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SOME NEW RELATIONSHIPS BETWEEN THE FRACTIONAL DERIVATIVES OF FIRST, SECOND, THIRD AND FOURTH CHEBYSHEV WAVELETS

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ABSTRACT

In this paper, the shifted first, second, third and fourth kind Chevelets wavelets $\Psi^1_{\tilde{n}m}(t)$,$\Psi^2_{nm}(t)$,$\Psi^3_{nm}(t)$ and $\Psi^4_{nm}(t)$ properties are presented.

The main aim is:

1. Generalize the first, second operational matrix to the fractional derivatives . In this approach, a truncated first, second matrix of fractional derivatives are used.
2. Presented a new proposal formula expressing of fractional derivative α>0 operational matrix of shifted first kind Chybeshev wavelets $D^\alpha\Psi^1_{\tilde{n}m}(t)$ interms of $D^\alpha\emptyset(x), \emptyset(x) = [T^*_0(x), T^*_1(x), ...,T^*_n(x)]^T$, and a formula expressing the fractional derivative α>0 of second kind Chebyshev wavelets $D^\alpha\Psi^2_{nm}(t)$ interms of $D^\propto\varphi(x), \varphi(x) = [U^*_0(x) , U^*_1(x) , ...,U^*_n(x)]^T$.
3. Presented a new proposal formula expressing of fractional derivative α>0 operational matrix of shifted third and fourth kind Chybeshev wavelets $D^\alpha\Psi^3_{nm}(t)$, $D^\alpha\Psi^4_{nm}(t)$interms of $\Psi^2_{nm}(t)$ and$\Psi^2_{nm-1}(t)$, and a formula expressing of fractional derivative α>0 of first kind Chebyshev wavelets $D^\alpha\Psi^1_{\tilde{n}m}(t)$interms of $\Psi^2_{nm}(t)$ and $\Psi^2_{nm-2}(t)$.

All the proposed results are of direct interest in many applications.

1. INTRODUCTION

The Chebyshev polynomials are one of the most useful polynomials, which are suitable in numerical analysis including polynomial approximation, integral and differential equations and spectral methods for partial differential equations [4, 9, 10, 17]. One of the attractive concepts in the initial and boundary value problems is differentiation and integration of fractional order [8, 16, 18, 19]. Many researchers extend classical methods in studies of differential and integral equations of integer order to fractional type of these problems [15, 21]. One of the wide classes of researches focuses to constructing the operational matrix of derivative in some spectral methods. Recently, a lot of attention has been devoted to construct operational matrix of fractional derivative[4,14,20].For example the fractional type first kind chebyshev polynomials are used to solving fractional diffusion equations[3,7] also are used to solve multi-order fractional equation,[12].

In this paper we use shifted chebyshev polynomials of first, second, third and fourth kind and recall some important properties. Next we used obtain the operational matrix of fractional derivative. Wavelets theory is a relatively new emerging in mathematical research [5, 6, 11, 23]. It has been applied in a wide range of engineering disciplines, particularly, shifted first kind chebyshev wavelets play an important role in establishing algebraic methods for the solution of multi-order fractional differential equations, [22] and initial and boundary values problems of fractional order,[13].New Spectral Second Kind Chebyshev Wavelets Algorithm for Solving Linear and Nonlinear Second-Order Differential Equations,[2].

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2. PROPOSAL OPERATIONAL MATRIX OF CHEBYSHEV'S FRACTIONAL DERIVATIVE

A. It well know the first kind chebyshev polynomial $T_n(z)$ of degree n, [22], which defined on [-1,1] by:

$T_n(z) = \mathrm{Cos}(n\,\theta)$ where $z= \mathrm{Cos}(\theta)\theta \in [0,\pi]$

and can be determined with the aid of the following recurrence formula:

$T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z)$.   n=1, 2, 3,…..

$T_0(z) =1,\ T_1(z) =z$

The analytic form of chebyshev polynomial $T_n(x)$ of degree (n) is given by:

$$T_n(z) = \sum_{i=0}^{n/2}(-1)^i\ 2^{n-2i-1}\ \frac{n(n-i-1)!}{i!(n-2!)!}\ z^{n-2i} \qquad n=2,.. \tag{1}$$

and are orthogonal on [1-,1] with respect to the weight function $\omega(t) = 1/\sqrt{1-z^2}$, that is:

$$\int_{-1}^{1}\frac{T_i(z).T_j(z)}{\sqrt{1-z^2}}dz = \begin{cases}\pi & i=j=0\\ \frac{\pi}{2} & i=j\neq 0\\ 0 & i\neq j\end{cases}$$

In order to use these polynomials on the interval [0, 1], we define the shifted chebyshev polynomials by introducing the change variable z=2x-1, then $T_n^*(x)$ can be obtained as follows:

$T_{i+1}^*(x) = 2(2x-1)\ T_i^*(x) - T_{i-1}\ (x)$ i=1, 2,…

where$T_0^*(x)=1, T_1^*(x) =2x-1$ and the analytic form is $T_n^*(x) = \sum_{i=0}^{n}(-1)^i\ 2^{2n-2i}\ \frac{n(2n-i-1)!}{i!(2n-2!)!}\ x^{n-I}$n=2, 3,…and are orthogonal with respect to the weight function $\omega(x)= 1/\sqrt{x-x^2}$, that is:

$$\int_{0}^{1}\frac{T_i^*(x).T_j^*(x)}{\sqrt{x-x^2}}dx = \begin{cases}\pi & i=j=0\\ \frac{\pi}{2} & i=j\neq 0\\ 0 & i\neq j\end{cases}$$

The function u(x) square integrable in [0, 1], may be expressed in the term of shifted first kind chebyshev polynomial as:

$$\sum_{i=0}^{\infty} c_i\ T_i^*(x)$$

where the coefficients c_i are given by

$$c_i = \frac{\pi\sigma_i}{2}\int_0^1 u(x)T_i^*(x)dx,\ \sigma_i = \begin{cases}2 & i=0\\ 1 & i\neq 0\end{cases}$$

In practice, only the first (m+1)-terms shifted first kind chebyshev polynomial are considered**.** Then we have:

$$u_m(x) = \sum_{i=0}^{m} c_i\ T_i^*(x) = C^T\emptyset(x)$$

where the shifted first kind chebyshev coefficient vector C and the shifted first kind chebyshev vector $\emptyset(x)$ are given by:$C^T = [c_0\ (x), c_1\ (x), \ldots, c_m\ (x)]$, $\emptyset(x) = [T_0^*(x), T_1^*(x), \ldots, T_m^*(x)]^T$.

For the Caputo's derivative we have, [22]: $D^\alpha C =0$, C is a constant.

$$D^\alpha x^n = \begin{cases}\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}x^{n-\alpha} & n \geq \lceil\alpha\rceil \ \text{ for } n\epsilon\mathbb{N}_0\\ 0 & n<\lceil\alpha\rceil \ \text{ for } n\epsilon\mathbb{N}_0\end{cases} \tag{2}$$

for $\mathbb{N}_0 = \{0,1,1,\ldots\}$.

In the following theorem we will define the fractional derivative of the vector $\emptyset(x)$.

Theorem 2.1: Let $\emptyset(x)$ be shifted first kind chebyshev vector defined as $\emptyset(x) = [T_0^*(x), T_1^*(x), \ldots\ldots, T_n^*(x)]^T$ and also suppose $\alpha>0$ then $D^\alpha\emptyset(x) = \Delta^{(\alpha)}\emptyset(x)$

where $\Delta^{(\alpha)}$ is (m+1) X (m+1) is an operational matrix of fractional derivative of order $\alpha>0$ in the caputo sense and is defined as follows :

$$\Delta^{\alpha}=\begin{bmatrix} 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \\ w^{(1)}_{0,0,i} & w^{(1)}_{0,1,i} & \cdots & w^{(1)}_{0,m,i} \\ \vdots & \vdots & & \vdots \\ \sum_{i=0}^{n-\lceil\alpha\rceil} w^{(1)}_{n-\lceil\alpha\rceil,0,i} & \sum_{i=0}^{n-\lceil\alpha\rceil} w^{(1)}_{n-\lceil\alpha\rceil,1,i} & \cdots & \sum_{i=0}^{n-\lceil\alpha\rceil} w^{(1)}_{n-\lceil\alpha\rceil,m,i} \\ \vdots & \vdots & & \vdots \\ \sum_{i=0}^{m} w^{(1)}_{m,0,i} & \sum_{i=0}^{m_1} w^{(1)}_{m,1,i} & \cdots & \sum_{i=0}^{m_1} w^{(1)}_{m,m,i} \end{bmatrix}$$

and

$w^{(1)}_{n-\lceil\alpha\rceil,j,i}$ is given by:

$$w^{(1)}_{n-\lceil\alpha\rceil,j,i}=\frac{\sigma_j}{\sqrt{\pi}}\sum_{k=0}^{j}(-1)^{k+i}2^{2j-2k+2n-2i}\frac{n(2n-i-1)!\,j(2j-k-1)!\,\Gamma(n-i-\alpha+j-k+\frac{1}{2})}{i!(2n-2!)!\,\Gamma(n-i-\alpha+1)k!(2j-2k)!\,\Gamma(n-i-\alpha+j-k+1)}$$

where n=$\lceil\alpha\rceil$ ...m and $\sigma_j=\begin{cases}1 & j=0\\ 2 & j\neq 0\end{cases}$.

