a sub group of $G$.
(iii) For an abelian group $(G, \ast)$ $F_{f_i}$ the set of all fixed points of $f_i$, is a normal subgroup of $G$.
(iv) For any group $(G, \ast)$, the self map $f_i$ on $G$ is a homomorphism and $F_{f_i}$ and ker $f_i$ are such that ker $f_i$ is a sub group of $F_{f_i}$ iff ker $f_i = \{e\}$.
(v) If $x$ is a fixed point of $f_i$ and $f_j$ then $x$ is also a fixed point of $f_i \circ f_j$.
(vi) $x$ is a fixed point of $f_i$ iff $o(x) \mid i - 1$.

Throughout this paper, For any group $G$ under multiplication, let $f_i : G \rightarrow G$ be a self map on $G$ defined by $f_i(g) = g^i$ for each $g \in G$, and $F_{f_i}$ be the set of all fixed points of $f_i$. The following results on cyclic groups are established.

Lemma 1: If $G$ is a cyclic group of order $n$, then $g$ is a fixed point of $f_i$ where $i < n$ implies $i - 1 \mid n$.

Proof: $g$ is a fixed point of $f_i \Rightarrow f_i(g) = g$

$\Rightarrow g^i = g$

$\Rightarrow g^{i-1} = e$

$\Rightarrow i - 1 \mid n$ (since $o(G) = n$, $o(g) \mid o(G)$).

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Lemma 2: If G is a cyclic group of order n and G = < g >, and if i-1 \mid n and \frac{n}{i-1} = r an integer, then g^r is a fixed point of f_i.

Proof: Suppose G = < g > and o(G) = n, then g^n = e.

Now, i-1 \mid n \Rightarrow n = (i-1) r (\frac{n}{i-1} = r \text{ forsome integer})

\Rightarrow n+r = ir
\Rightarrow gn+r = gr i
\Rightarrow gn.gr = (gr) i
\Rightarrow gr = (gr) i = f_i (gr) (\text{since } g^n = e)

Therefore g^r is a fixed point of f_i where \frac{n}{i-1} = r.

Theorem 3: If G is a cyclic group of order n and G = < g > and o(G) = n, for i < n, g is a fixed point of f_i iff i-1 = n.

Proof: If g is a fixed point of f_i, f_i (g) = g

\Rightarrow g^{i-1} = g
\Rightarrow g^i = g
\Rightarrow f_i (g) = g

Conversly, i-1 = n \Rightarrow g^{i-1} = g^n = e (\text{since G = < g>, o (G) = n})

\Rightarrow g^i = g
\Rightarrow f_i (g) = g.

Therefore g is a fixed point of f_i.

Example 4: Let G = < i > = { 1, -1, i, -i }.Then G is a cyclic group of order 4 and i^2 is the fixed point of f_3, and i is fixed point of f_5.

For, 3-1 \mid 4 and \frac{4}{4} = 2, an integer, f_3 (i^2) = i^6 = i^2.

5-1 \mid 4 and \frac{4}{4} = 1, an integer, f_5 (i) = i^5 = i.

Lemma 5: If G is a cyclic group of order n, then every element of G is a fixed point of f_{n+1}.

Proof: f_{n+1} ( g ) = g^{n+1} = g^n .g = e. g = g for each g in G.

Therefore f_{n+1} ( g ) = g \forall g \in G.

Lemma 6: Let G be a group. Then
(i) If G is abelian then f_i is a homomorphism on G,
(ii) If G is a cyclic group of order i then ker f_i = G iff G is cyclic group of order i.

Proof:
(i) If G is abelian then f_i is a homomorphism.

f_i (ab) = (ab)^i = a^i b^i = f(a).f(b)

Therefore f_i is a homomorphism.

(ii) Suppose G is a cyclic group of order i.

Let x \in G. Then x^i = e

\Rightarrow f_i (x) = e
\Rightarrow x \in ker f_i
\Rightarrow G \subseteq ker f_i.
\Rightarrow ker f_i = G.

On the other hand suppose ker f_i = G.

That is { x \in G \mid f_i (x) = e } = G.\n
Then f_i (x) = x^i = e \forall x \in G.

\Rightarrow G is a cyclic group of order i.
Lemma 7: The set \{f_i : G \rightarrow G \mid i \in Z_+\} is a commutative monoid under composition of mappings.

Proof:
(i) commutativity: For any \(i, j \in Z_+\),
\[ f_i \circ f_j(x) = f_i(f_j(x)) = x^{ij} = x^{ji} = f_j \circ f_i(x) \]
\[ f_j \circ f_i(x) = f_j(f_i(x)) = x^{ji} = x^{ij} = f_i \circ f_j(x) \]
\(\therefore f_j \circ f_i = f_i \circ f_j \forall i, j \in Z_+\).

(ii) associativity: It is easy to observe for any \(i, j, k \in Z_+\),
\[ (f_j \circ f_i) \circ f_k = f_i \circ (f_j \circ f_k) \]
\[ = f_j \circ (f_i \circ f_k) \]
\[ = f_j \circ (f_k \circ f_i) \]
\[ = f_j \circ f_k \circ f_i \]
\[ = f_j \circ f_k \circ f_i \]
\[ = f_i \circ (f_j \circ f_k) \]
\[ = f_i \circ f_j \circ f_k \]
\[ = f_i \circ f_j \circ f_k \]
\[ = f_i \circ f_j \circ f_k \]
\[ = f_i \circ f_j \circ f_k \]
\(\therefore f_i \circ (f_j \circ f_k) = (f_i \circ f_j) \circ f_k \)
\(\forall i, j, k \in Z_+\).

