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# ON RETRACT EXTENSION OF ORDERED SET–A CONSTRUCTIVE POINT OF VIEW[1]

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*ABSTRACT*

*In this paper a concept of the retract extensions of ordered sets with apartness is given by Bishop's constructive mathematics point of view.*

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***Key words and phrases**: Constructive mathematics, order and anti-order relation, ideal and anti-ideal of ordered set, order isotone and order reverse isotone function, anti-order isotone and anti-order reverse isotone function.*

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## 1. INTRODUCTION

The extension problem for groups is as follows: Given two groups H and K, construct all groups G which have a normal subgroup N such that N is isomorphic to H, in symbol $N \cong H$ and $G/N \cong K$, where G/N is the quotient of G by N. G is called the simply the extension of H by K. Given a semigroup S and a semigroup Q with zero, a semigroup V is called an ideal extension of P by Q if there exists an ideal A of V such that A is isomorphic to P and the Rees quotient V/(A,B) is isomorphic to B. The extension problem for semigroups, or ordered semigroups, is as follows: Given a semigroup S and a semigroup Q with zero, S and Q are disjoint, construct all the semigroups V which are extensions of S by Q. For the definition of the Rees quotient for semigroups and ordered semigroups, we refer to [17] and [12], respectively. Ideal extensions of semigroups have been considered in [6] with a detailed exposition of the theory appearing in [7], [17]. Extensions of weakly reductive semigroups, strict and pure extensions, retract extensions, dense extensions, equivalent extensions have been also considered in [17]. Ideal extensions of totally ordered semigroups have been studied in [9] and [10], of topological semigroups in [5] and [8]. For the ideal extensions of lattices we refer to [11]. For the ideal extensions of ordered semigroups we refer to [12]. Inspired by semigroups, ideal extensions of partially ordered sets have been studied in [13].

In the classical theory, ideal extensions of ordered sets have been considered by Kehayopulu in [13]. In Bishop's constructive mathematics extension of ordered set (under compatible order and anti-order relations) with apartness is presented in our forthcoming paper [23]. In the present paper we continue that investigation. This time, we introduce and study the concept of the retract extensions of ordered sets (under compatible order and anti-order relations) with apartnesses.

If X and Y are two disjoint ordered sets, an ordered set V is called an extension of X by Y if there is an ideal A and an anti-ideal B of V such that A is isomorphic to X and B is isomorphic to Y. The ideal extension problem for ordered sets is as follows. Given two disjoint ordered sets X and Y, construct the ordered sets V which are ideal extensions of X by Y. Mathematicians are often interested in building more complex semigroups, lattices, ordered sets, ordered or topological semigroups out of some of 'simpler' structure and this can be sometimes achieved by constructing the ideal extensions.

For undefined notions and notations in classical mathematics we referred to books [1], [3], [7], [15] and [17] and papers [5] – [6], [8] – [14] and [24] and of constructive items in books [2], [4], [16] and [25] and the author's papers [18] – [23].

## 2 PRELIMINARIES:

Let $(S, =, \neq)$ be an ordered set with apartness under order relation $\leq$ and under anti-order relation $\Theta$.

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(1) An *order ideal* of S is a subset I of S such that

$$(\forall x,y)(x\in I \wedge y \leq x \Rightarrow y\in I)$$

(2) An *order anti-ideal* of S is a subset K of S such that

$$(\forall x,y)(y\in K \Rightarrow y\Theta x \vee x\in K)$$

For an order ideal I and order anti-ideal K we say that they are *compatible* if and only if $I \subseteq \neg K$.

We say that $(S,=_S,\neq_S,\leq_S,\Theta_S)$ is an *ordered substructure* of $(T,=_T,\neq_T,\leq_T,\Theta_T)$ if S is a subset of T and the order on S is the restriction to S of the order on T.

Let $(V,=,\neq,\leq,\Xi)$ be an ordered set. The following lemma shows some basic properties of ordered set:

**Lemma 1:** ([23], Lemma 3.1) *Each nonempty subset* Z *of an ordered set* $(V,=,\neq,\leq,\Xi)$ *with the relations* $=_Z$, $\neq_Z$, $\leq_Z$, $\xi_Z$ *on* Z *defined by*

$$=_Z = =_V \cap(Z\times Z),\ \neq_Z = \neq_V \cap( Z\times Z),\ \leq_Z = \leq_V \cap(Z \times Z),\ \xi_Z = \Xi_V \cap (Z\times Z),$$