Note that in Δ^{α}, the first $\lceil\alpha\rceil$ rows, are all zero.

Proof: Let $T^*_m(x)$ be shifted first kind chebyshev polynomial then by using (1) and (2) we can find that:
$D^{\alpha}T^*_n(x)=0$, $n<\lceil\alpha\rceil$ and for $n\geq\lceil\alpha\rceil$

$$D^{\alpha}T^*_n(x)=\sum_{i=0}^{n}(-1)^i\,2^{2n-2i}\frac{n(2n-i-1)!}{i!(2n-2!)!}D^{\alpha}x^{n-i}$$
$$=\sum_{i=0}^{n-\lceil\alpha\rceil}(-1)^i\,2^{2n-2i}\frac{n(2n-i-1)!}{i!(2n-2!)!\,\Gamma(n-i-\alpha+1)}\,x^{n-i-\alpha}$$

Now, approximate ($x^{n-i-\alpha}$) by (m+1)-terms of shifted first kind chebyshev polynomial, we have
$x^{n-i-\alpha}=\sum_{j=0}^{m}d_{n-i,j}\,T^*_j(x)$ where $d_{n-i,j}=\frac{\sigma_j}{\pi}\int_0^1\frac{x^{n-i-\alpha}}{\sqrt{x-x^2}}T^*_j(x)\,dx$

$T^*_j(x)=\sum_{k=0}^{j}(-1)^k\,2^{2j-2k}\frac{j(2j-k-1)!}{k!(2j-2k)!}\,x^{j-k}$, then

$$d_{n-i,j}=\frac{\sigma_j}{\pi}\sum_{k=0}^{j}(-1)^k\,2^{2j-2k}\frac{j(2j-k-1)!}{k!(2j-2k)!}\int_0^1\frac{x^{n-i-\alpha+j-k}}{\sqrt{x-x^2}}\,dx$$
$$=\frac{\sigma_j}{\pi}\sum_{k=0}^{j}(-1)^k2^{2j-2k}\frac{j(2j-k-1)!\,\Gamma(n-i-\alpha+j-k+\frac{1}{2})\sqrt{\pi}}{k!(2j-2k)!\,\Gamma(n-i-\alpha+j-k+1)}$$

where,
$\sigma_j=\begin{cases}1 & j=0\\ 2 & j\neq 0\end{cases}$, then

$$D^{\alpha}T^*_n(x)=\sum_{i=0}^{n-\lceil\alpha\rceil}\sum_{j=0}^{m}(-1)^i\,2^{2n-2i}\frac{n(2n-i-1)!(n-i)!}{i!(2n-2!)!\,\Gamma(n-i-\alpha+1)}d_{n-i,j}T^*_n(x)$$
$$=\sum_{j=0}^{m}\Big[\sum_{i=0}^{n-\lceil\alpha\rceil}w^{(1)}_{n,j,i}\Big]T^*_n(x),\ \text{for}\ n\geq\lceil\alpha\rceil.$$
$$=\Big[\sum_{i=0}^{n-\lceil\alpha\rceil}w^{(1)}_{n-\lceil\alpha\rceil,i,0},\ \sum_{i=0}^{n-\lceil\alpha\rceil}w^{(1)}_{n-\lceil\alpha\rceil,i,1},\ldots,\sum_{i=0}^{n-\lceil\alpha\rceil}w^{(1)}_{n-\lceil\alpha\rceil,i,m}\Big]\emptyset(x),\ \text{for}\ n\geq\lceil\alpha\rceil$$

and $D^{\alpha}T^*_n(x)=[\,0,\ldots\ldots0\,]\,\emptyset(x)$, $n<\lceil\alpha\rceil$.

B. The second kind of degree (n), [6], which defined on the interval $[-1,1]$ as:
$U_n(z)=\frac{\sin(n+1)\theta}{\sin\theta}$ where $z=\cos\theta$, $\theta\neq n\pi+2k\pi$. B

These polynomials satisfy the following recurrence relation
$U_0(z)=1, U_1(z)=2z,$
$U_n(z)=2zU_{n-1}(z)-U_{n-2}(z), n=2,3,\cdots$

For using this polynomials on interval (0,1) which called shifted chebyshev polynomials,[6], by introducing the change variable $(2x-1)$ and satisfy the following $U^*_n(x)=(4x-2)U^*_{n-1}(x)-U^*_{n-2}(x)$, $n=2,3,\cdots$

where $U^*_0(x)=1, U^*_1(x)=4x-2$

and the analytic form of shifted chebyshev polynomials $u_n^*(x)$ of degree (n) is given by

$$U_n^*(x) = \sum_{r=0}^{n+1} r\,(-1)^{n+1-r} \frac{(n+r)!\,2^{2r-1}}{(n+1-r)!\;\;2r!} x^{r-1} \tag{3}$$

and are orthogonal with respect to the weight function $\omega(x)= 1/\sqrt{x-x^2}$, that is:

$$\int_0^1 \frac{U_m^*(x)\cdot U_n^*(x)}{\sqrt{x-x^2}}dx = \begin{cases} \frac{\pi}{8} & m=n \\ 0 & m\neq n \end{cases}$$

The function f(x) square integrable in [0, 1], may be expressed in the term of shifted first kind chebyshev polynomial as:

$\sum_{i=0}^{\infty} c_i\, U_i^*(x)$

where the coefficients c_i are given by

$c_i = \frac{8}{\pi}\int_0^1 fx) T_i^*(x)dx,$

In practice, only the first (m+1)-terms shifted first kind chebyshev polynomial are considered**.** Then we have:

$f_m(x) = \sum_{i=0}^{m} c_i\, T_i^*(x) = C^T\varphi(x)$

where the shifted first kind chebyshev coefficient vector C and the shifted first kind chebyshev vector $\varphi(x)$ are given by:$C^T = [c_0\,(x), c_1\,(x), \dots, c_m\,(x)]$, $\emptyset(x) = [U_0^*(x), U_1^*(x), \dots, U_m^*(x)]^T$.

In the following theorem we will define the fractional derivative of the vector $\varphi(x)$.

Theorem 2.2: Let $\varphi(x)$ be shifted second kind chebyshev vector defined in $\varphi(x) = [U_0^*(x), U_1^*(x), \cdots, U_n^*(x)]^T$ also suppose $\propto > 0$ then

$D^{\propto}\varphi(x) = \Delta^{(\propto)}\varphi(x)$

where $\Delta^{\propto}$ is the $(m_1+1)\times(m_1+1)$ operational Matrix of fractional derivative of order $\propto$ in the Caputo sense and defined as follows:

$$\Delta^{\propto} = \begin{bmatrix} 0 & 0 & & 0 \\ . & . & & . \\ 0 & 0 & & 0 \\ \sum_{r=\lceil\propto\rceil+1}^{\lceil\propto\rceil+1} w^{(2)}_{\lceil\propto\rceil+1,0,r} & \sum_{r=2}^{2} w^{(2)}_{\lceil\propto\rceil+1,1,r} & \cdots & \sum_{r=2}^{2} w^{(2)}_{\lceil\propto\rceil+1,m_1,r} \\ \vdots & \vdots & & \vdots \\ \sum_{r=\lceil\propto\rceil+1}^{n+1} w^{(2)}_{n+1,0,r} & \sum_{r=\lceil\propto\rceil+1}^{n+1} w^{(2)}_{n+1,1,r} & \cdots & \sum_{r=\lceil\propto\rceil+1}^{n+1} w^{(2)}_{n+1,m_1,r} \\ \vdots & \vdots & & \vdots \\ \sum_{r=\lceil\propto\rceil+1}^{m_1} w^{(2)}_{m_1,0,r} & \sum_{r=\lceil\propto\rceil+1}^{m_1} w^{(2)}_{m_1,1,r} & \cdots & \sum_{r=\lceil\propto\rceil+1}^{m_1} w^{(2)}_{m_1,m_1,r} \end{bmatrix}$$

and

$$w^{(2)}_{n+1,p,r} = \frac{8}{\sqrt{\pi}}\sum_{\ell=0}^{p+1} \frac{r.\ell(-1)^{n+p+2-(\ell+r)}(n+r)!\,(p+\ell)!\,(r-1)!\;\;2^{2(\ell+r)-2}\,\lceil\left(r-\propto+\ell+\frac{1}{2}\right)}{(n+1-r)!\;\;2r!\;\lceil(r-\propto)(p+1-\ell)!\;\;2\ell!\;\lceil(r-\propto+\ell+2)}$$

Note that in Δ^{α}, the first$\lceil\alpha\rceil+1$ rows, are all zero

Proof: Let $U_m^*(x)$ be shifted scondt kind chebyshev polynomial then by using (2) and (3) we can find that:

