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ON RINGS AND MODULES WITH DESCENDING CHAIN CONDITIONS ON THEIR SUBSTRUCTURES

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ABSTRACT

The aim of this paper our attempt is to study fuzzy aspect of rings and modules with descending chain conditions on their fuzzy subsrtuctures and find out some new results about fuzzy ideals. Fuzzy min-E module is defined and study some characteristics of such modules.

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1. INTRODUCTION

American Cyberneticist L.A. Zadeh in [8] described the notion of fuzzy subsets for the first time in 1965. In 1971 Rosenfeld [7] studied fuzzy subgroup of a group and opened a new direction towards the fuzzy algebra. The concept of fuzzy ideals of ring was introduced and examined by Liu [3] around 1982. Negoita and Ralescu first introduced fuzzy submodules of modules over a ring in [5]. In this paper we study rings with descending chain condition (D.C.C) on their fuzzy substructures and various results are established. With the help of essential fuzzy submodule introduced by M. Kalita and H.K. Saikia [1] fuzzy min-E module is defined and study some characteristics of modules with descending chain condition on fuzzy submodules.

2. PRELIMINARIES

Throughout this paper R is a non commutative ring with unity and M is an R-module. By a fuzzy subset of ring R, we mean any mapping μ from R to [0, 1]. $[0,1]^R$ denotes the set of all fuzzy subsets of R. For each fuzzy subset $\mu \in R$ and $t \in [0,1]$, the set $\mu_t = \{x \in R | \, \mu(x) \geq t\}$ is called a t-level subset of μ and $\mu^* = \{x \in R | \, \mu(x) > 0\}$ is called the support of μ.

Definition 2.1 [4]: A fuzzy subset μ of R is called a fuzzy left ideal if it satisfies the following properties:
(i) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$, for all $x, y \in R$
(ii) $\mu(xy) \geq \mu(y)$, for all $x, y \in R$

Definition 2.2 [4]: A fuzzy subset μ of R is called a fuzzy right ideal if it satisfies the following properties:
(i) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$, for all $x, y \in R$
(ii) $\mu(xy) \geq \mu(x)$, for all $x, y \in R$

Definition 2.3 [4]: A fuzzy subset μ of R is called a fuzzy ideal if it satisfies the following properties:
(i) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$, for all $x, y \in R$
(ii) $\mu(xy) \geq \mu(x) \vee \mu(y)$, for all $x, y \in R$

Definition 2.4 [4]: A fuzzy subset μ of M is called a fuzzy submodule of M if the following conditions are satisfied:
(i) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$, for all $x, y \in M$.
(ii) $\mu(rx) \geq \mu(x)$, for all $r \in R, x \in M$.
(iii) $\mu(0) = 1$

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The set of all fuzzy submodules is denoted by F(M).

Definition 2.5 [1]: A fuzzy submodule μ of M is called an essential fuzzy submodule of M, denoted by $\mu \subseteq_e M$ if for every nonzero fuzzy submodule ν of $M, \mu \cap \nu \neq \chi_\theta$.

Definition 2.6 [1]: Let μ and σ be two nonzero fuzzy submodules of M such that $\mu \subseteq \sigma$. Then μ is called fuzzy essential in σ, denoted by $\mu \subseteq_e \sigma$ if for every nonzero fuzzy submodule ν of M satisfying $\nu \subseteq \sigma, \mu \cap \nu \neq \chi_\theta$.

Definition 2.7 [6]: A fuzzy submodule δ of M is said to be fuzzy simple submodule if $\mu \subseteq \delta$ where $\mu \in F(M)$ implies either $\mu = \chi_\theta$ or $\mu = \delta$.

Definition 2.8 [6] If μ is fuzzy submodule of M then the socle of μ, denoted by $soc\mu$ is defined as the sum of all fuzzy simple submodules of μ. If μ has no fuzzy simple submodules then $soc\mu = \chi_\theta$.

Lemma 2.1 [6]: Let μ and σ be two nonzero fuzzy submodules of M such that μ is fuzzy essential in σ. Then for any fuzzy submodule ν of $M, \mu \cap \nu \subseteq_e \sigma \cap \nu$.

