International Journal of Mathematical Archive-2(6), June - 2011, Page: 904-909
IJMA Available online through www.ijma.info ISSN 2229 – 5046

SOME RESULTS ON COMMON FIXED POINT OF MAPPINGS SATISFYING *A*-CONTRACTION TYPE CONDITION IN SYMMETRIC SPACES

[1]Azhar Ali Zafar*, [2]Muhammad Akram

Department of Mathematics, G. C. University, Lahore 54000, Pakistan

E-mail: [1]adolfaaz@yahoo.com, [2]makram71@yahoo.com

(Received on: 13-04-11; Accepted on: 26-04-11)

ABSTRACT

In this paper some common fixed point theorems for self mappings and for converse commuting mappings of a symmetric space satisfying A-contractions (introduced in [2] by the present authors) type conditions are proved.

Keywords: *A-contractions, Symmetric spaces, Fixed point, Weakly compatible, Converse commuting maps.*

AMS(2000) Classification: *Primary 54H25, Secondary 47H10.*

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1. INTRODUCTION:

Banach contraction principle is a fundamental result in fixed point theory, which has been used and extended in many different directions. However it has been observed in [8] that some of the defining properties of the metric are not needed in the proofs of certain fixed point (common fixed point) theorems. Motivated by this fact, Hicks [8] established some common fixed point theorems in symmetric spaces and proved that very general probabilistic structure admit a compatible symmetric or semi-metric. In [2] present authors with A. A. Siddiqui introduced a general class of contractions, called *A*-contractions. This class properly includes contractions originally studied by R. Kannan [10], M. S. Khan [12], Bianchini [6], Reich [13] for details see [2]. In [1] M. Aamiri and D. El Moutawakil presented some fixed point theorems under strict contractive conditions in the symmetric spaces. In this paper, we prove some fixed point theorems for self mappings satisfying *A*-contractions type conditions in the symmetric spaces. Some more fixed point theorems for *A*-contractions in complete metric spaces can be found for example in [3], [4] and [5].

2. PRELIMINARIES:

Definition: 2.1 *A symmetric on a set X is a real valued function d on $X \times X$ such that*

(i) $d(x, y) \geq 0$

(ii) $d(x, y) = 0$ *iff* $x = y$

(iii) $d(x, y) = d(y, x)$.

Example: 2.2 *Let* $X =]-\infty, \infty[$ *and* $d : X \times X \to R$ *defined by* $d(x, y) = e^{|x-y|}$ *for all* $x, y \in X$. *Obviously* d *is symmetric on* X *, but it is not metric on X.*

Let d be a symmetric on a set X and for $r > 0$ and any $x \in X$, let $B(x, r) = \{y \in X : d(x, y) < r\}$. A topology $\tau(d)$ on X is given by $U \in \tau(d)$ if and only if for each $x \in U, B(x, r) \subset U$ for some $r > 0$. A symmetric d is a semi-metric if for each $x \in X$ and each $r > 0$, $B(x, r)$ is a neighborhood of x in the topology $\tau(d)$. Note that

$$\lim_{n\to\infty} d(x_n, x) = 0$$

if and only if $x_n \to x$ in the topology $\tau(d)$.

The following two axioms are given in Wilson [15]. Let (X, d) be a symmetric space.

(A.1) Given $\{x_n\}, x$ and $y \in X$,

$$\lim_{n\to\infty} d(x_n, x) = 0$$

***Corresponding author:** [1]*Azhar Ali Zafar*, ***E-mail:** adolfaaz@yahoo.com*

and

$$\lim_{n\to\infty} d(x_n, y) = 0$$

imply $x = y$.

(A.2) Given that $\{x_n\},\{y_n\}$, and $x\in X$,

$$\lim_{n\to\infty} d(x_n, x) = 0$$

and

$$\lim_{n\to\infty} d(x_n, y_n) = 0$$

imply that

$$\lim_{n\to\infty} d(y_n, x) = 0.$$

(A.3) If given $\{x_n\},\{y_n\}$, and $x\in X$,

$$\lim_{n\to\infty} d(x_n, x) = 0$$

and

$$\lim_{n\to\infty} d(y_n, x) = 0$$

implies

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Two self mappings S and T on a symmetric space (X,d) are said to satisfy the property

(P.1) If there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} d(Sx_n, t) = \lim_{n\to\infty} d(Tx_n, t) = 0 \text{ for some } t\in X.$$

Definition: 2.3 *Two self mappings S and T of a symmetric space (X,d) are said to be Compatible if* $\lim_{n\to\infty} d(STx_n, TSx_n) = 0,$ whenever $\{x_n\}$ is sequence in X such that

$$\lim_{n\to\infty} d(Sx_n, t) = \lim_{n\to\infty} d(Tx_n, t) = 0, \text{ for some } t\in X.$$

Definition: 2.4 Two self mappings S and T of a symmetric space (X,d) are said to be weakly compatible if they commute at their coincidence points i.e. if $Tu = Su$ for some $u\in X$, then $TSu = STu$.