$D^{\propto}U_n^*(x) = 0 n \le \lceil\propto\rceil$ and for $(n = \lceil\propto\rceil + 1 \dots m)$ we have

$$\begin{aligned} D^{\propto}U_n^*(x) &= \sum_{r=0}^{n+1} r\,(-1)^{n+1-r}\frac{(n+r)!\,2^{2r-1}}{(n+1-r)!\;\;2r!} D^{\propto}x^{r-1} \\ &= \sum_{r=\lceil\propto\rceil+1}^{n+1} r\,(-1)^{n+1-r}\frac{(n+r)!\,2^{2r-1}(r-1)!}{(n+1-r)!\;\;2r!\,\lceil(r-\propto)} x^{r-\propto-1} \end{aligned}$$

Now, approximate ($x^{r-\propto-1}$) by (m + 1)-terms of shifted second kind chebyshev series, we have

$$x^{r-\propto-1} = \sum_{p=0}^{m_1} d_{r-1,p}\, U_p^*(x)$$

$$d_{r-1,p} = \frac{8}{\pi}\int_0^1 x^{r-\propto-1}\sqrt{x-x^2}\,U_p^*(x)\,dx$$
$$= \frac{8}{\pi}\sum_{\ell=0}^{p+1}\frac{\ell(-1)^{p+1-\ell}(p+\ell)!\;2^{2\ell-1}}{(p+1-\ell)!\;2\ell!}\int_0^1\sqrt{x-x^2}\,x^{r-\propto-1}\;dx$$
$$= \frac{8}{\pi}\sum_{\ell=0}^{p+1}\frac{\ell(-1)^{p+1-\ell}(p+\ell)!\;2^{2\ell-1}\Gamma\left(r-\propto+\ell+\frac{1}{2}\right)\cdot\sqrt{\pi}}{(p+1-\ell)!\;2\ell!\,\Gamma(r-\propto+\ell+2)}$$

$$D^{\propto}U_n^*(x) = \sum_{r=\lceil\propto\rceil+1}^{n+1}\sum_{p=0}^{m_1} r(-1)^{n+1-r}\frac{(n+r)!\,2^{2r-1}(r-1)!}{(n+1-r)!\;2r!\,\lceil(r-\propto)}d_{r-1,p}U_n^*(x)$$
$$= \sum_{p=0}^{m_1}\left[\sum_{r=\lceil\propto\rceil+1}^{n+1} w_{n+1,p,r}^{(2)}\right]U_n^*(x)$$
$$= \left[\sum_{r=\lceil\propto\rceil+1}^{n+1} w_{n+1,0,r}^{(2)}, \sum_{r=\lceil\propto\rceil+1}^{n+1} w_{n+1,1,r}^{(2)}, \cdots, \sum_{r=\lceil\propto\rceil+1}^{n+1} w_{n+1,m_1,r}^{(2)}\right]\varphi(x),$$

for n=$\lceil\propto\rceil + 1\ldots m_1$ and $D^{\propto}U_n^*(x) = [0,\cdots,0]\varphi(x) n \le \lceil\propto\rceil$.

3. PROPOSAL OPERATIONAL MATRIX OF CHEBYSHEVWAVELETS FRACTIONAL DERIVATIVE

A. This shifted first kind chebyshev wavelets $\Psi_{nm}^1(t) = \Psi^1(k,\tilde{n},m,t)$ have four arguments; $k\in\mathbb{N}$, n=1,2,…,2^{k-1} and $\tilde{n} = 2n-1$; moreover, m is the order of the chebyshev polynomials of the first kind and t is the normalized time, and they are defined on the interval [0,1) as, [12]:

$$\Psi_{\tilde{n}m}^1(t) = \begin{cases} 2^{k/2}T_m^*(2^k t-\tilde{n}) & \frac{\tilde{n}-1}{2^k}\le t\le\frac{\tilde{n}}{2^k}\\ 0 & o.w\end{cases}$$

$$\text{where}\quad T_m^* = \begin{cases}\frac{1}{\sqrt{\pi}}T_m & m = 0\\ \sqrt{\frac{2}{\pi}}T_m & m>0\end{cases}$$

m = 0, 1 ….M-1, n = 1, 2,……2^{k-1} and ñ=0,1,……2^k-1, the weight function

$\tilde{w} = w(2t-1)$ *and* $w_{\tilde{n}}(t) = w(2^k t - \tilde{n})$ *where* $w(2^k t-\tilde{n}) = \frac{1}{\sqrt{(2k\,t-\tilde{n})-(2k\,t-\tilde{n})^2}}$.

A function *f(t) defined* over [0,1) my be expanded as follows
$f(t) = \sum_{\tilde{n}=0}^{\infty}\sum_{m=0}^{\infty} c_{\tilde{n}m}\Psi_{\tilde{n}m}^1(t)$ *where*

$c_{nm} = (f(t), \Psi_{\tilde{n}m}^1(t))_w = \int_0^1 \tilde{w}(t)\,f(t).\Psi_{\tilde{n}m}^1(t)\,dt$ *and* $f(t) = \sum_{\tilde{n}=1}^{\infty}\sum_{m=0}^{\infty} c_{\tilde{n}m}\,\Psi^1_{\tilde{n}m}(t) = c^T\Psi^1_{\tilde{n}m}(t)$
where $C = [\,c_{00}, c_{01}, \ldots\ldots c_{2^k-1,M}, \ldots\ldots c_{2^k-1,1}, \ldots\ldots c_{2^k-1,M}]^T$

Thus,
$\Psi^1_{\tilde{n}m}(t) = [\Psi_{0,0}^1, \Psi_{0,1}^1, \ldots, \Psi_{0,M}^1, \ldots, \Psi^1_{2^k-1,M}, \ldots, \Psi^1_{2^k-1,1}, \ldots, \Psi^1_{2^k-1,M}]^T$

Theorem 3.1: *Let* $\Psi_{\tilde{n}m}^1(t)$ *be shifted of first kind chebyshev vector and also suppose α > 0) then*
$D^{\alpha}\Psi_{\tilde{n}m}^1(t)(t) = D^{\alpha}\left(\frac{c_m\,.2^{k/2}}{\sqrt{\pi}}T_m^*((2^k\,t-\tilde{n})\right) = \Delta^{(\propto)}\Psi_{nm}^1(t)$, *such that* $C_m = \begin{cases}1 & m=0\\ \sqrt{2} & m>0\end{cases}$, *where* $\Delta^{\propto}$ *is the (m+1)x(m+1) operational matrix derivative of order (α) in the caputo sense and is defined as follows:*

$$\Delta^{\propto} = .\begin{bmatrix} 0 & 0 & & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & & 0\\ w_{0,0,i}^{\sim}\,w_{0,1,i}^{\sim} & & \cdots & w_{0,M,i}^{\sim}\\ \vdots & \vdots & & \vdots\\ \sum_{i=0}^{m-\lceil\alpha\rceil} w_{m-\lceil\alpha\rceil,0,i}^{\sim} & \sum_{i=0}^{m-\lceil\alpha\rceil} w_{m-\lceil\alpha\rceil,1,i}^{\sim} & \cdots & \sum_{i=0}^{m-\lceil\alpha\rceil} w_{m-\lceil\alpha\rceil,M,i}^{\sim}\\ \vdots & \vdots & & \vdots\\ \sum_{i=0}^{m_1} w_{M,0,i}^{\sim}\,\sum_{i=0}^{m_1} w_{M,1,i}^{\sim} & & \cdots & \sum_{i=0}^{m_1} w_{M,M,i}^{\sim}\end{bmatrix}$$

And $w_{m-\lceil\alpha\rceil,j,i}^{\sim}$ *is given by*

$$w^{\sim}_{m-\lceil\alpha\rceil,j,i} = \frac{\sigma_j}{\pi}\sum_{k=0}^{j}(-1)^{k+i}\, 2^{2(j+m)-2(k+i)} \cdot \frac{m(2m-i-1)!(m-i)!j(2j-k-1)!\ \Gamma(m-i-\alpha+j-k+\frac{1}{2})}{i!(2m-2i)!\ \Gamma(m-i-\alpha+1)k!(2j-2k)!\ \Gamma(m-i-\alpha+k+1)}$$

where,

$$\sigma_j = \begin{cases} 1 & j=0 \\ 2 & j\neq 0 \end{cases}$$

Note that in Δ^{α}, the first$\lceil\alpha\rceil$ rows, are all zero.