Lemma 2.2 [4]: Let $\nu \in F(M)$ and A be a submodule of M. Define $\xi \in [0,1]^{\frac{M}{A}}$ as follows:

$$\xi([x]) = \vee\{\nu(u) \mid u \in [x]\}, \text{for all } x \in M$$

where $\frac{M}{A}$ denotes the quotient module of M with respect to A and [x] represents the coset $x + A$. Then $\xi \in F\left(\frac{M}{A}\right)$.

Let $\mu, \nu \in F(M)$ be such that $\mu \subseteq \nu$. Then both μ^* and ν^* are submodules of M. Clearly, $\mu^* \subseteq \nu^*$. Thus μ^* is a submodule of ν^*. Moreover, it is also clear that $\nu|_{\nu^*} \in F(\nu^*)$. Therefore, it follows from the above lemma that if we define $\xi \in F\left(\frac{\nu^*}{\mu^*}\right)$ as follows:

$$\xi([x]) = \vee\{\nu(z) \mid z \in [x]\},$$

$\forall\, x \in \nu^*$, where $[x]$ denotes the coset $x + \nu^*$, then $\xi \in F\left(\frac{\nu^*}{\mu^*}\right)$. The fuzzy submodule ξ is called the quotient of ν with respect to μ and written as $\frac{\nu}{\mu}$.

Lemma 2.3[6]: If $\mu \in F(M)$ and ξ is the intersection of all essential fuzzy sub- modules of μ then $soc\mu = \xi$.

Lemma 2.4[1]: Let μ, ν and σ be nonzero fuzzy submodules of M such that $\mu \subseteq \nu \subseteq \sigma$. Then $\mu \subseteq_e \sigma$ if and only if $\mu \subseteq_e \nu \subseteq_e \sigma$.

Definition 2.8: A ring in which every strictly descending chain of fuzzy left ideals is finite is called a fuzzy left Artinian ring. Also a fuzzy subset μ of a ring R is called fuzzy left Artinan if every strictly descending chain of fuzzy left ideals of μ is finite.

Definition 2.9: A fuzzy submodule μ of M is called a fuzzy min-E module if every descending chain on essential fuzzy submodules of μ is finite.

Theorem 2.1: If A is any ideal of a ring R, $A \neq R$, then the fuzzy subset μ of R defined by

$$\mu(x) = \begin{cases} \text{s, if x} \in \text{A} \\ t, if\ x \in R - A \end{cases}$$

where $s, t \in [0, 1], s > t$ is a fuzzy ideal of R.

3 MAIN RESULTS

Theorem 3.1: If a ring R with unity is fuzzy left Artinian then every fuzzy left ideal on R has finite number of values.

Proof: Let R be fuzzy left Artinian.Suppose there exists a fuzzy left ideal μ of R such that $Im\mu = \{\mu(x): x \in R\}$ is infinite. Since $Im\mu \subseteq 0, 1], Im\mu$ is a bounded set. Hence there is an infinite sequence {tn} of elements of Imμ such that either $t_1 < t_2 < t_3 \ldots$ or $t_1 > t_2 > t_3 \ldots$.

Case - I: Suppose $t_1 < t_2 < t_3 \ldots$

Define

$$\mu_r(x) = \begin{cases} 1 - \mathrm{t_r}, & \text{if } \mu(\mathrm{x}) \geq \mathrm{t_r} \\ 0, & otherwise \end{cases}$$

Then μ_r is a fuzzy left ideal of R. As $t_r < t_{r-1}$, we have for any $x \in R$, $\mu_{r-1}(x) < \mu_r(x)$ i.e. $\mu_r \subset \mu_{r-1}$. Thus we obtain a strictly descending sequence $\mu_1 \supset \mu_2 \supset \mu_3 \supset \ldots$ of fuzzy left ideals of R. This contradicts that R is fuzzy left Artinian.

Case - II: Suppose $t_1 > t_2 > t_3 \ldots$.