*is an ordered set.*

Let $(X,=_X,\neq_X,\leq_X,\alpha_X)$, $(Y,=_Y,\neq_Y,\leq_Y,\alpha_Y)$ be ordered sets and $X \rhd\!\lhd Y$. An ordered set $(V,=_V,\neq_V,\leq_V ,\Xi)$ is called an *ideal extension* of X by Y if there exists an ideal A and an anti-ideal B of V such that

$$(X,=_X,\neq_X,\leq_X,\alpha_X) \cong (A,=_A,\neq_A,\leq_A,\alpha),\ (B,=_B,\neq_B\leq_B,\beta) \cong (Y,=_Y,\neq_Y,\leq_Y,\alpha_Y)$$

Where

$$=_A = =_V \cap(A\times A),\ \neq_A = \neq_V \cap(A\times A),\ \leq_A = \leq_V \cap(A \times A),\ \alpha = \Xi_V \cap (A\times A), \text{ and}$$

$$=_B = =_V \cap(B\times B),\ \neq_B = \neq_V \cap(B\times B),\ \leq_B = \leq_V \cap(B \times B),\ \beta = \Xi_V \cap (B\times B).$$

If $(V,=,\neq,\leq_V ,\Xi)$ is an extension of X by Y, we always denote by $\varphi$ and $\psi$ the isomorphisms

$$\varphi : (X,=_X,\neq_X,\leq_X,\alpha_X) \to (A,=_A,\neq_A,\leq_A,\alpha),$$

$$\psi : (Y,=_Y,\neq_Y,\leq_Y,\alpha_Y) \to (B,=_B,\neq_B\leq_B,\beta).$$

We denote by $\Theta$ and $\Omega$ sets defined by

$$\Theta = \{(a,b)\in X\times Y : \varphi(a) \leq \psi(b)\} \text{ and } \Omega = \{(a,b)\in X\times Y : \varphi(a)\Xi\psi(b)\}.$$

The next theorem gives our first result on extension of ordered sets: The main Theorem of the ideal extensions of ordered sets given in [23] is the following:

**Proposition 1:** ([23], Theorem 4.1) *Let* $(V,=_V,\neq_V,\leq_V ,\Xi)$ *be an extension of* $(X,=_X,\neq_X,\leq_X,\alpha_X)$ *by* $(Y,=_Y,\neq_Y,\leq_Y,\alpha_Y)$. *Then the set* $X \cup Y$, *endowed with the relations* "=", "≠", "≤" and "Σ" *defined by*

$$\neq = \neq_X \cup \neq_Y \cup (X\times Y)\cup(Y\times X),\quad \leq = \leq_X \cup \leq_Y \cup \Theta,\quad \Sigma = \alpha_X \cup \alpha_Y \cup \Omega,$$

*is an ordered set and there exists strongly extensional, embedding, injective, order isotone and reverse isotone, anti-order isotone and reverse isotone function*

$$f : (X \cup Y, =, \neq, \leq, \Sigma) \to (V,=_V, \neq_V, \leq_V, \Xi).$$

We give the main theorem of extensions: if $(X,=_X,\neq_X,\leq_X,\alpha_X)$ and $(Y,=_Y,\neq_Y,\leq_Y,\alpha_Y)$ are two aparted ordered sets, $\theta$ an arbitrary subset of $X \times Y$,

$$\Theta(\theta) = \{(a,b)\in X \times Y \mid (\exists(x,y)\in \theta \subseteq X\times Y)(a \leq_X x \wedge y \leq_Y b\},$$

and

$$\Omega(\theta) = c((\Theta(\theta)^C) \cap ((X \times Y)\cup(Y\times X)),$$

then the set $V = X \cup Y$ endowed with the order "≤" defined by $\leq = \leq_X \cup \leq_Y \cup \Theta$ and with the antiorder "Σ" defined by $\Sigma = (\alpha_X\cup\alpha_Y) \cup \Omega(\theta)$ is an ordered set and it is an extension of X by Y.

**Proposition 2:** ([23], Theorem 4.2) *Let* $(X,=_X,\neq_X,\leq_X,\alpha_X)$ *and* $(Y,=_Y,\neq_Y,\leq_Y,\alpha_Y)$ *be ordered sets such that* $X \rhd\lhd Y$. *Let* $\theta \subseteq X \times Y$ *and* $V = X \cup Y$. *Define relations* "=", "≠", "≤" *and* "Σ" *on* V *by*

$$\neq\ = \neq_X \cup \neq_Y \cup (X\times Y)\cup(Y\times X), \quad \leq\ = \leq_X \cup \leq_Y \cup \Theta, \quad \Sigma = \alpha_X \cup \alpha_Y \cup \Omega,$$

*Then Let* $(V, =_V, \neq_V, \leq_V, \Sigma)$ *is an ordered set and it is an extension of* X *by* Y.