Proof: *$D^{\alpha} T_m^*$ $(2^k$ t - ñ) = 0 m <$\lceil\alpha\rceil$ and for (m $\geq \lceil\alpha\rceil$)*

$$D^{\alpha} T_m^*\ (2^k\ t - \tilde{n}) = \sum_{i=0}^{m}(-1)^i\, 2^{2m-2i}\, \frac{m(2m-i-1)!}{i!(2m-2i)!} D^{\alpha}\ (2^k\ t - \tilde{n})^{m-i}$$
$$= \sum_{i=0}^{m-\lceil\alpha\rceil}(-1)^i\, 2^{2m-2i}\, \frac{m(2m-i-1)!}{i!(2n-2i)!}\ (2^k\ t - \tilde{n})^{m-i-\alpha}$$

Now, approximate $(2^k\ t - \tilde{n})^{m-i-\alpha}$ by (m+1) terms of shifted first kind chebyshev wavelet , we have

$$(2^k\ t - \tilde{n})^{m-i-\alpha} = \sum_{j=0}^{m} d_{n-i,j}\ \ T_j^*\ (2^k\ t - \tilde{n})$$

$$d_{m-i,j} = \frac{\sigma_j}{\pi}\int_0^1 (2^k\ t\ -\ \tilde{n})^{n-i-\alpha}\, \frac{T_j^*\ (2^k\ t-\tilde{n})}{\sqrt{(2^k\ t-\tilde{n})-(2^k\ t-\tilde{n})^2}}\, dt$$
$$= \frac{\sigma_j}{\pi}\sum_{k=0}^{j}(-1)^k\, 2^{2j-2k}\, \frac{j(2j-k-1)!}{k!(2j-2k)!}\int_0^1 \frac{(2^k\ t-\tilde{n})}{\sqrt{(2^k\ t-\tilde{n})-(2^k\ t-\tilde{n})^2}}dt$$

where,

$$\sigma_j = \begin{cases} 1 & j=0 \\ 2 & j\neq 0 \end{cases}, \text{ then}$$

$$d_{m-i,j} = \frac{\sigma_j}{\pi}\sum_{k=0}^{j}(-1)^k\, 2^{2j-2k}\, \frac{j(2j-k-1)!\ \Gamma(m-i-\alpha+j-k+\frac{1}{2})}{k!(2j-2k)!\ \Gamma(m-i-\alpha+k+1)}$$

Therefore,

$$D^{\alpha}\ T_m^*\ (2^k\ t - \tilde{n}) = \sum_{i=0}^{m-\lceil\alpha\rceil}\sum_{j=0}^{m_1}(-1)^i\, 2^{2m-2i}\, \frac{m(2m-i-1)!(m-i)!}{i!(2m-2i)!\ \Gamma(m-i-\alpha+1)}\, d_{m-i,j}\ \ T_m^*\ (2^k\ t - \tilde{n})$$
$$= \sum_{j=0}^{m_1}\Big[\sum_{i=0}^{m-\lceil\alpha\rceil} w_{m,j,i}\ \Big]\ \ T_m^*\ (2^k\ t - \tilde{n}).$$

$$D^{\alpha}\Psi^1_{\tilde{n}m}(t) = \Big[\sum_{i=0}^{m-\lceil\alpha\rceil} w^{\sim}_{m-\lceil\alpha\rceil,i,0},\quad \sum_{i=0}^{m-\lceil\alpha\rceil} w^{\sim}_{m-\lceil\alpha\rceil,i,1}, \ldots, \sum_{i=0}^{m-\lceil\alpha\rceil} w^{\sim}_{m-\lceil\alpha\rceil,i,M}\Big]\Psi^1_{\tilde{n}m}(t), \text{ for } m \geq \lceil\alpha\rceil$$

and $D^{\alpha}\Psi^1_{\tilde{n}m}(t) = [0,\ldots.0]\ \Psi^1_{\tilde{n}m}(t)$ for m $<\lceil\alpha\rceil$.

B. Second kindchebyshev wavelets $\Psi^2_{nm}(t) = \Psi^2$ (k, n, m, t) have four argumentsk, n can assume any positive integer, m is the order of second kind chebyshev polynomials, and t is the normalized time. They are defined on the interval [0, 1] by: [2]

$$\Psi^2_{nm}(t) = \begin{cases} \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}} u_m^*\left(2^k\ t - n\right) & t\ \in\ [\frac{n}{2^k}, \frac{n+1}{2^k}] \\ 0 & \text{o.w} \end{cases} \quad m=0,1,\ldots\ldots M, n=0,1,\ldots\ldots 2^k-1$$

and w(2t-1) has to be dilated and translated as follows:

$w_n(2^k\ t\text{-}n) = \sqrt{(2^k\ t\ -\ n)-(2^k\ t\ -\ n)^2}$ and the function approximation

A function (t) defined over [0, 1] may be expanded in terms of second kind Chebyshevwavelets as

$$f(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{nm}\ \Psi^2_{nm}(t)$$

$$c_{nm} = (f(t)\ ,\Psi^2_{nm}(t)) = \int_0^1 w(t).f(t).\Psi^2_{nm}(t)dt \text{ and}$$

$(t) = \sqrt{t-\ t^2}$ If the infinite series is truncated, then (t) can be approximated as

$$f(t) = \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} c_{nm}\ \Psi_{nm}(t) = c^T\Psi^2_{nm}(t)$$

where C and $\Psi(t)$ are 2^k $(M$ + 1) × 1 matrices defined by

$c = [\, c_{00}\, , c_{01}\, , \ldots\ldots c_{2^k-1,M}\, , \ldots\ldots c_{2^k-1,1}\, ,\ldots. c_{2^k-1,M}]^T$

$\Psi^2_{nm}(t) = [\Psi^2_{0,0}, \Psi^2_{0,1}\, , \ldots., \Psi^2_{0,M}, \ldots, \Psi^2{}_{2^k-1,M}, \ldots, \Psi^2{}_{2^k-1,1}\, , \ldots., \Psi^2{}_{2^k-1,M}\,]$

Theorem 3.2: *Let $\Psi^2_{nm}(t)$ be second kind chebyshev wavelets and suppose*

$\alpha > 0$ then $D^{\propto}\Psi^2_{nm}(t) = \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}} D^{\propto}(u^*_m(2^k\, t-n) = \Delta^{(\propto)}\Psi^2_{nm}(t)$

where $\Delta^{\propto}$ is (M+1)×(M+1) operational matrix derivative of order $\propto$ in the caputo sense and defined as follow:

$$\Delta^{\propto} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ . & . & & . \\ 0 & 0 & \cdots & 0 \\ \sum_{r=\lceil\propto\rceil+1}^{\lceil\propto\rceil+1} w^{\sim}_{\lceil\propto\rceil+1,0,r} & \sum_{r=2}^{2} w^{\sim}_{\lceil\propto\rceil+1,1,r} & \cdots & \sum_{r=2}^{2} w^{\sim}_{\lceil\propto\rceil+1,M,r} \\ \vdots & \vdots & & \vdots \\ \sum_{r=\lceil\propto\rceil+1}^{m+1} w^{\sim}_{m+1,0,r} & \sum_{r=\lceil\propto\rceil+1}^{n+1} w^{\sim}_{m+1,1,r} & \cdots & \sum_{r=\lceil\propto\rceil+1}^{n+1} w^{\sim}_{m+1,M,r} \\ \vdots & \vdots & & \vdots \\ \sum_{r=\lceil\propto\rceil+1}^{M} w^{\sim}_{M,0,r} & \sum_{r=\lceil\propto\rceil+1}^{M} w^{\sim}_{M,1,r} & \cdots & \sum_{r=\lceil\propto\rceil+1}^{M} w^{\sim}_{M,M,r} \end{bmatrix}$$

and

$w^{\sim}_{m+1,p,r} = \frac{8}{\sqrt{\pi}} \sum_{\ell=0}^{p+1} \frac{r\ell(-1)^{m+p+2-(\ell+r)}(r-1)!(p+\ell)!\; 2^{2(\ell+r)-2}\Gamma\left(r-\propto+\ell+\frac{1}{2}\right)\cdot}{(m+1-r)!\;\; 2r!\; \lceil(r-\alpha(p+1-\ell)!\; 2\ell!\,\Gamma(r-\propto+\ell+2)}$

Proof: From ( ), we have that,

$U^*_m(2^k\, t-n) = \sum_{r=0}^{m+1} r\, (-1)^{m+1-r} \frac{(m+r)!\, 2^{2r-1}}{(m+1-r)!\;\; 2r!} (2^k\, t-n)^{\, r-1}$

Also, we have that

$D^{\propto}U^*_m\, (2^k\, t-n)$ = $n \le \lceil\propto\rceil$ and for ($n = \lceil\propto\rceil + 1 \ldots M$) we have

$D^{\propto}U^*_m\, (2^k\, t-n) = \sum_{r=0}^{m+1} r\, (-1)^{m+1-r} \frac{(m+r)!\, 2^{2r-1}}{(m+1-r)!\;\; 2r!} D^{\propto}(2^k\, t-n)^{\, r-1}$

$= \sum_{r=\lceil\propto\rceil+1}^{m+1} r\, (-1)^{m+1-r} \frac{(m+r)!\, 2^{2r-1}(r-1)!\, (2^k\, t-n)^{\, r-\alpha-1}}{(m+1-r)!\;\; 2r!\; \lceil(r-\alpha)}$

Now, approximate $(2^k\, t-n)^{\, r-\alpha-1}$ *by (M+1) – term of shifted second kind chebyshev wavelets, we have:*

$(2^k\, t-n)^{\, r-\alpha-1} = \sum_{p=0}^{M} d_{r-1,p}\, U^*_p\, (2^k\, t-n)$

$d_{r-1,p} = \frac{8}{\pi}\int_0^1 (2^k\, t-n)\, U^*\, (2^k\, t-n).\sqrt{(2^k\, t\, -\, n) - (2^k\, t\, -\, n)^2}\, dt$

