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ON THE PERFECT AND SUPERPERFECT GROUPS

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ABSTRACT
In this paper, we extended the notion of perfect and superperfect numbers to finite groups. We provide some general theorem and present examples of perfect and superperfect groups. Also, we prove some related results.

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1. INTRODUCTION

The study of perfect numbers has been in progress for as long as many other important mathematical fields. Although it is unknown when the study of perfect numbers first began, there is clear evidence of perfect numbers being studied as early as Pythagoras. Euclid produced the first significant mathematical result on perfect numbers. He provided a form for a set of even perfect numbers using the formula for the sum of a geometric progression. It is a well-known unsolved problem whether or not an odd perfect number exists and we know that perfect numbers are only "even". A natural number n is said to be perfect number if it is equal to the sum of its proper divisors. Nicomachus stated that perfect numbers will be arranged in regular order; that is, only one among the units, one among the tens, one among the hundreds, and one among the thousands; for example, 6, 28, 496, 8128 are the only perfect numbers in the corresponding intervals between 1, 10, 100, 1000, 10000; and the last digit of the successive perfect numbers is alternately 6 and 8. These statements of Nicomachus imply that (1) all perfect numbers are even; (2) the nth perfect number has n digits; (3) all perfect numbers end in 6 and 8 alternately; (4) Euclid's formula provides all perfect numbers; (5) there are infinitely many perfect numbers. With the test of time, it has been discovered that some of Nicomachus's assertions are correct, some are incorrect and some are still open questions. Let $\sigma(n)$ the sum of the divisors of n. For any positive number n, n is called perfect if $\sigma(n) = 2n$. Also, n is called superperfect if $\sigma(\sigma(n)) = 2n$. ([1, 4, 9, 10, 11])

Leinster in [2] extended the notion of perfect numbers to finite groups. He called a finite group is perfect (FPG) if its order is equal to the sum of the orders of all normal subgroups of the group. In the other words, G is called perfect group if $\sigma(G) = \sum_{N\trianglelefteq G} |N| = 2|G|$. ([6, 7, 8, 12, 13, 14])

Example 1.1: Let $M_{n,t} = < x, y \mid x^n = y^{2^t} = 1, y^{-1}xy = x^{-1} > = C_n : C_{2^t}$. The group $M_{n,t}$ is a Leinster group if and only if t = 1 and n is an odd perfect number, or n = $2^t - 1$ is a Mersenne prime. ([8])

Example 1.2: (Cyclic Groups) Let C_n be the cyclic group of order n. Then C_n has one normal subgroup of order d for each divisor d of n, and no others, so $\sigma(C_n) = \sigma(n)$ and C_n is perfect just when n is perfect. For example, C_6, C_{28}.

Proposition 1.3: (**Dihedral Groups**) Let E_{2n} be the dihedral group of order 2n: that is, the group of all isometries of a regular n-sided polygon. Of the 2n isometries, n are rotations (forming a cyclic subgroup of order n) and n are reflections. If n is odd number then $\sigma(E_{2n}) = \sigma(C_n)+2n$. Therefore, E_{2n} is perfect group if and only if n is a perfect number. If n is even number then $\sigma(E_{2n}) \geq 1 + n + n + 2n > 4n$. Hence, E_{2n} is not perfect group.

Question: We do not know how many perfect numbers there are. We do know that there are an infinite number of prime numbers, which means there is a very high chance that there are an infinite number of perfect numbers because there is a strong link between perfect numbers and a certain kind of prime number (the Mersenne primes). Is it true that there are infinitely many perfect groups?

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Definition 1.4: Let G be a finite group. Then G is called superperfect group if $\sigma(\sigma(G)) = 2|G|$. ([13])

Example 1.5: Let $Q_8 = < a, b: a^4 = e, a^2 = b^2, ba = a^{-1}b >$ be a quaternion group of order eight. And H = $< e, a, a^2, a^3 >$ be a normal subgroup of Q_8. Then H is a superperfect group. ([13])

Example 1.6: Let D_8 ba e dihedral group of order eight. And H = $< e, a, a^2, a^3 >$ be a normal subgroup of D_8. Then H is a superperfect group. ([13])

Example 1.7: Let G be a abelian finite group of order 2 or 4. Then G is a superperfect group. ([13])

Theorem 1.8: Let G be a perfect group such that $|G| \neq 1$. Then G is not superperfect group. ([13])

Definition 1.9: Let G be a finite group. G is called simple if G is non-trivial and does not have any proper non-trivial normal subgroup. For any prime p, the group Z_p is simple because it has no proper nontrivial subgroups by Lagrange's Theorem (and therefore has no proper nontrivial normal subgroups).

Lemma 1.10: Let F be a field. Assume that $|F| \geq 4$. Let H be a normal subgroup of G=$SL_2(F)$. Then either H$\leq$ Z or H=Z. ([3])

Lemma 1.11: Let G_1 and G_2 are groups whose orders are coprime then $\sigma(G_1 \times G_2) = \sigma(G_1)\sigma(G_2)$. This means σ is multiplicative. ([8])

Lemma 1.12: If n is composite number, then σ(n) > n+$\sqrt{n}$. [(16)]

2. ON THE PERFECT AND SUPERPERFECT GROUPS

Lemma 2.1: If F is a field with at least four elements then the group $PSL_2(F)$ is not perfect.

Proof: The normal subgroups of $PSL_2(F)$ are the projections of the normal subgroups of $SL_2(F)$ which contain Z. From Lemma 1.10 it follows that every normal subgroup of $PSL_2(F)$ is either trivial or all of $PSL_2(F)$. Hence $PSL_2(F)$ is simple group. Therefore, this group is not perfect group.