(iii) Identity: For \(1 \in Z_+\) we have
\[ f_1 \circ f_i = f_1 = f_i \circ f_1 \]
\(\therefore f_1 \) is the identity element of \{\(f_i \mid i \in Z_+\}\).

Lemma 8: If \(x\) is a fixed point of \(f_i\) or \(f_j\) then \(x\) is also a fixed point of \(f_{\text{lcm}(i-1,j-1)+1}\).

Proof: \(x \in F_{f_i} \cup F_{f_j} \Rightarrow x \in F_{f_i} \) or \(x \in F_{f_j}\)
\[ f_i(x) = x \text{ or } f_j(x) = x \]
\[ \Leftrightarrow x = x \circ x = x \circ x \]
\[ = o(x) \mid i-1 \text{ or } o(x) \mid j-1 \]
\[ = o(x) \mid \text{lcm}(i-1,j-1) \]
\[ = o(x) \mid \text{lcm}(i-1,j-1) + 1 - 1 \]
\(\therefore x \in F_{f_{\text{lcm}(i-1,j-1)+1}}, \) that is, \(x\) is a fixed point of \(f_{\text{lcm}(i-1,j-1)+1}\). (From (vi))

Corollary 9: In general if \(x\) is a fixed point of \(f_{i_1}, f_{i_2}, \ldots, f_{i_n}\) then \(x\) is a fixed point of \(f_{\text{lcm}(i_1-1,i_2-1,\ldots,i_n-1)+1}\).

Theorem 10: If \(G\) is a cyclic group of order \(n\), then \(F_{f_i}\) is a cyclic subgroup of \(G\).

Proof: Since \(F_{f_i} \subseteq G\), and a subgroup of \(G\) is cyclic (subgroup of a cyclic group is cyclic).
Also \(F_{f_i}\) is abelian (Every cyclic group is abelian).

Now, we establish some results of fixed points on normal subgroups. We know that if \(N\) is a normal subgroup of a group \(G\), then \(G/N \:= \{xN \mid x \in G \}\) is a group under the operation on \(G\).

Theorem 11: Let \(N\) be a normal subgroup of \(G\), and \(x\) is a fixed point of \(f_i : G \rightarrow G\) by \(f_i(x) = x^i\), then \(xN\) is a fixed point of \(g_i : G/N \rightarrow G/N\) defined by \(g_i(xN) = x^iN\) iff \(x^{i-1} \in N\).

Proof: \(xN\) is a fixed point of \(g_i \Leftrightarrow x^iN = xN\)
\[ \Leftrightarrow x^iN = xN \]
\[ \Leftrightarrow x^{i-1}N = N \]
\[ \Leftrightarrow x^{i-1} \in N \]

In [3] if \(M, N\) are two normal subgroups of a group \(G\), \(M \cap N = \{e\}\) then \(MN = NM\) and hence \(MN\) is a subgroup of \(G\).
We use this result in the following theorem.

Theorem 12: If \(M, N\) are two normal subgroups of \(G\) such that \(M \cap N = \{e\}\) and \(x\) is a fixed point of \(f_i \mid M\) and \(y\) is a fixed point of \(f_i \mid N\), then \(xy\) is a fixed point of \(f_i \mid MN\).

Proof: Let \(M, N\) be normal subgroups of \(G\) such that \(M \cap N = \{e\}\). Then \(MN\) is a subgroup of \(G\) and every element of \(M\) commutes with every element of \(N\).

Now \((xy)^2 = (xy)(xy)\)
\[ = xyyx \]
\[ = x^2y^2 = x^{2i} \]
Therefore \((xy)^i = x^iy^i\) for any positive integer \(i\).

Let \(h : MN \rightarrow MN\) defined by \(h(xy) = (xy)^i\).

Then \(h(xy) = (xy)^i = x^iy^i = xy\)
Therefore $xy$ is a fixed point of $f_i|MN$.

Now we observe that to prove the converse of the above it is needed that at least one of $o(x) | i-1$ or $o(y) | j-1$.

**Corollary 13:** If $M$, $N$ are two normal subgroups of $G$ such that $M \cap N = \{e\}$ if $o(x) | i-1$ or $o(y) | j-1$ then $xy$ is a fixed point of $f_i|MN$, iff $x$ is a fixed point of $f_i|M$, $y$ is a fixed point of $f_i|N$.

**Proof:** If $x$ is a fixed point of $f_i|M$, $y$ is a fixed point of $f_i|N$, then $xy$ is a fixed point of $f_i|MN$, was proved in the above theorem.

On the other hand suppose $xy$ is a fixed point of $f_i|MN$.

Then $(xy)^i = xy$

$\Rightarrow x^iy^i = xy$

$\Rightarrow x^iy^i = e$

$\Rightarrow x^i = y^{-1} \in M \cap N = \{e\}$

$\Rightarrow x^i = e, y^{-1} = e$

$\Rightarrow x = x, y = y$

Therefore $x$ is a fixed point of $f_i|M$ and $y$ is a fixed point of $f_i|N$.

**REFERENCES**


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