$= \frac{8}{\pi}\sum_{\ell=0}^{p+1} \frac{\ell\, (-1)^{p+1-\ell}(p+\ell)!\, 2^{2\ell-1}}{(p+1-\ell)!\;\; 2\ell!\lceil\, (r+\ell-\alpha)} \int_0^1 \sqrt{(2^k\, t\, -\, n) - (2^k\, t\, -\, n)^2}\, (2^k\, t\, -\, n)dt$

$= \frac{8}{\pi}\sum_{\ell=0}^{p+1} \frac{\ell\, (-1)^{p+1-\ell}(p+\ell)!\, 2^{2\ell-1}\, \lceil(r-\alpha+\ell+\frac{1}{2})\sqrt{\pi}}{(p+1-\ell)!\;\; 2\ell!\, \lceil(r+\ell-\alpha+2)}$

$D^{\propto}U^*_m\, (2^k\, t-n) = \sum_{r=\lceil\propto\rceil+1}^{m+1} \quad \sum_{p=0}^{M} r\, (-1)^{m+1-r} \frac{(m+r)!\, 2^{2r-1}(r-1)!}{(m+1-r)!\;\; 2r!\; \lceil(r-\alpha)} .d_{r-1,p}.\, U^*_m\, (2^k\, t-n)$

$= \sum_{p=0}^{M}\left[\sum_{r=\lceil\propto\rceil+1}^{m+1} w^{\sim}_{n+1,p,r}\right] U^*_m\, (2^k\, t-n)$

$D^{\propto}\Psi^2_{nm}(t) = \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}} \sum_{p=0}^{M}\left[\sum_{r=\lceil\propto\rceil+1}^{m+1} w^{\sim}_{n+1,p,r}\right] U^*_m\, (2^k\, t-n)$, *thus*

$D^{\propto}\Psi^2_{nm}(t) = [\sum_{r=\lceil\propto\rceil+1}^{m+1} w^{\sim}_{m+1,0,r}, \sum_{r=\lceil\propto\rceil+1}^{m+1} w^{\sim}_{m+1,1,r}, \ldots, \sum_{r=\lceil\propto\rceil+1}^{m+1} w^{\sim}_{m+1,M,r}\,].\Psi^2_{nm}(t)$

and $D^{\propto}\Psi^2_{nm}(t) = \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}} D^{\propto}U^*_m\, (2^k\, t-n) = [\, 0,0,\ldots,0\,]\, \Psi^2_{nm}(t) \qquad n \le \lceil\propto\rceil$.

4. Operational Matrices of Fractional Derivative Chebyshev wavelets

The third and fourth – kind chebyshev wavelets $\Psi^3_{nm}(t) = \Psi^4_{nm}(t)=\Psi(k,n,m,t)$ has four argument, k, n∈ N, m is the order of the polynomial $V^*_m(t)$ or $W^*_m(t)$ and t is the normalized time . they are defined explicitly on the interval [0,1] as: [1]

$\Psi^3_{nm} = \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} v^*_n(2^k\ t-n, \Psi^4_{nm} = \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} w^*_n(2^k\ t-n)$ for $t \in [\frac{n}{2^k}, \frac{n+1}{2^k}]$, m=0,1,…M, n=0,1,…, 2^k -1 and $\Psi^3_{nm} = \Psi^4_{nm} = 0$ otherwise. and the weight function:

$w^*_1 = \sqrt{\frac{(2^k\ t-n)}{1-(2^k\ t-n)}}$, $w^*_2 = \sqrt{\frac{1-(2^k\ t-n)}{(2^k\ t-n)}}$

A function nd the function f(t) defined over [0,1] may be expanded in terms of chebyshev wavelets as:

$f(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{nm}\ \Psi^{3,4}_{nm}(t),$

where

$c_{nm} = \int_0^1 w^*_{1,2} f(t)\ .\ \Psi^{3,4}_{nm}(t)\ dt$

$f(t) = \sum_{n=0}^{2^k-1}\sum_{m=0}^{m} c_{nm}\ \Psi^{3,4}_{nm}(t)$

$c = [c_{00}, c_{01}, \ldots\ c_{0,M}, \ldots\ c_{2^k-1,1}, \ldots\ldots c_{2^k-1,M}]^T$

$\Psi^{3,4}_{nm}(t) = [\Psi^{3,4}_{0,0}, \Psi^{3,4}_{0,1}, ..\Psi^{3,4}_{2^k-1,M}, \ldots\Psi^{3,4}_{2^k-1,0}, \Psi^{3,4}_{2^k-1,1}, \ldots.\Psi^{3,4}_{2^k-1,M}]^T$

4.1. New Relation between Operational Matrices of Fractional Derivative for $\Psi^2_{nm}(t)$ and $\Psi^3_{nm}(t)$

$$\Psi^2_{nm}(t) = \begin{cases} \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} U^*_m(2^k\ t-n) & t \in [\frac{n}{2^k}, \frac{n+1}{2^k}] \\ 0 & 0.w \end{cases}$$

m=0, 1,……..m, n=0,1,………2^k -1 , $U^*_m(t) = \sqrt{\frac{2}{\pi}} U_m(t)$

$$\Psi^3_{nm}(t) = \begin{cases} \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} V^*_m(2^k\ t-n) & t \in [\frac{n}{2^k}, \frac{n+1}{2^k}] \\ 0 & 0.w \end{cases}$$

m=0,1,……..M, n=0,1,………2^k -1

$V^*_m(t) = \frac{1}{\sqrt{\pi}} V_m(t)$

$$V_m(2^k\ t-n) = U_m(2^k\ t-n) - U_{m-1}(2^k\ t-n)$$

$$\sqrt{\frac{2}{\pi}} V_m(2^k\ t-n) = \sqrt{\frac{2}{\pi}} U_m(2^k\ t-n) - \sqrt{\frac{2}{\pi}} U_{m-1}(2^k\ t-n)$$

$$\sqrt{2} V^*_m(2^k\ t-n) = U^*_m(2^k\ t-n) - U^*_{m-1}(2^k\ t-n)$$

$$\sqrt{2}\frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}} V^*_m(2^k\ t-n) = \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}} U^*_m(2^k\ t-n) - \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}} U^*_{m-1}(2^k\ t-n)$$

$$2\sqrt{2}\Psi^3_{nm}(t) = \Psi^2_{nm}(t) - \Psi^2_{nm-1}(t)$$

Theorem 4.1.1: *Let $\Psi^3_{nm}(t)$ be third kind chebyshev vector and suppose ($\alpha>0$) then:*

$$\left(\sum_{r=\lceil\alpha\rceil+1}^{m+1} w^{\sim}_{m+1,M,r} - \sum_{r=\lceil\alpha\rceil+1}^{m} w^{\approx}_{m,M,r}\right) D^{\alpha}\Psi^3_{nm}(t) = \frac{2^{\frac{k+3}{2}}}{2\sqrt{2}.\sqrt{\pi}} D^{\alpha}\left(U^*_m(2^k\ t-n) - U^*_{m-1}(2^k\ t-n)\right) = \Delta^{\alpha}\Psi^3_{nm}(t)$$

where Δ^{α} is the (M+1)x(M+1) operational matrix derivative of order α in the caputo sense and defined as follow

$$\Delta^{\propto} = \frac{1}{2\sqrt{2}.}\begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ (\sum_{r=\lceil\propto\rceil+1}^{\lceil\propto\rceil+1} w_{\lceil\propto\rceil+1,0,r}^{\sim} - \sum_{r=\lceil\propto\rceil+1}^{\lceil\propto\rceil+1} w_{\lceil\propto\rceil+1,0,r}^{\approx}) & (\sum_{r=\lceil\propto\rceil+1}^{\lceil\propto\rceil+1} w_{\lceil\propto\rceil+1,1,r}^{\sim} - \sum_{r=\lceil\propto\rceil+1}^{\lceil\propto\rceil+1} w_{\lceil\propto\rceil+1,1,r}^{\approx}) \cdots & (\sum_{r=\lceil\propto\rceil+1}^{\lceil\propto\rceil+1} w_{\lceil\propto\rceil+1,M,r}^{\sim} - \sum_{r=\lceil\propto\rceil+1}^{\lceil\propto\rceil+1} w_{\lceil\propto\rceil+1,M,r}^{\approx}) \\ \vdots & \vdots & \vdots \\ (\sum_{r=\lceil\propto\rceil+1}^{m+1} w_{m+1,0,r}^{\sim} - \sum_{r=\lceil\propto\rceil+1}^{m} w_{m,0,r}^{\approx}) & (\sum_{r=\lceil\propto\rceil+1}^{m+1} w_{m+1,1,r}^{\sim} - \sum_{r=\lceil\propto\rceil+1}^{m} w_{m,1,r}^{\approx}) \cdots & (\sum_{r=\lceil\propto\rceil+1}^{m+1} w_{m+1,M,r}^{\sim} - \sum_{r=\lceil\propto\rceil+1}^{m} w_{m,M,r}^{\approx}) \\ \vdots & \vdots & \vdots \\ (\sum_{r=\lceil\propto\rceil+1}^{M} w_{M,0,r}^{\sim} - \sum_{r=\lceil\propto\rceil+1}^{M} w_{M,0,r}^{\approx}) & (\sum_{r=\lceil\propto\rceil+1}^{M} w_{M,1,r}^{\sim} - \sum_{r=\lceil\propto\rceil+1}^{M} w_{M,1,r}^{\approx}) \cdots & (\sum_{r=\lceil\propto\rceil+1}^{M} w_{M,M,r}^{\sim} - \sum_{r=\lceil\propto\rceil+1}^{M} w_{M,M,r}^{\approx}) \\ & \vdots & \end{bmatrix}$$