Definition: 2.5 Two self mappings S and T of a symmetric space (X,d) will be non-compatible if there exists at least one sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} d(Sx_n, t) = \lim_{n\to\infty} d(Tx_n, t) = 0 \text{ for some } t\in X, \text{ but } \lim_{n\to\infty} d(STx_n, TSx_n)$$

is either non-zero or does not exist. Two self mappings S and T of a symmetric space (X,d) are said to be weakly commuting if $d(STx, TSx) \le d(Sx, Tx)$ for all $x\in X$.

Definition: 2.6 *Let f, g be single valued mapping from X into itself. Two mappings f and g are called converse commuting if forall x in X, $fgx = gfx$ implies $fx = gx$. A point t in* X is said to be commuting point of f and g if

$$fgt = gft.$$

The notion of weakly commuting maps was introduced by S. Sessa in [14]. It is established in [9] that two weakly commuting mappings are compatible but the converse is not true. It is also easy to see that two compatible maps are weakly compatible but the converse is not true.

3. SOME FIXED POINT THEOREMS:

Let R_+ denotes the set of all non-negative real numbers and A be the set of functions $\alpha : R_+^3 \to R_+$ satisfying

(i) α is continuous on the set R_+^3 (with respect to the Euclidean metric on R^3)

(ii) $a \le kb$ for some $k \in [0,1)$ whenever $a \le \alpha(a,b,b)$ or $a \le \alpha(b,a,b)$ or $a \le \alpha(b,b,a)$ for all $a,b \in R_+^3$.

Remark: 3.1 *If in addition we take α non-decreasing then we have the following observations:*
(I) In (ii) given above $a \le kb < b$ and $a = b$ is possible only when $a = b = 0$.
(II) $a \le \alpha(a,0,0)$ or $a \le \alpha(0,a,0)$ or $a \le \alpha(0,0,a)$ or $a \le \alpha(a,a,0)$ or $a \le \alpha(a,0,a)$ or $a \le \alpha(0,a,a)$ or $a \le \alpha(0,0,0)$ implies $a \le \alpha(a,a,a)$ and by (ii) given above we have $a \le ka$ for some $k \in [0,1)$, which is only possible when $a = 0$.

Theorem: 3.2 *Let d be a symmetric on X that satisfies (A.1) and (A.3). Let A and B be two weakly compatible self mappings of (X,d) such that*

(i) $d(Ax,By) \le \alpha(d(Bx,By), d(Bx,Ay), d(Ay,By))$ for all $x, y \in X$ and for some $\alpha \in \mathrm{A}$ with α non-decreasing.
(ii) A and B satisfy the property (P.1)
(iii) $AX \subset BX$.

If the range of A or B is a complete subspace of X then, A and B have a unique common fixed point.

Proof: Since A and B satisfy the property (P.1), therefore there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} d(Ax_n,t) = \lim_{n\to\infty} d(Bx_n,t) = 0 \text{ for some } t \in X. \text{ By (A.3), we have}$$
$$\lim_{n\to\infty} d(Ax_n, Bx_n) = 0.$$

Suppose that BX is complete subspace of X then $t = Bu$ for some $u \in X$. We claim that $Au = Bu$. Indeed, by (i), we have

$$d(Au, Bx_n) \le \alpha(d(Bu, Bx_n), d(Bu, Ax_n), d(Ax_n, Bx_n)).$$

Taking limit as $n \to \infty$ and using continuity of α , we have

$$\lim_{n\to\infty} d(Au, Bx_n) \le \alpha(\lim_{n\to\infty} d(Bu, Bx_n), \lim_{n\to\infty} d(Bu, Ax_n), \lim_{n\to\infty} d(Ax_n, Bx_n)).$$

