PSEUDO PROJECTIVELY FLAT ALMOST PSEUDO RICCI-SYMMETRIC MANIFOLDS

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ABSTRACT

The object of the present paper is to study pseudo projectively flat almost pseudo Ricci symmetric manifolds.


Keywords and Phrases: Pseudo Ricci-symmetric manifolds, almost pseudo Ricci-symmetric manifolds, pseudo-projective curvature tensor, Scalar curvature, con-circular vector field.

1. INTRODUCTION

As an extended class of pseudo Ricci symmetric manifolds, very recently M.C.Chaki and T. Kawaguchi [1] introduced the notation of almost pseudo Ricci-symmetric manifolds. A Riemannian manifold \((M^n, g)\) is called an almost pseudo Ricci-symmetric manifold if its Ricci tensor \(S\) of type \((0,2)\) is not identically zero and satisfies a relation

\[
(D_X S)(Y, Z) = (A(X) + B(X))S(Y, Z) + A(Y)S(X, Z) + A(Z)S(Y, X),
\]

where \(D\) denotes the operator of covariant differentiation with respect to the Riemannian metric \(g\) and \(A, B\) are nowhere vanishing 1 - forms such that \(g(X, \rho) = A(X)\) and \(g(X, \mu) = B(X)\) for all \(X, \rho\) and \(\mu\) are called the basic vector fields of the manifold.

The one form \(A\) and \(B\) are called the associated 1–forms and \(n\)–dimensional manifold of this kind is denoted by \(A(PRS)_n\).

If, in particular \(B = A\), then (1) reduces to

\[
(D_X S)(Y, Z) = 2A(X)S(Y, Z) + A(Y)S(X, Z) + A(Z)S(Y, X)
\]

which represents a pseudo Ricci-symmetric manifold [2]. In [1], Chaki and Kawaguchi also studied conformally flat \(A(PRS)_n\). Recently Shaikh and Hui [5] studied the properties of quasi-conformally flat almost pseudo Ricci-symmetric manifold. In [6], Prasad defined and studied a tensor field \(\hat{P}\) of type \((1,3)\) which is the generalisation of Weyl projective curvature tensor, called pseudo-projective curvature tensor. The present paper deals with a study of pseudo-projectively flat \(A(PRS)_n\).

The paper is organized as follows. Section 2 concerned with preliminaries. Section 3 devoted to the study of pseudo-projectively flat \(A(PRS)_n\) and proved that the vector fields \(\mu\) and \(\xi\) are co-directional. It is shown that in a Pseudo-projectively flat \(A(PRS)_n\) the integral curves of the generator \(\lambda\) defined by \(g(X, \lambda) = T(X)\) are geodesic and the vector field \(\lambda\) is a unit proper con-circular vector field. Also it is shown that in this manifold the Ricci tensor is Codazzi type and the vector field \(\lambda\) is a unit parallel vector field.

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2. PRELIMINARIES

Let \( Q \) be the symmetric endomorphism of the tangent space at any point of the manifold corresponding to the Ricci tensor \( S \), i.e., \( S(X, Y) = g(QX, Y) \) for all vector fields \( X, Y \) and \( \{e_i\}, i = 1, 2, 3 \ldots \ n \) be an orthonormal basis of tangent space at any point of the manifold. Then by setting \( Y = Z = e_i \) in (1) and then taking summation over \( i, 1 \leq i \leq n \), we obtain

\[
dr(X) = r(A(X) + B(X)) + 2A(QX)
\]

where \( r \) is the scalar curvature of the manifold.

Again from (1), we get

\[
(D_s S)(Y, Z) - (D_s S)(X, Z) = B(X)S(Y, Z) - B(Y)S(X, Z)
\]

(4)

Setting \( Y = Z = e_i \) in (4) and then taking summation over \( i \), for \( 1 \leq i \leq n \), we obtain

\[
dr(X) = 2rB(X) - 2B(QX)
\]

(5)

If the scalar curvature \( r \) is constant, then

\[
dr(X) = 0, \text{ for all } X.
\]

(6)

By virtue of (6), (5) yields,

\[
rB(X) = B(QX)
\]

(7)

i.e., \( S(X, \mu) = r g(X, \mu) \)

(8)

Proposition 1: In an \( A(PRS)_n \) of constant scalar curvature, \( r \) is an eigen value of the Ricci tensor \( S \) corresponding to the eigen vector \( \mu \).