where

$$w_{m+1,p,r}^{\sim} = \frac{8}{\sqrt{\pi}}\sum_{\ell=0}^{p+1}\frac{r.\ell(-1)^{p+m+2-(\ell+r)}(r-1)!\,(m+r)!(p+\ell)!\;2^{2(r+\ell)-2}\left\lceil\left(r-\propto+\ell+\frac{1}{2}\right)\right.}{(m+1-r)!\,2r!\lceil(r-\propto)(p+1-\ell)!\;2\ell!\,\lceil(r+\ell-\propto+2)}$$

$$w_{m,p,r}^{\approx} = \frac{8}{\sqrt{\pi}}\sum_{\ell=0}^{p+1}\frac{r.\ell(-1)^{p+m+1-(\ell+r)}(r-1)!\;(m-1+r)!\,(p+\ell)!\;\;2^{2(r+\ell)-2}\left\lceil\left(r-\propto+\ell+\frac{1}{2}\right)\right.}{(m-r)!\;2r!\lceil(r-\propto)(p+1-\ell)!\;\;2\ell!\;\lceil(r+\ell-\propto+2)}$$

Note that in Δ^{α} , the first$\lceil\alpha\rceil+1$ rows, are all zero .

Proof: Fromtheorem(2.2), we have that

$$D^{\propto}U_m^*\left(2^k\,t-n\right) = \sum_{r=\lceil\alpha\rceil+1}^{m+1}\sum_{p=0}^{M}\frac{8}{\sqrt{\pi}}\sum_{\ell=0}^{p+1}\frac{r.\ell(-1)^{p+m+2-(\ell+r)}(r-1)!\,(m+r)!(p+\ell)!\;2^{2(r+\ell)-2}\left\lceil\left(r-\propto+\ell+\frac{1}{2}\right)\right.}{(m+1-r)!\,2r!\lceil(r-\propto)(p+1-\ell)!\;2\ell!\,\lceil(r+\ell-\propto+2)} \text{ and}$$

$$D^{\propto}U_{m-1}^*(2^k\,t-n) = \sum_{r=\lceil\alpha\rceil+1}^{m}\sum_{p=0}^{M}\frac{8}{\sqrt{\pi}}\sum_{\ell=0}^{p+1}\frac{r.\ell(-1)^{p+m+1-(\ell+r)}(r-1)!\;(m-1+r)!\,(p+\ell)!\;\;2^{2(r+\ell)-2}\left\lceil\left(r-\propto+\ell+\frac{1}{2}\right)\right.}{(m-r)!\;2r!\lceil(r-\propto)(p+1-\ell)!\;\;2\ell!\;\lceil(r+\ell-\propto+2)}$$

Then

$$D^{\propto}U_m^*\left(2^k\,t-n\right)-D^{\propto}U_{m-1}^*\left(2^k\,t-n\right)=$$

$$\left(\sum_{p=0}^{M}\sum_{r=\lceil\alpha\rceil+1}^{m+1} w_{m+1,p,r}^{\sim}\right)U_m^*(2^k t-n)-\left(\sum_{p=0}^{M}\sum_{r=\lceil\alpha\rceil+1}^{m} w_{m,p,r}^{\sim}\right)U_m^*(2^k t-n),$$

Thus,

$$D^{\propto}\Psi_{nm}^3(t)=\frac{2^{\frac{k+3}{2}}}{2\sqrt{2}.\sqrt{\pi}}\sum_{p=0}^{M}\left[\sum_{r=\lceil\propto\rceil+1}^{m+1} w_{m+1,p,r}^{\sim}-\sum_{r=\lceil\propto\rceil+1}^{m} w_{m,p,r}^{\sim}\right]U_m^*\;(2^k\;t-n),\text{ then}$$

$$D^{\propto}\Psi_{nm}^3(t)=\frac{1}{2\sqrt{2}}\Big[\,(\sum_{r=\lceil\propto\rceil+1}^{m+1} w_{m+1,0,r}^{\sim}-\sum_{r=\lceil\propto\rceil+1}^{m} w_{m,0,r}^{\sim}),\,(\sum_{r=\lceil\propto\rceil+1}^{m+1} w_{m+1,1,r}^{\sim}-\sum_{r=\lceil\propto\rceil+1}^{m} w_{m,1,r}^{\sim}),\ldots,(\sum_{r=\lceil\propto\rceil+1}^{m+1} w_{m+1,M,r}^{\sim}-\sum_{r=\lceil\propto\rceil+1}^{m} w_{m,M,r}^{\sim})\;\Big].\Psi_{nm}^3(t)\text{ and}$$

$$D^{\propto}\Psi_{nm}^3(t)=\frac{2^{\frac{k+3}{2}}}{2\sqrt{2}.\sqrt{\pi}}\left(D^{\propto}U_m^*(2^k\;t-n)-D^{\propto}U_{m-1}^*(2^k\;t-n)\right)$$
$$=\frac{1}{2\sqrt{2}}[\,0,0,\ldots,0\,]\,\Psi_{nm}^3(t)\qquad n\le\lceil\propto\rceil$$

4.2. New Relation between Operational Matrices of Fractional Derivative for $\Psi_{nm}^2(t)$ and $\Psi_{nm}^4(t)$

$$\Psi_{nm}^2(t)=\begin{cases}\dfrac{2^{\frac{k+3}{2}}}{\sqrt{\pi}}U_m^*(2^k t-n) & t\in\left[\dfrac{n}{2^k},\dfrac{n+1}{2^k}\right]\\ 0 & o.w\end{cases}$$

$m=0,1,\cdots,M$,$n=0,1,\cdots,2^k-1$,

$$U_m^*(t) = \sqrt{\frac{2}{\pi}}\, U_m(t)$$

$$\Psi_{nm}^4(t) = \begin{cases} \dfrac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} W_m^*(2^k t - n) & t \in \left[\dfrac{n}{2^k}, \dfrac{n+1}{2^k}\right] \\ 0 & o.w. \end{cases}$$

$m = 0, 1, \cdots, M, n = 0, 1, \cdots, 2^k - 1,$

$$W_m^*(t) = \frac{1}{\sqrt{\pi}} W_m(t)$$

$$W_m(2^k t - n) = U_m(2^k t - n) + U_{m-1}(2^k t - n)$$

$$\sqrt{\frac{2}{\pi}}\, W_m(2^k t - n) = \sqrt{\frac{2}{\pi}}\, U_m(2^k t - n) + U_{m-1}(2^k t - n)$$

$$\sqrt{2}\, W_m^*(2^k t - n) = U_m^*(2^k t - n) + U_{m-1}^*(2^k t - n)$$

$$\sqrt{2} \cdot \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}}\, W_m^*(2^k t - n) = \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}} U_m^*(2^k t - n) + \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}} U_{m-1}^*(2^k t - n)$$

$$2\sqrt{2} \cdot \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}\, W_m^*(2^k t - n) = \Psi_{nm}^2(t) - \Psi_{nm-1}^2(t)$$

$$2\sqrt{2}\Psi_{nm}^4(t) = \Psi_{nm}^2(t) - \Psi_{nm-1}^2(t)$$

Theorem 4.2.1: *Let $\varphi_{nm}^4(t)$ be fourth kind chebyshev vector and suppose $\propto > 0$, Then:*

$$D^{\propto}\Psi_{nm}^4(t) = \frac{2^{\frac{k+3}{2}}}{2\sqrt{2}\cdot\sqrt{\pi}} D^{\propto}\left(U_m^*(2^k t - n) + U_{m-1}^*(2^k t - n)\right) = \Delta^{(\propto)}\Psi_{nm}^4(t)$$

where $\Delta^{(\propto)}$ is the $(M+1)\times(M+1)$ operational Matrix derivative of order $\propto$ in the Caputoense and defined as follow:

$$\Delta^{(\propto)} = \frac{1}{2\sqrt{2}.}\begin{bmatrix} 0 & 0 & & 0 \\ \cdot & \cdot & & \cdot \\ 0 & 0 & & 0 \\ \left(\sum_{r=\lceil\propto\rceil+1}^{\lceil\propto\rceil+1} w_{\widetilde{\lceil\propto\rceil+1},0,r} - \sum_{r=\lceil\propto\rceil+1}^{\lceil\propto\rceil+1} w_{\overset{\approx}{\lceil\propto\rceil+1},0,r}\right) & \left(\sum_{r=\lceil\propto\rceil+1}^{\lceil\propto\rceil+1} w_{\widetilde{\lceil\propto\rceil+1},1,r} - \sum_{r=\lceil\propto\rceil+1}^{\lceil\propto\rceil+1} w_{\overset{\approx}{\lceil\propto\rceil+1},1,r}\right) & \cdots & \left(\sum_{r=\lceil\propto\rceil+1}^{\lceil\propto\rceil+1} w_{\widetilde{\lceil\propto\rceil+1},M,r} - \sum_{r=\lceil\propto\rceil+1}^{\lceil\propto\rceil+1} w_{\overset{\approx}{\lceil\propto\rceil+1},M,r}\right) \\ \vdots & \vdots & & \vdots \\ \left(\sum_{r=\lceil\propto\rceil+1}^{m+1} w_{\widetilde{m+1},0,r} - \sum_{r=\lceil\propto\rceil+1}^{m} w_{\overset{\approx}{m},0,r}\right) & \left(\sum_{r=\lceil\propto\rceil+1}^{m+1} w_{\widetilde{m+1},1,r} - \sum_{r=\lceil\propto\rceil+1}^{m} w_{\overset{\approx}{m},1,r}\right) & \cdots & \left(\sum_{r=\lceil\propto\rceil+1}^{m+1} w_{\widetilde{m+1},M,r} - \sum_{r=\lceil\propto\rceil+1}^{m} w_{\overset{\approx}{m},M,r}\right) \\ \vdots & \vdots & & \vdots \\ \left(\sum_{r=\lceil\propto\rceil+1}^{M} w_{\widetilde{M},0,r} - \sum_{r=\lceil\propto\rceil+1}^{M} w_{\overset{\approx}{M},0,r}\right) & \left(\sum_{r=\lceil\propto\rceil+1}^{M} w_{\widetilde{M},1,r} - \sum_{r=\lceil\propto\rceil+1}^{M} w_{\overset{\approx}{M},1,r}\right) & \cdots & \left(\sum_{r=\lceil\propto\rceil+1}^{M} w_{\widetilde{M},M,r} - \sum_{r=\lceil\propto\rceil+1}^{M} w_{\overset{\approx}{M},M,r}\right) \\ & \vdots & & \end{bmatrix}$$

where

$$w_{\widetilde{m+1},p,r} = \frac{8}{\sqrt{\pi}}\sum_{\ell=0}^{p+1}\frac{r.\ell(-1)^{p+m+2-(\ell+r)}(r-1)!\,(m+r)!(p+\ell)!\;\;2^{2(r+\ell)-2}\lceil\left(r-\propto+\ell+\frac{1}{2}\right)}{(m+1-r)!\;2r!\lceil(r-\propto)(p+1-\ell)!\;\;2\ell!\;\lceil(r+\ell-\propto+2)}$$

$$w_{\overset{\approx}{m},p,r} = \frac{8}{\sqrt{\pi}}\sum_{\ell=0}^{p+1}\frac{r.\ell(-1)^{p+m+1-(\ell+r)}(r-1)!\;(m-1+r)!\,(p+\ell)!\;\;2^{2(r+\ell)-2}\lceil\left(r-\propto+\ell+\frac{1}{2}\right)}{(m-r)!\;\;2r!\lceil(r-\propto)(p+1-\ell)!\;\;2\ell!\;\lceil(r+\ell-\propto+2)}$$

Note that in Δ^{α} , the first$\lceil\alpha\rceil + 1$ rows, are all zero .

4.3. New Relation between Operational Matrices of Fractional Derivative for$\boldsymbol{\Psi^2_{nm}(t)}$, $\boldsymbol{\Psi^3_{nm}(t)}$ and $\boldsymbol{\Psi^4_{nm}(t)}$

$$\Psi^2_{nm}(t) = \begin{cases} \dfrac{2^{\frac{k+3}{2}}}{\sqrt{\pi}} U^*_m(2^k t - n) & t \in \left[\dfrac{n}{2^k}, \dfrac{n+1}{2^k}\right] \\ 0 & o.w. \end{cases}$$

$m = 0, 1, \cdots, M, n = 0, 1, \cdots, 2^k - 1,\ U^*_m(t) = \sqrt{\frac{2}{\pi}}\, U_m(t)$

$$\Psi^3_{nm}(t) = \begin{cases} \dfrac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} V^*_m(2^k t - n) & t \in \left[\dfrac{n}{2^k}, \dfrac{n+1}{2^k}\right] \\ 0 & o.w. \end{cases}$$

$m = 0, 1, \cdots, M, n = 0, 1, \cdots, 2^k - 1, \quad V^*_m(t) = \frac{1}{\sqrt{\pi}} V_m(t)$

$$\Psi^4_{nm}(t) = \begin{cases} \dfrac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} W^*_m(2^k t - n) & t \in \left[\dfrac{n}{2^k}, \dfrac{n+1}{2^k}\right] \\ 0 & O.W. \end{cases}$$

$m = 0, 1, \cdots, M, n = 0, 1, \cdots, 2^k - 1\ , \quad W^*_m(t) = \frac{1}{\sqrt{\pi}} W_m(t)$

$$2U_m(2^k t - n) = V_m(2^k t - n) + W_m(2^k t - n)$$

$$\sqrt{2} \cdot \sqrt{\frac{2}{\pi}}\, U_m(2^k t - n) = \frac{1}{\sqrt{\pi}} V_m(2^k t - n) + \frac{1}{\sqrt{\pi}} W_m(2^k t - n)$$

$$\sqrt{2}\,.\frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}} U^*_m(2^k t - n) = \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}} V^*_m(2^k t - n) + \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}} W^*_m(2^k t - n)$$

$$\frac{2^{\frac{k+3}{2}}}{\sqrt{2}\sqrt{\pi}} U^*_m(2^k t - n) = \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} V^*_m(2^k t - n) + \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} W^*_m(2^k t - n)$$

$$\frac{1}{\sqrt{2}} \Psi^2_{nm}(t) = \Psi^3_{nm}(t) + \Psi^4_{nm}(t)$$

Theorem 4.3.1: *Let* $\Psi^2_{nm}(t)$,$\Psi^3_{nm}(t)$*and*$\Psi^4_{nm}(t)$ *are shifted second, third and fourth kind chebyshev vector respectively and suppose* $\propto > 0$, *Then:*

$$D^{\propto}\Psi^2_{nm}(t) = \Delta^{(\propto)}\Psi^2_{nm}(t) = \frac{2^{\frac{k+1}{2}}}{2\sqrt{2}\cdot\sqrt{\pi}} D^{\propto}(v^*_m(2^k t - n) + w^*_m(2^k t - n))$$
$$= \frac{1}{2\sqrt{2}\cdot} (\Delta^{(\propto)}\varphi^3_{nm}(t) + \Delta^{(\propto)}\Psi^4_{nm}(t))$$

where $\Delta^{(\propto)}$ *is the* $(M+1) \times (M+1)$ *operational Matrix derivative of order* $\propto$ *in the Caputo derivative.*

4.4. New Relation between Operational Matrices of Fractional Derivative for $\boldsymbol{\Psi^1_{\hat{n}m}(t)}$ and $\boldsymbol{\Psi^2_{nm}(t)}$

$$\Psi^1_{\hat{n}m}(t) = \begin{cases} \dfrac{2^{k/2}}{\sqrt{\pi}} T^*_m(2^k t - \hat{n}) & t \in \left[\dfrac{\hat{n}-1}{2^k}, \dfrac{\hat{n}}{2^k}\right] \\ 0 & o.w. \end{cases}$$

where

$$T^*_m(t) = \begin{cases} \dfrac{1}{\sqrt{\pi}} T_m(t) & m = 0 \\ \sqrt{\dfrac{2}{\pi}}\, T_m(t) & m > 0 \end{cases}$$

$$\Psi^2_{nm}(t) = \begin{cases} \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}} U^*_m(2^k t - n) & t \in \left[\frac{n}{2^k}, \frac{n+1}{2^k}\right] \\ 0 & o.w. \end{cases}$$

$$U^*_m(t) = \sqrt{\frac{2}{\pi}} U_m(t)$$

$$2\, T_m(2^k t - n) = U_m(2^k t - n) + U_{m-2}(2^k t - n) m = 2, 3, \cdots$$

$$2\sqrt{\frac{2}{\pi}}\, T_m(2^k t - n) = \sqrt{\frac{2}{\pi}} U_m(2^k t - n) - \sqrt{\frac{2}{\pi}} U_{m-2}(2^k t - n)$$

$$2\, T^*_m(2^k t - n) = U^*_m(2^k t - n) + U^*_{m-2}(2^k t - n)$$

$$2 \cdot \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}}\ T^*_m(2^k t - n) = \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}} U^*_m(2^k t - n) + \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}} U^*_{m-2}(2^k t - n)$$

$$c_{\mathrm{m}} = \begin{cases} 1 & \mathrm{m} = 0 \\ \sqrt{2} & \mathrm{m} \neq 0 \end{cases}$$

$$\frac{4\sqrt{2}}{C_m} \Psi^1_{\tilde{n}m}(t) = \Psi^2_{nm}(t) - \Psi^2_{nm-2}(t) \quad m = 2, 3, \cdots$$