Define

$$\mu_r(x) = \begin{cases} \mathrm{t_r}, & \text{if } \mu(\mathrm{x}) \geq 1 - \mathrm{t_r} \\ 0, & otherwise \end{cases}$$

Then μ_r is a fuzzy left ideal of R. Also as $t_r > t_{r-1}$, we have for any $x \in R$, $\mu_{r-1}(x) > \mu_r(x)$ i.e. $\mu_{r-1} \supset \mu_r$. Thus in this case also we get a strictly descending chain $\mu_1 \supset \mu_2 \supset \mu_3 \supset \ldots$ of fuzzy left ideals of R, which is also a contradiction. Thus in either case, every fuzzy left ideal of R has finite number of values.

Using $theorem$ 2 of Sen *et al* [2] and $theorem$ 3.1 we get the following result.

Theorem 3.2: A ring R with unity is fuzzy left Artinian if and only if every fuzzy left ideal on R has finite number of values.

Proof: Suppose every fuzzy ideal of R has finite number values and R is not fuzzy left Artinian. Then there exist a strictly descending sequence $\mu_0 \supset \mu_1 \supset \mu_2 \supset \ldots$ of fuzzy left ideals of R. So we have $(\mu_0)_t \supset (\mu_1)_t \supset (\mu_2)_t \supset \ldots$.Now continuing as in [2] we can get the required result.

Theorem 3.3: Let R and S be two fuzzy left Artinian rings with unity 1. Then $R \times S$ is also fuzzy left Artinian.

Proof: Let μ be a fuzzy left ideal of $R \times S$. Define fuzzy left ideals μ_1 and μ_2 of R and S respectively by $\mu_1(x) = \mu(x, 0)$ for all $x \in R$ and $\mu_2(y) = \mu(0, y)$ for all $y \in S$. Now by hypothesis, $\mu(x, 0) = \mu\{(1,0)(x,y)\} \geq \mu(x,y)$, $\forall y \in S$ and $(0, y) = \mu\{(0,1)(x,y)\} \geq \mu(x,y)$, $\forall x \in R$. Let $(x, y) \in R \times S$. Then we have, $(x, y) \leq min\{\mu1(x), \mu2(y)\}$. Again $\mu(x, y) = \mu((x, 0) + (0, y)) \geq min\{\mu(x, 0), \mu(0, y)\} = min\{\mu1(x), \mu2(y)\}$. Hence $\mu(x, y) = min\{\mu1(x), \mu2(y)\}$. Now R and S are fuzzy Artinian rings with 1. So, $Im\mu_1$ and $Im\mu_2$ are finite subsets of $[0, 1]$. Hence $\mu(x, y) = min\{\mu1(x), \mu2(y)\}$ implies that $Im\mu$ is also a finite subset of $[0, 1]$ and consequently $R \times S$ is fuzzy left Artinian.

Note: For the last two results we assume that equality of support implies the equality of the fuzzy sets.

Theorem 3.4: A fuzzy R -module μ is a fuzzy min-E module if and only if $\frac{\mu}{soc\mu}$ is fuzzy Artinian.

Proof: First suppose, μ is a fuzzy min-E module. Then because finite intersection of essential fuzzy submodules is fuzzy essential, $soc\mu$ is fuzzy essential in μ. Also for any submodule μ_i of μ containing $soc\mu$, we have $soc\mu \subseteq \mu_i \subseteq \mu$. Thus any submodule of μ containing $soc\mu$ is essential. Let $\frac{\mu_1}{soc\mu} \supseteq \frac{\mu_2}{soc\mu} \supseteq \frac{\mu_3}{soc\mu} \supseteq \cdots$ be a descending chain. Then $as\ soc\mu \subseteq \mu_i$; $\forall i$ we have each μ_i is fuzzy essential. Then $\mu_1 \supset \mu_2 \supset \mu_3 \supset \ldots$is a descending chain of essential fuzzy submodules of μ, which is a fuzzy min-E module and this gives $\mu_k = \mu_{k+1} = \cdots$ for some positive integer k. This gives $\frac{\mu_k}{soc\mu} = \frac{\mu_{k+1}}{soc\mu} = \cdots$. Hence $\frac{\mu}{soc\mu}$ is fuzzy Artinian.