**2. THE RESULT:**

**Definition:** Let X and Y be ordered sets under order and anti-order relations, and let θ be an arbitrary subset of X×Y. An extension V of X by Y is called *retract extension* if there is an strongly extensional order-isotone and anti-order reverse isotone function $\phi : Y \to X$ such that $(a,b)\in\theta$ implies $a \leq_X \phi(b)$ and $(a,b)\in\Omega$ implies $a\ \alpha_X\ \phi(b)$.

**Theorem: 1** *An extension* V *of* X *by* Y *is retract if and only if there is a strongly extensional order isotone and anti-order reverse isotone mappings* $g : A\cup B \to X$ *such that* $(\forall a\in A)(g(a) = \varphi^{-1}(a))$.

**Proof:**

**(1)** Let $(X,=_X,\neq_X,\leq_X,\alpha_X)$ and $(Y,=_Y,\neq_Y,\leq_Y,\alpha_Y)$ be ordered sets under order and anti-order relations, and let θ be an arbitrary subset of X×Y, and let V be a retract extension of X by Y. Then there is an strongly extensional order-isotone function $\phi : Y \to X$ such that $(a,b)\in\theta$ implies $a \leq_X \phi(b)$ and $(a,b)\in\Omega$ implies $a\ \alpha_X\ \phi(b)$.

We consider the mapping $g : A\cup B \to X$ defined by $g(a) = \varphi^{-1}(a)$ if $a\in A$ and $g(b) = \phi(\psi^{-1}(b))$ if $b\in B$.

1.1 The mapping g is a function from A∪B into X. In fact:

$$\begin{aligned} a\in A \wedge a'\in A \wedge a =_A a' &\Rightarrow \varphi^{-1}(a) =_X \varphi^{-1}(a') \wedge a\in A \wedge a'\in A \\ &\Rightarrow g(a) =_X g(a') \wedge a\in A \wedge a'\in A; \\ a\in B \wedge a'\in B \wedge a =_B a' &\Rightarrow \psi^{-1}(a) =_Y \psi^{-1}(a') \wedge a\in B \wedge a'\in B \\ &\Rightarrow \phi(\psi^{-1}(a)) =_Y \phi(\psi^{-1}(a')) \wedge a\in B \wedge a'\in B \\ &\Rightarrow g(a) =_Y g(a') \wedge a\in B \wedge a'\in B; \end{aligned}$$

The case $a\in A \wedge a'\in B \wedge a = a'$ is impossible.

1.2 The mapping g is a strongly extensional function. Indeed:

$$\begin{aligned} a\in A \wedge a'\in A \wedge g(a) \neq_X g(a') &\Rightarrow a\in A \wedge a'\in A \wedge \varphi^{-1}(a) \neq_X \varphi^{-1}(a') \\ &\Rightarrow a\in A \wedge a'\in A \wedge a \neq_A a' \\ &\Rightarrow a \neq a'; \\ a\in B \wedge a'\in B \wedge g(a) \neq_X g(a') &\Rightarrow a\in B \wedge a'\in B \wedge \phi(\psi^{-1}(a)) \neq_X \phi(\psi^{-1}(a')) \\ &\Rightarrow a\in B \wedge a'\in B \wedge \psi^{-1}(a) \neq_Y \psi^{-1}(a') \\ &\Rightarrow a\in B \wedge a'\in B \wedge a \neq_B a' \\ &\Rightarrow a \neq a'; \end{aligned}$$

$$a\in A \wedge a'\in B \wedge g(a) \neq_X g(a') \Rightarrow a \neq b \text{ (because } A \subseteq B^C)$$

1.3 The mapping g is order-isotone:

Let $a\in V$, $b\in V$ such that $a \leq_V b$. Then $g(a) \leq_X g(b)$. In fact:

(a) Let $b\in A$. Since $V\ni a \leq_V b\in A$ and A is an ideal of V, we have $a\in A$. Since $a,b\in A$, we have $g(a) = \varphi^{-1}(a)$ and $g(b) = \varphi^{-1}(b)$ and $g(a) = \varphi^{-1}(a) \leq_X \varphi^{-1}(b) = g(b)$ because $\varphi^{-1}$ is order isotone.