Theorem 4.4.1: *Let $\Psi^1_{\tilde{n}m}(t)$ be fourth kind chebyshev vector and suppose $\propto > 0$, Then:*

$$D^{\propto}\Psi^1_{\tilde{n}m}(\mathrm{t})(t) = \frac{C_m 2^{\frac{k+3}{2}}}{4\sqrt{2} \cdot \sqrt{\pi}} D^{\propto}\left(U^*_m(2^k t - n) + U^*_{m-1}(2^k t - n)\right) = \Delta^{(\propto)}\Psi^2_{nm}(t)$$

where $\Delta^{(\propto)}$ is the $(M+1) \times (M+1)$ operational Matrix derivative of order $\propto$ in the Caputo sense and defined as follow:

$$\Delta^{(\propto)} = \frac{C_m}{4\sqrt{2}.} \begin{bmatrix} 0 & 0 & 0 \\ . & . & . \\ 0 & 0 & 0 \\ \left(\sum_{r=\lceil\propto\rceil+1}^{\lceil\propto\rceil+1} w_{\lceil\propto\rceil+1,0,r}^{\sim} - \sum_{r=\lceil\propto\rceil+1}^{\lceil\propto\rceil+1} w_{\lceil\propto\rceil+1,0,r}^{\approx}\right) & \left(\sum_{r=\lceil\propto\rceil+1}^{\lceil\propto\rceil+1} w_{\lceil\propto\rceil+1,1,r}^{\sim} - \sum_{r=\lceil\propto\rceil+1}^{\lceil\propto\rceil+1} w_{\lceil\propto\rceil+1,1,r}^{\approx}\right) & \cdots\left(\sum_{r=\lceil\propto\rceil+1}^{\lceil\propto\rceil+1} w_{\lceil\propto\rceil+1,M,r}^{\sim} - \sum_{r=\lceil\propto\rceil+1}^{\lceil\propto\rceil+1} w_{\lceil\propto\rceil+1,M,r}^{\approx}\right) \\ \vdots & \vdots & \vdots \\ \left(\sum_{r=\lceil\propto\rceil+1}^{m+1} w_{m+1,0,r}^{\sim} - \sum_{r=\lceil\propto\rceil+1}^{m-1} w_{m-1,0,r}^{\approx}\right) & \left(\sum_{r=\lceil\propto\rceil+1}^{m+1} w_{m+1,1,r}^{\sim} - \sum_{r=\lceil\propto\rceil+1}^{m-1} w_{m-1,1,r}^{\approx}\right)\cdots & \left(\sum_{r=\lceil\propto\rceil+1}^{m+1} w_{m+1,M,r}^{\sim} - \sum_{r=\lceil\propto\rceil+1}^{m-1} w_{m-1,M,r}^{\approx}\right) \\ \vdots & \vdots & \vdots \\ \left(\sum_{r=\lceil\propto\rceil+1}^{M} w_{M,0,r}^{\sim} - \sum_{r=\lceil\propto\rceil+1}^{M} w_{M,0,r}^{\approx}\right) & \left(\sum_{r=\lceil\propto\rceil+1}^{M} w_{M,1,r}^{\sim} - \sum_{r=\lceil\propto\rceil+1}^{M} w_{M,1,r}^{\approx}\right) \cdots & \left(\sum_{r=\lceil\propto\rceil+1}^{M} w_{M,M,r}^{\sim} - \sum_{r=\lceil\propto\rceil+1}^{M} w_{M,M,r}^{\approx}\right) \\ & \vdots & \end{bmatrix}$$

where

$$w_{m+1,p,r}^{\sim} = \frac{8}{\sqrt{\pi}} \sum_{\ell=0}^{p+1} \frac{r.\ell(-1)^{p+m+2-(\ell+r)}(r-1)!\,(m+r)!(p+\ell)!\ 2^{2(r+\ell)-2}\Gamma\left(r-\propto+\ell+\frac{1}{2}\right)}{(m+1-r)!\ 2r!\Gamma(r-\propto)(p+1-\ell)!\ 2\ell!\ \Gamma(r+\ell-\propto+2)}$$

$$w_{m-2,p,r}^{\approx} = \frac{8}{\sqrt{\pi}} \sum_{\ell=0}^{p+1} \frac{r.\ell(-1)^{p+m-(\ell+r)}(r-1)!\ (m-2+r)!\,(p+\ell)!\ \ 2^{2(r+\ell)-2}\ \Gamma\left(r-\propto+\ell+\frac{1}{2}\right)}{(m-1-r)!\ 2r!\,\Gamma(r-\propto)(p+1-\ell)!\ \ 2\ell!\ \Gamma(r+\ell-\propto+2)}$$

Note that in Δ^{α}, the first$\lceil\alpha\rceil + 1$ rows, are all zero .

Proof:

$$D^{\propto} U_m^*\left(2^k\ t - n\right) = \sum_{r=\lceil\alpha\rceil+1}^{m+1} \sum_{p=0}^{M} \frac{8}{\sqrt{\pi}} \sum_{\ell=0}^{p+1} \frac{r.\ell(-1)^{p+m+2-(\ell+r)}(r-1)!\,(m+r)!(p+\ell)!\ 2^{2(r+\ell)-2}\Gamma\left(r-\propto+\ell+\frac{1}{2}\right)}{(m+1-r)!\,2r!\Gamma(r-\propto)(p+1-\ell)!\ 2\ell!\,\Gamma(r+\ell-\propto+2)} \text{ and}$$

$$D^{\propto} U_{m-2}^*\left(2^k\ t - n\right) = \sum_{r=\lceil\alpha\rceil+1}^{m-2} \sum_{p=0}^{M} \frac{8}{\sqrt{\pi}} \sum_{\ell=0}^{p+1} \frac{r.\ell(-1)^{p+m-(\ell+r)}(r-1)!\,(m-2+r)!(p+\ell)!\ 2^{2(r+\ell)-2}\Gamma\left(r-\propto+\ell+\frac{1}{2}\right)}{(m-1-r)!\,2r!\Gamma(r-\propto)(p+1-\ell)!\ 2\ell!\,\Gamma(r+\ell-\propto+2)}$$

Then

$$D^{\propto} U_m^*\left(2^k\ t - n\right) - D^{\propto} U_{m-2}^*\left(2^k\ t - n\right) = \left(\sum_{p=0}^{M} \sum_{r=\lceil\alpha\rceil+1}^{m+1} \mathrm{w}_{m+1,p,r}^{\sim}\right) U_m^*\left(2^k t - n\right) - \left(\sum_{p=0}^{M} \sum_{r=\lceil\alpha\rceil+1}^{m-1} \mathrm{w}_{m-1,p,r}^{\sim}\right) U_m^*\left(2^k t - n\right),$$

Thus,

$$D^{\propto}\Psi_{\tilde{n}m}^1(t) = \frac{C_m\, 2^{\frac{k+3}{2}}}{4\sqrt{2}.\sqrt{\pi}} \sum_{p=0}^{M} \left[\sum_{r=\lceil\propto\rceil+1}^{m+1} \mathrm{w}_{m+1,p,r}^{\sim} - \sum_{r=\lceil\propto\rceil+1}^{m-1} \mathrm{w}_{m-1,p,r}^{\sim}\right] U_m^*\ (2^k\ t - n), \text{ then}$$

$$D^{\propto}\Psi_{\tilde{n}m}^1(t) = \frac{C_m}{4\sqrt{2}}\left[\ \left(\sum_{r=\lceil\propto\rceil+1}^{m+1} w_{m+1,0,r}^{\sim} - \sum_{r=\lceil\propto\rceil+1}^{m-1} \mathrm{w}_{m-1,0,r}^{\sim}\right), \left(\sum_{r=\lceil\propto\rceil+1}^{m+1} w_{m+1,1,r}^{\sim} - \sum_{r=\lceil\propto\rceil+1}^{m-1} \mathrm{w}_{m-1,1,r}^{\sim}\right), \ldots, \left(\sum_{r=\lceil\propto\rceil+1}^{m+1} w_{m+1,M,r}^{\sim} - \sum_{r=\lceil\propto\rceil+1}^{m-1} \mathrm{w}_{m-1,M,r}^{\sim}\right.\ \right].\Psi_{nm}^2(t) \text{ and}$$

$$D^{\propto}\Psi_{\tilde{n}m}^1(t) = \frac{C_m\, 2^{\frac{k+3}{2}}}{4\sqrt{2}.\sqrt{\pi}}\left(D^{\propto} U_m^*\left(2^k\ t - n\right) - D^{\propto} U_{m-1}^*\left(2^k\ t - n\right)\right) = \frac{C_m}{4\sqrt{2}}[0, 0, \ldots, 0\ ]\ \Psi_{\mathrm{nm}}^2(\mathrm{t}) \qquad n \leq \lceil\propto\rceil$$

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