The pseudo-projective curvature tensor \( \hat{P} \) of type \((1, 3)\) is defined by \([6]\)

\[
\hat{P}(X, Y)Z = -(n - 1)b P(X, Y)Z + [a + (n - 1)b)C(X, Y)Z
\]

(9)

where \( a \) and \( b \) are arbitrary constants not simultaneously zero and \( P, C \) are respectively Weyl projective and concircular curvature tensors. It bridges the gap between the Weyl projective and concircular curvature tensors. Its tensorial relation is given by

\[
\hat{P}(X, Y)Z = a R(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y] - \frac{r}{n} \left[ \frac{a}{n-1} + b \right] [g(Y, Z)X - g(X, Z)Y]
\]

(10)

\[
(div \hat{P})(X, Y)Z = a(div R)(X, Y)Z + b[(D_s S)(Y, Z) - (D_s S)(X, Z)] - \frac{1}{n-1} \left[ \frac{a}{n-1} + 2b \right] [g(Y, Z)drX - g(X, Z)drY]
\]

(11)

where \( div \) denotes divergence. Again it is known that in a Riemannian manifold, we have

\[
(div R)(X, Y)Z = [(D_s S)(Y, Z) - (D_s S)(X, Z)]
\]

(12)

Consequently by the virtue of above equation (11) takes the form

\[
(div \hat{P})(X, Y)Z = (a + b)[(D_s S)(Y, Z) - (D_s S)(X, Z)] - \frac{1}{n-1} \left[ \frac{a}{n-1} + 2b \right] [g(Y, Z)drX - g(X, Z)drY]
\]

(12)

3. PSEUDO PROJECTIVELY FLAT \( A(PRS)_n \)

Let us consider a pseudo projectively flat \( A(PRS)_n \), then we have

\[
(div \hat{P})(X, Y)Z = 0.
\]

(13)

and hence (12) yields

\[
(a + b)[(D_s S)(Y, Z) - (D_s S)(X, Z)] = - \frac{1}{n-1} \left[ \frac{a}{n-1} + 2b \right] [g(Y, Z)drX - g(X, Z)drY]
\]

(14)

By virtue of (3) and (5), it follows from (14) that,

\[
(a + b)[B(X)S(Y, Z) - B(Y)S(X, Z)] = 2 \left[ \frac{a + (n - 1)b}{n(n-1)} \right]
\]

\[
[r(g(Y, Z)B(X) - g(X, Z)B(Y)) - (g(Y, Z)B(QX) - g(X, Z)B(QY))]
\]

(15)

Provided that \( a + b \neq 0 \). Putting \( Z = \mu \) in (15), we obtain

\[
B(X)B(QY) - B(Y)B(QX) = 0.
\]

(16)

Provided that \( a + b \neq 0 \) and \( (n + 1)a - (n - 1)b \neq 0 \).
Let \( B(QX) = g(QX, \mu) = P(X) = g(X, \xi) \), for all \( X \).

Then from (16), we get,
\[
B(X)P(Y) = B(Y)P(X),
\]
(17)
which shows that the vector fields \( \mu \) and \( \xi \) are co-directional. Hence we can state the following

**Theorem 3.1:** In a pseudo-projectively flat \( A(PRS)_n \) (\( n > 2 \)), with \( a + b \neq 0 \) and \( (n + 1)a - (n - 1)b \neq 0 \), the vector fields \( \mu \) and \( \xi \) are co-directional.

If \( a + b = 0 \) and \( (n + 1)a - (n - 1)b \neq 0 \), then using (5) in (14), it can be easily shown that the relation (17) holds. Hence we can state the following

**Corollary 3.1:** In a pseudo-projectively flat \( A(PRS)_n \) (\( n > 2 \)), with \( a + b = 0 \) and \( (n + 1)a - (n - 1)b \neq 0 \), the vector fields \( \mu \) and \( \xi \) are co-directional.

Again if \( a + b \neq 0 \) and \( (n + 1)a - (n - 1)b = 0 \), then using (4) in (14), it can be easily shown that the relation (17) holds. Hence we can state the following

**Corollary 3.2:** In a pseudo-projectively flat \( A(PRS)_n \) (\( n > 2 \)), with \( a + b \neq 0 \) and \( (n + 1)a - (n - 1)b = 0 \), the vector fields \( \mu \) and \( \xi \) are co-directional.

It may be noted that in a pseudo-projectively flat \( A(PRS)_n \) (\( n > 2 \)), with \( a + b = 0 \) and \( (n + 1)a - (n - 1)b = 0 \) cannot hold simultaneously as \( a \) and \( b \) are not simultaneously zero.

Again setting \( Y = Z = e_i \) in (15) and then taking summation over \( i \), \( 1 \leq i \leq n \), then we obtain
\[
B(QX) = rB(X),
\]
(18)
Provided that \( a - b \neq 0 \), i.e.
\[
S(X, \mu) = rg(X, \mu)
\]
(19)
Hence we can state the following

**Theorem 3.2:** In a pseudo-projectively flat \( A(PRS)_n \) (\( n > 2 \)), with \( a + b \neq 0 \) and \( a - b \neq 0 \), \( r \) is an eigen value of the Ricci tensor \( S \) corresponding to the eigen vector \( \mu \).