Lemma 2.2: Let n ≥ 2, (n,q) ≠(2,2), (2,3). Then $PSL_n(q)$ is not perfect group. ([3]).

Lemma 2.3: Let G_1 and G_2 are groups whose $G_1 \cong G_2$ and G_1 be a perfect group then G_2 is a perfect group.

Example 2.4: Let G be a finite group. We define the function φ(G) =$\sum_{g\in G}$ o(g), where o(g) denotes the order of g ∈ G. Let G be any group of order 60. If φ(G) ≥ 211 and φ(G) = 211 then G ≅ A_5. Therefore, G is not perfect group. ([15])

Question: Is it true that "the order of any perfect group is even"?

Proposition 2.5: Let C_n be the cyclic group of order n. Then C_n has one normal subgroup of order d for each divisor d of n, and no others, so $\sigma(C_n)$ = σ(n). Therefore, C_n is superperfect just when n is superperfect. For example, C_4, C_{16}, C_{64}.

Question: We do not know how many superperfect numbers there are. We do know that there are an infinite number of prime numbers, which means there is a very high chance that there are an infinite number of superperfect numbers because there is a strong link between superperfect numbers and Mersenne primes because even superperfect numbers are $2^p - 1$, where $2^p - 1$ is a Mersenne prime. Is it true that there are infinitely many superperfect groups?

Conjecture 2.6: Is there an odd superperfect group? It is not known whether there are any odd superperfect numbers.

Notice 2.7: If $2^k - 1$ be a prime number then k is a prime number. ([1])

Proposition 2.8: Suppose $n_1, n_2, \ldots, n_k$ are relatively prime integers with product n = $n_1 n_2 \ldots n_k$.
Then $C_{n_1} \times C_{n_2} \times \ldots \times C_{n_k} \cong C_n$. ([5])

Theorem 2.9: Let G be a finite abelian group of order n. Then

1. G ≅ $\prod_{p \mid n} G(p)$ is the direct product of its nontrivial subgroups G(p).
2. For each prime p there exist unique positive integers $e_1 \geq e_2 \geq \cdots \geq e_k > 0$ such that $e_1 + e_2 + \ldots + e_k$ is the power of p dividing n and G(p) ≅ $C_{p^{e_1}} \times C_{p^{e_2}} \times \ldots \times C_{p^{e_k}}$. ([5])

Theorem 2.10: Let G be a group such that G=A×B. If A be a cyclic group of order $2^k - 1$, where $2^k - 1$ is a Mersenne prime, and B be a cyclic group of order 2^{k-1}then G is a perfect group.

Proof: By Lemma 1.11 and Proposition 2.8 and Theorem 2.9, we have

$$\sigma(G) = \sigma(C_{(2^k-1)(2^{k-1})}) = \sigma(C_{(2^k-1)})\,\sigma(C_{(2^{k-1})}). \tag{1.1}$$

But we have $\sigma(C_n) = \sigma(n)$, so $\begin{cases} \sigma(C_{(2^k-1)}) = \sigma(2^k - 1) \\ \sigma(C_{(2^{k-1})}) = \sigma(2^{k-1}) \end{cases}$. (1.2)

Therefore, by using (1.1), (1.2) we have $\sigma(G) = \sigma(2^k - 1)\sigma(2^{k-1})$. But $(2^k - 1, 2^{k-1}) = 1$, so $\sigma(G) = \frac{2^k-1}{2-1}(2^k - 1 + 1) = (2 \times 2^{k-1})(2^k - 1) = 2 \times (2^{k-1})(2^k - 1) = 2|G|$. This finishes the proof.

Theorem 2.11: Let E_{2n} be the dihedral group of order 2n. If n be an even number and $\sigma(E_{2n})$ be a composite number then E_{2n} is not superperfect group.

Proof: Let n is even number then $\sigma(E_{2n}) \geq 1 + n + n + 2n > 4n$. But $\sigma(n) > n+\sqrt{n}$.
Therefore, $\sigma(\sigma(E_{2n})) > \sigma(E_{2n}) + \sqrt{\sigma(E_{2n})} > 4n + \sqrt{4n} \neq 4n$.
Hence, $\sigma(\sigma(E_{2n})) \neq 4n = 2|E_{2n}|$. Therefore, E_{2n} is not superperfect group.

Proposition 2.12: Let G be a cyclic group of order p, where p is a prime number. Then G is not perfect group.

Proof: We know that $\sigma(C_n) = \sigma(n)$. Therefore, $\sigma(C_p) = \sigma(G) = 1+p \neq 2p$.

Proposition 2.13: Let G be a cyclic group of order pq, where p, q are two prime. Then G is not perfect group.

Proof: We know that $\sigma(C_n) = \sigma(n)$. Hence, $\sigma(C_{pq}) = \sigma(G) = \sigma(pq) = 1+p+q + pq = (1 + p)(1 + q) \neq 2pq$.
Therefore, C_{pq} is not perfect group.

Proposition 2.14: Let G be a cyclic group of order n = $p_1 p_2 \ldots p_k$, where the numbers p_i are odd primes. Then G is not perfect group.

Proof: We know that $\sigma(C_n) = \sigma(n)$. Hence, $\sigma(C_{p_1 p_2 \ldots p_k}) = \sigma(G) = \sigma(p_1 p_2 \ldots p_k) = \sigma(p_1)\sigma(p_2) \ldots \sigma(p_K) = (1 + p_1)(1 + p_2) \ldots (1 + p_k) \neq 2p_1 p_2 \ldots p_k$. Therefore, $C_{p_1 p_2 \ldots p_k}$is not perfect group.**"**

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