This gives

$$\lim_{n\to\infty} d(Au, Bx_n) \le \alpha(0,0,0),$$

by using Remark 3.1. we have

$$\lim_{n\to\infty} d(Au, Bx_n) = 0.$$

Hence by (A.1) we have $Au = t = Bu$. Since A and B are weakly compatible therefore A and B commute at their coincidence point i.e. $ABu = BAu$ i.e. $At = Bt$. Now we prove that t is a common fixed point of A and B . In view of (i), it follows

$$\begin{aligned} d(At,t) = d(At,Bu) &\le \alpha(d(Bt,Bu), d(Bt,Au), d(Au,Bu)) \\ &\le \alpha(d(At,t), d(At,t), d(t,t)) \\ &\le \alpha(d(At,t), d(At,t), 0) \\ &\le \alpha(d(At,t), d(At,t), d(At,t)). \end{aligned}$$

By Remark 3.1. we get $d(At,t) = 0$ and therefore t is a common fixed point of A as will as of B . The proof is similar when AX is assumed to be a complete subspace of X, since $AX \subset BX$. For uniqueness, suppose contrary that t is not unique common fixed point of A and B. Let $Av = Bv = v$ i.e. v be a another common fixed point of A and B such that $v \neq t$, then (i) gives

$$\begin{aligned} d(At, Av) = d(At, Bv) &\leq \alpha(d(Bt, Bv), d(Bt, Av), d(Bv, Av)) \\ &\leq \alpha(d(At, Av), d(At, Av), d(Bv, Av)) \\ &\leq \alpha(d(At, Av), d(At, Av), 0) \\ &\leq \alpha(d(At, Av), d(At, Av), d(At, Av)). \end{aligned}$$

Again by Remark 3.1. we have $d(At, Av) = 0$. Therefore t=At=Av=v and the common fixed point is unique.

Theorem: 3.3 *Let d be a symmetric on X that satisfies (A.1), (A.2) and (A.3). Let A, B, T and S be self mappings of (X, d) such that*

(i) $d(Ax, By) \leq \alpha(d(Sx, Ty), d(Sx, By), d(Ty, By))$ *for all $x, y \in X$ and for some $\alpha \in \mathrm{A}$ with α non-decreasing.*
(ii) *The pairs (A, T) and (B, S) are weakly compatible.*
(iii) *The pair (A, S) or (B, T) satisfies the property (P.1).*
(iv) $AX \subset TX$ *and* $BX \subset SX$.

If the range of any one of the four mappings A, B, T and S is a complete subspace of X then, A, B, T and S have a unique common fixed point.

Proof: Suppose that the pair of mappings (B,T) satisfies the property (P.1). Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} d(Bx_n, t) = \lim_{n\to\infty} d(Tx_n, t) = 0,$ for some $t \in X$. By using (A.3)we have $\lim_{n\to\infty} d(Tx_n, Bx_n) = 0.$

Since, $BX \subset SX$, then there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$, hence

$\lim_{n\to\infty} d(Sy_n, t) = 0$. Similarly, $AX \subset TX$, then there exists a sequence $\{y_n\}$ in X such that $Ay_n = Tx_n$, therefore $\lim_{n\to\infty} d(Ay_n, t) = 0.$

Suppose that SX is complete subspace of X. Let $t = Su$ for some $u \in X$. From above we have,

$$\lim_{n\to\infty} d(Ay_n, Su) = \lim_{n\to\infty} d(Bx_n, Su) = \lim_{n\to\infty} d(Tx_n, t) = \lim_{n\to\infty} d(Sy_n, t) = 0$$

From (i), it follows
$$d(Au, Bx_n) \leq \alpha(d(Su, Tx_n), d(Su, Bx_n), d(Tx_n, Bx_n)).$$

Taking limit as $n \to \infty$, we have
$$\lim_{n\to\infty} d(Au, Bx_n) \leq \alpha(0, 0, 0).$$
which implies
$$\lim_{n\to\infty} d(Au, Bx_n) = 0.$$
Since
$$\lim_{n\to\infty} d(t, Bx_n) = 0,$$

so by (A.1), we have $Au = t = Su.$ The weak compatibility of A and S implies that $ASu = SAu$ and then $AAu = ASu = SAu = SSu$. On the other hand, since $Ax \subset TX$, there exists $v \in X$ such that $Au = Tv$. We claim that $Au = Bv$. To justify our claim, we proceed as follows. From (i) we have,

$$\begin{aligned} d(Au,Bv) &\le \alpha(d(Su,Tv),d(Su,Bv),d(Tv,Bv)) \\ &\le \alpha(d(Au,Tv),d(Au,Bv),d(Au,Bv)) \\ &\le \alpha(0,d(Au,Bv),d(Au,Bv)) \\ &\le \alpha(d(Au,Bv),d(Au,Bv),d(Au,Bv)) \end{aligned}$$