If \( a + b = 0 \), then it follows from (15) that (19) holds provided that \( a + (n - 1)b \neq 0 \).

Hence, we can state the following

**Corollary 3.3:** In a pseudo-projectively flat \( A(PRS)_n \) (\( n > 2 \)), with \( a + b = 0 \) and \( a - (n - 1)b \neq 0 \), \( r \) is an eigen value of the Ricci tensor \( S \) corresponding to the eigen vector \( \mu \).

Also for \( a + b \neq 0 \) and \( a + (n - 1)b = 0 \), then we can state the following

**Corollary 3.4:** In a pseudo-projectively flat \( A(PRS)_n \) (\( n > 2 \)), with \( a + b \neq 0 \) and \( a + (n - 1)b = 0 \), \( r \) is an eigen value of the Ricci tensor \( S \) corresponding to the eigen vector \( \mu \).

In view of (18), (15) yields
\[
B(X)S(Y, Z) = B(Y)S(X, Z).
\]
(20)
Setting \( X = \mu \) in (20), we get
\[
S(Y, Z) = \frac{1}{B(\mu)}B(Y)B(QZ).
\]
(21)
In view of (18), (21) yields
\[
S(Y, Z) = rT(Y)T(Z),
\]
(22)
where \( T(X) = g(X, \lambda) = \frac{1}{\sqrt{B(\mu)}}B(X) \), \( \lambda \) being a unit vector field associated with the nowhere vanishing 1-form \( T \).

From (22), it follows that if \( r = 0 \), then \( S(Y, Z) = 0 \), which is inadmissible by the definition of \( A(PRS)_n \). Hence we can state the following

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Theorem 3.3: In a pseudo-projectively flat \( A(PR S)_n \) \((n > 2)\), with \( a + b \neq 0 \) and \( a - b \neq 0 \), the scalar curvature cannot vanish and the Ricci tensor is of the form (22).

Again from (22), we have
\[
(D_S)(Y, Z) = dr(X)T(Y)T(Z) + r((D_T)(Y)T(Z) + (D_T)(Z)T(Y)).
\]

Using (23) in (14) we obtain
\[
(a + b)[(dr(T)(Y)T(Z) - dr(Y)T(Z))]r((D_T)(Y)T(Z) + (D_T)(Z)T(Y) - (D_T)(X)T(Z) - (D_T)(Z)T(X))
\]
\[
= \frac{1}{n} \left[ \frac{a}{n-1} + 2b \right] (g(Y, Z)drX - g(X, Z)drY).
\]

Setting \( Y = Z = e_i \) in (24) and then taking summation over, \( 1 \leq i \leq n \), then we obtain
\[
(a + b) \left[ dr(\lambda)T(X) + r((D_T)(X)T(X) + \sum_{i=1}^{n-1} (D_{e_i}T)(e_i)) \right] = \left( \frac{(n-1)a + b}{n} \right) dr(X).
\]

Again putting \( Y = Z = \lambda \) in (24), we get
\[
r(a + b)(D_T)(X) = \left( \frac{(n^2 - n - 1)a + (n - 1)b}{n(n-1)} \right) (dr(X) - T(X)dr(\lambda)).
\]

Using (26) in (25), we get
\[
r(a + b)T(X) \sum_{i=1}^{n-1} (D_{e_i}T)(e_i) + E[(n - 2)dr(X) + dr(\lambda)T(X)] = 0,
\]
where \( E = \frac{a(n-1)b}{n(n-1)} \). Substituting \( X = \lambda \) in (27), we get
\[
r(a + b) \sum_{i=1}^{n-1} (D_{e_i}T)(e_i) = -(n - 1)Edr(\lambda),
\]

From (27) and (28), we have
\[
dr(X) = dr(\lambda)T(X)
\]

Provided that \( a + (n - 1)b \neq 0 \). Again putting \( Z = \lambda \) in (24) and then using (29), we get
\[
r(a + b)((D_T)(Y) - (D_T)(X)) = 0,
\]

which implies that
\[
(D_T)(Y) = (D_T)(X) = 0,
\]

because \( r \neq 0 \) and \( a + b \neq 0 \). The relation (30) implies that the 1-form \( T \) is closed.

In view of (29), it follows from (26) that
\[
(D_T)(X) = 0,
\]

provided \( a + b \neq 0 \), which implies that \( D_T\lambda = 0 \). Hence we can state the following

Theorem 3.4: In a pseudo-projectively flat \( A(PR S)_n \) \((n > 2)\), with \( a + b \neq 0 \) and \( a - b \neq 0 \) and \( a + (n - 1)b \neq 0 \), the integral curve of the generator \( \lambda \) are geodesics.

Also setting \( Y = \lambda \) in (24), we obtain by virtue of (29) and (31) that
\[
(D_T)(Z) = \frac{E}{r(a + b)} dr(\lambda)(T(X)T(Z) - g(X, Z)),
\]

provided that \( a + b \neq 0 \).