(b) Let $b\in B$. Then $g(b) = \phi(\psi^{-1}(b))$. In the matter of fact: If $a\in A$, then $g(a) = \varphi^{-1}(a)$.
On the other hand, we have $(\varphi^{-1}(a), \psi^{-1}(b))\in X\times Y$. Moreover, since $a \leq_V b$, we have that $\varphi(\varphi^{-1}(a)) \leq_V \psi(\psi^{-1}(b))$ because $a = \varphi(\varphi^{-1}(a))$ and $b = \psi(\psi^{-1}(b))$.

Thus $(\varphi^{-1}(a), \psi^{-1}(b))\in\Theta$. By hypothesis, we have $\varphi^{-1}(a) \leq_X \phi(\psi^{-1}(b))$. Then, we have $g(a) \leq_X g(b)$.

Let $a\in B$. Then $g(a) = \phi(\psi^{-1}(a))$. On the other hand, the implication $a \leq_B b \Rightarrow \psi^{-1}(a) \leq_Y \psi^{-1}(b)$ holds because $\psi^{-1}$ is order isotone function. Since $\phi$ is order isotone, we have $g(a) = \phi(\psi^{-1}(a)) \leq_X \phi(\psi^{-1}(b)) = g(b)$.

1.4 The mapping g is anti-order reverse isotone: If $g(a)\ \alpha_X\ g(b)$, then:

$$a\in A \wedge b\in A \wedge g(a)\ \alpha_X\ g(b) \Rightarrow a\in A \wedge b\in A \wedge \varphi^{-1}(a)\ \alpha_X\ \varphi^{-1}(b)$$
$$\Rightarrow a\in A \wedge b\in A \wedge a\alpha b\ \ (\varphi^{-1} \text{ is reverse isotone})$$
$$\Rightarrow (a,b)\in \alpha = \Xi \cap (A\times A) \subseteq \Xi;$$

$$a\in A \wedge b\in B \wedge g(a)\ \alpha_X\ g(b) \Rightarrow a\in B \wedge b\in B \wedge \phi(\psi^{-1}(a))\ \alpha_X\ \phi(\psi^{-1}(b))$$
$$\Rightarrow a\in B \wedge b\in B \wedge \psi^{-1}(a)\ \alpha_Y\ \psi^{-1}(b)$$
$$\Rightarrow a\in B \wedge b\in B \wedge a\ \beta\ b$$
$$\Rightarrow (a,b)\in \beta = \Xi \cap (B\times B) \subseteq \Xi;$$

$$a\in A \wedge b\in B \wedge g(a)\ \alpha_X\ g(b) \Rightarrow a\in A \wedge b\in B \wedge \varphi^{-1}(a)\ \alpha_X\ \phi(\psi^{-1}(b))$$
$$\Rightarrow a\in A \wedge b\in B \wedge (\varphi^{-1}(a), \psi^{-1}(b))\in \Omega \subseteq \Sigma$$
$$\Rightarrow a\in A \wedge b\in B \wedge (a,b)\in \Xi;$$

The case $a\in B \wedge b\in A \wedge g(a)\ \alpha_X\ g(b)$ is impossible.

**(2)** Let $g : A\cup B \to X$ be an strongly extensional order-isotone and anti-order reverse isotone mapping such that $(\forall a\in A)(g(a) = \varphi^{-1}(a))$. We consider the mapping

$\phi : Y \to X$ by $\phi(y) = g(\psi(y))$. It is easily to see that $\phi$ is a strongly extensional order isotone and anti-order reverse isotone mapping from Y into X.

If $(a,b)\in\Theta$, then $a \leq_X \phi(b)$. Indeed:

$$(x,y)\in\Theta\ = \{(u,v)\in X\times Y\!: \varphi(u) \leq_V \psi(v)\} \Rightarrow\ \varphi(x)\in A \wedge \psi(y)\in B \wedge \varphi(x) \leq_V \psi(y)$$
$$\Rightarrow g(\varphi(x)) = \varphi^{-1}(\varphi(x)) = x\in X \wedge y\in Y \wedge g(\varphi(x)) \leq_X g(\psi(y))$$
$$\Rightarrow x\in X \wedge y\in Y \wedge x \leq_X \phi(y);$$

Let $x\in X \wedge y\in Y \wedge x\ \alpha_X\ \phi(y)$. Then $\varphi(x)\in A$, $x = g(\varphi(x)) = \varphi^{-1}(\varphi(x))\in X$, $\phi(y) = g(\psi(y))$ and $g(\varphi(x))\ \alpha_X\ g(\psi(y))$. Since g is anti-order reverse isotone, then $\varphi(x)\Xi\psi(y)$. At the other hand, from $x\in X$, $y\in Y$ and $(\varphi(x),\psi(y))\in\Xi$, we conclude that must be $(x,y)\in\Omega$ by definition of $\Omega$.

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