Conversely, suppose $\frac{\mu}{soc\mu}$ is fuzzy Artinian. Let $\mu_1 \supset \mu_2 \supset \mu_3 \supset \ldots$is a descending chain of essential fuzzy submodules of μ. As $soc\mu$ is the intersection of all essential fuzzy submodules of μ, we get $soc\mu \subseteq \mu_i \subseteq \mu$. Thus we get, $\frac{\mu_1}{soc\mu} \supseteq \frac{\mu_2}{soc\mu} \supseteq \frac{\mu_3}{soc\mu} \supseteq \cdots$ be a descending chain of fuzzy submodules of $\frac{\mu}{soc\mu}$, which is fuzzy Artinian. So we have, $\frac{\mu_n}{soc\mu} = \frac{\mu_{n+j}}{soc\mu}$ for some $n \in I^+$, $j \geq 1$. This implies $\frac{(\mu_n)^*}{(soc\mu)^*} = \frac{(\mu_{n+j})^*}{(soc\mu)^*}$ and this gives $(\mu_n)^* = (\mu_{n+j})^*$; for some $n \in I^+$, $j \geq 1$. So we have $\mu_n = \mu_{n+j}$. Hence the result follows.

Theorem 3.5: Let μ and σ be fuzzy submodules of M. If σ is a fuzzy min-E module and $\frac{\mu}{\sigma}$ is fuzzy Artinian, then μ is also a fuzzy min-E module.

Proof: We consider the descending sequence, $\mu_1 \supset \mu_2 \supset \mu_3 \supset \ldots$of essential fuzzy submodules of M contained in μ, and the sequence, $\frac{\mu_1+\sigma}{\sigma} \supset \frac{\mu_2+\sigma}{\sigma} \supset \frac{\mu_3+\sigma}{\sigma} \ldots$ of fuzzy submodules of $\frac{\mu}{\sigma}$ as well as a sequence, $\mu_1 \cap \sigma \supset \mu_2 \cap \sigma \supset \mu_3 \cap \sigma \ldots$ of essential fuzzy submodules of σ. But both these sequences are stationary, say after n-steps.

So we have, $\mu_n \cap \sigma = \mu_{n+1} \cap \sigma = \mu_{n+2} \cap \sigma \ldots$ and $\frac{\mu_n+\sigma}{\sigma} = \frac{\mu_{n+1}+\sigma}{\sigma} = \frac{\mu_{n+2}+\sigma}{\sigma} \ldots$. From the first relation we have, $(\mu_n \cap \sigma)^* = (\mu_{n+1} \cap \sigma)^* = (\mu_{n+2} \cap \sigma)^* \ldots$, this implies $\left({\mu_n}^* \cap \sigma^*\right) = \left({\mu_{n+1}}^* \cap \sigma^*\right) = \cdots$ and from the second relation we have, $\left(\frac{\mu_n+\sigma}{\sigma}\right)^* = \left(\frac{\mu_{n+1}+\sigma}{\sigma}\right)^* = \left(\frac{\mu_{n+2}+\sigma}{\sigma}\right)^* \ldots$ which gives $\frac{(\mu_n+\sigma)^*}{\sigma^*} = \frac{(\mu_{n+1}+\sigma)^*}{\sigma^*} = \frac{(\mu_{n+2}+\sigma)^*}{\sigma^*} = \cdots$ and this follows ${\mu_n}^* + \sigma^* = {\mu_{n+1}}^* + \sigma^* = {\mu_{n+2}}^* + \sigma^* = \cdots$.

Now, ${\mu_n}^* = {\mu_n}^* \cap ({\mu_n}^* + \sigma^*) = {\mu_{n+1}}^* + ({\mu_n}^* \cap \sigma^*) = {\mu_{n+1}}^* + ({\mu_{n+1}}^* \cap \sigma^*) = {\mu_{n+1}}^*$. Thus $\mu_n = \mu_{n+1} = \cdots$. Hence the result follows.

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