Let us now consider a non zero scalar function \( f = \frac{E}{r(a + b)}dr(\lambda) \), where the scalar curvature \( r \) is non constant. Then we have
\[
D_X f = \frac{E}{r^2(a + b)}[dr(\lambda)dr(X) - rd^2r(\lambda, X)]
\]

From (29) it follows that
\[
d^2r(X, Y) = d^2r(\lambda, Y)T(X) + dr(\lambda)(D_T)(X).
\]

Again in a Riemannian manifold, the second covariant derivative of any function \( h \in C^\infty(M) \) is defined by
\[
d^2h(X, Y) = X(Yh) - (D_X Y)(h),
\]

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for all \(X,Y \in \mathfrak{X}(M)\), which implies that
\[ d^2h(X,Y) = d^2h(Y,X), \]
for all \(X,Y \in \mathfrak{X}(M)\), and hence (34) implies that
\[ d^2r(\lambda,Y) = d^2r(\lambda,X)T(Y), \]  \(\text{(35)}\)
replacing \(Y\) by \(\lambda\) in (35) we have
\[ d^2r(\lambda,X) = d^2r(\lambda,\lambda)T(X) = \psi T(X) \]
where \(\psi = d^2r(\lambda,\lambda)\) is a scalar function. Using (29) and (36) in (33), we obtain
\[ D_\lambda f = \sigma T(X), \]
where \(\sigma = \frac{E}{r^2(a+b)^2} \left( r^2 - (dr(\lambda))^2 \right)\) is a non zero scalar.

We now consider a 1-form \(\omega\), given by
\[ \omega(X) = \frac{E}{r(a-b)} dr(\lambda)T(X) = f T(X). \]
Then by the virtue of (30), (37) and above relation, we have
d\(\omega(X,Y) = 0\).

Hence 1-form \(\omega\) is closed, therefore (32) can be written as
\[ (D_\lambda T)(Z) = -fg(X,Z) + \omega(X)T(Z), \]  \(\text{(38)}\)
which implies that the vector field \(\lambda\) corresponding to the 1-form \(T\) defined by \(g(X,\lambda) = T(X)\) is a proper concircular vector field [4], [7].

Hence we can state the following

**Theorem 3.5:** In a pseudo-projectively flat \(A(PRS)\) \((n > 2)\) of non constant scalar curvature with \(a + b \neq 0\) and \(a - b \neq 0\) and \(a + (n - 1)b \neq 0\), the vector field \(\lambda\) is a unit proper concircular vector field.

If in particular, \(a + b = 0\), then from (14), we get
\[ dr(X)g(Y,Z) - dr(Y)g(X,Z) = 0, \]
which yields,
\[ dr(X) = 0, \]  \(\text{(39)}\)
for all \(X\), provided that \(a + (n - 1)b \neq 0\). This means that the scalar curvature of the pseudo-projectively flat \(A(PRS)\) is constant.

Hence we can state the following

**Theorem 3.6:** In a pseudo-projectively flat \(A(PRS)\) \((n > 2)\), the scalar curvature \(r\) is constant provided \(a + b = 0\) and \(a + (n - 1)b \neq 0\).

Again if \(a + b \neq 0\) and \(a + (n - 1)b = 0\), then from (14), we get
\[ (D_\lambda S)(Y,Z) = (D_\lambda S)(X,Z) \]
\(\text{(40)}\)
for all \(X,Y,Z\), which means that the Ricci tensor is of Codazzi type [3] and hence
\[ dr(X) = 0, \]
\(\text{(41)}\)
for all \(X\), hence (24) takes the form
\[ (D_\lambda T)(Y)T(Z) - (D_\lambda T)(X + T(Y))T(Z)(D_\lambda T)(Z) - T(X)(D_\lambda T)(Z) = 0. \]

Putting \(Z = \lambda\) in (42), we get (30). Also for \(Y = \lambda\) (30) implies that
\[ (D_\lambda T)(X) = 0, \]
for all \(X\). Using this relation we obtain from (42) (for \(Y = \lambda\)) that \( (D_\lambda T)(Z) = 0, \) for all \(X,Z\). This implies that
\[ g(Z,D_\lambda \lambda) = 0. \]

For all \(X,Z\). Since \(g\) is non degenerate, the last relation yields \(D_\lambda \lambda = 0, \) for all \(X\), which means that \(\lambda\) is a parallel vector field. Hence we can state the following
Theorem 3.7: In a pseudo-projectively flat $A(PRS)_n$ ($n > 2$) with $a + (n - 1)b = 0$ and provided $a + b \neq 0$, the Ricci tensor is of Codazzi type and the vector field $\lambda$ is a unit parallel vector field.

REFERENCES


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