therefore by using Remark 3.1. $d(Au,Bv)=0$. Hence $Au=Su=Tv=Bv$. The weak compatibility of B and T implies that $BTv=TBv$ and $TTv=TBv=BTv=BBv$. Now, we show that Au is common fixed point of $A,B,T and S$. Suppose that $AAu \ne Au$ then

$$\begin{aligned} d(AAu,Au)=d(AAu,Bv) &\le \alpha(d(SAu,Tv),d(SAu,Bv),d(Tv,Bv)) \\ &\le \alpha(d(AAu,Au),d(AAu,Au),0) \\ &\le \alpha(d(AAu,Au),d(AAu,Au),AAu,Au). \end{aligned}$$

Again by Remark 3.1. $d(AAu,Au)=0$. Therefore $Au=AAu=ASu=SAu$ and $Au=t$ is a common fixed point of A and S i.e. $At=t$ and $Bt=t$. Similarly, we can prove that Bv is a common fixed point of B and T. Since $Au=Bv$, we conclude that t is a common fixed point of A,B,T *and* S. The proof is similar when TX is assumed to be complete subspace of X. The case in which AX and BX is a complete subspace of X are similar to the case in which TX or SX respectively is complete, as $Ax \subset TX$ and $BX \subset SX$. For uniqueness let $Aw=Bw=Tw=Sw=w$ be an other common fixed point of $A,B,T and S$ such that $t \ne w$, then from (i) we have,

$$\begin{aligned} d(t,w)=d(At,Bw) &\le \alpha(d(St,Tw),d(St,Bw),d(Tw,Bw)) \\ &\le \alpha(d(t,w),d(t,w),0) \end{aligned}$$

Again by Remark 3.1. $d(t,w)=0$. Therefore $t=w$ and the common fixed point is unique.

Following result is related to converse commuting and commuting point of the map

Theorem: 3.4 *Let d be a symmetric on X and f_1, f_2, g_1, and g_2 are self maps on X satisfying the inequality*

$$d(f_1x,g_1y) \le \alpha(d(f_2x,g_2y),d(f_1x,g_2y),d(g_1y,f_2x))$$

for all $x,y \in X$ and for some $\alpha \in \mathrm{A}$ with α non-decreasing. If the pair of mappings (f_1,f_2) and (g_1,g_2) are converse commuting maps and further if these pair of mappings have a commuting point then there exists a common fixed point of f_1, f_2, g_1,and g_2.

Proof: Suppose that u be the commuting point of (f_1,f_2) and v be the commuting point of (g_1,g_2). Also the pair (f_1,f_2) is converse commuting so we have $f_1f_2u=f_2f_1u$ implies $f_1u=f_2u$, similarly we have $g_1g_2v=g_2g_1v$ implies $g_1v=g_2v$. We claim that $f_1u=g_1v$. To justify our claim we proceeds as,

$$\begin{aligned} d(f_1u,g_1v) &\le \alpha(d(f_2u,g_2v),d(f_1u,g_2v),d(g_1v,f_2u)) \\ &\le \alpha(d(f_1u,g_1v),d(f_1u,g_1v),d(g_1v,f_1u)) \end{aligned}$$

this implies $d(f_1u,g_1v)=0$, that is $f_1u=g_1v$

Now we show that f_1u is the fixed point of f_1 as follows,

$$\begin{aligned} d(f_1f_1u,f_1u)=d(f_1f_1u,g_1v) &\le \alpha(d(f_2f_1u,g_2v),d(f_1f_1u,g_2v),d(g_1v,f_2f_1u)) \\ &\le \alpha(d(f_1f_1u,g_1v),d(f_1f_1u,g_1v),d(g_1v,f_1f_1u)). \end{aligned}$$

This implies $d(f_1f_1u,f_1u)=0$ i.e. $f_1f_1u=f_1u$.

Similarly we have $g_1 v = g_1 g_1 v$ of $f_1 u = g_1 f_1 u$ and $f_1 u$ is the fixed point of g_1. On the other hand, $f_1 u = g_1 v = g_2 g_1 v = g_2 f_1 u$ and $f_1 f_1 u = f_1 f_2 u = f_2 f_1 u$. Hence $f_1 u$ is a common fixed point of f_1, g_1, f_2, g_2.

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