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MICRO STIFF FLUID COSMOLOGICAL CONSTANT MODEL IN GENERAL RELATIVITY

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ABSTRACT

In the present paper, Bianchi type-1 micro cosmological model for perfect fluid distribution in the presence of cosmological constant is investigated in Einstein's general relativity. For solving the field equations, the role of cosmological constant Λ is studied for three different cases i.e $\Lambda = 0$, $\Lambda > 0$ and $\Lambda < 0$. In first case i.e. when $\Lambda = 0$, the solutions of the field equations generate a anisotropic micro stiff fluid model of the universe. In second case, i.e when $\Lambda > 0$, it is observed that the real physical model of the universe does not survive. However in third case i.e when $\Lambda < 0$, the perfect fluid characterized by the equation of state $p = \rho$ degenerates homogeneous, inflationary and isotropic universe.

Keywords: *Einstein's general relativity, cosmological constant, scalar field, perfect fluid.*

1. INTRODUCTION

General theory of relativity developed by Einstein (1916) is the only coordinate invariant theory which laid foundation for constructing mathematical models of the universe. In the year 1917, Einstein introduced the cosmological constant Λ to modify his own developed field equations of general relativity. Now this Λ term remains a focal point of interest in the context of quantum field theories, quantum gravity, super gravity theories, Kaluza-Klein theories and also in the inflationary universe scenario. A number of observations suggest that the universe possess a non-zero cosmological constant (Krauss and Turner, 1995). The cosmological term which is a measure of the energy of empty space, provides a repulsive force opposing the gravitational pull between the galaxies. If the cosmological term exists, then the energy it represents counts as mass because Einstein has shown that mass and energy are equivalent. Further, if Λ posses a large value then the energy involved due to the matter in the universe can sum up to the number that inflation predicts. In this inflationary era it is quite important to study about the cosmological constant. But recent research suggests that the cosmological term corresponds to a very small value of the order 10^{-58} cm^{-2} (Jhori and Chandra, 1983).

The scalar meson field which represent matter field with spin less quanta are two types. The first type scalar field is zero rest mass scalar field and the second type scalar field is massive scalar field. The zero rest mass scalar field describes long range interactions, whereas massive scalar field describes short range interactions. The study of scalar meson field in general relativity has drawn the attention of the researchers due to its physical importance in particle physics. The massless scalar field in relativistic mechanics yields some significant results as regards to the singularities. The scalar meson field being a field of a single variable 'v' (say). It is is the special case of general field and the expression given by

$$T_{ij} = v_i v_j - \tfrac{1}{2} g_{ij}\left(v_k v^k - m^2 v^2\right)$$

is the energy-momentum tensor of Yukawa (1935) fields (spin zero meson particle) for the metric of (+2) signature in flat space time .

The Klein-Gorden equation takes the form

$$g^{ij} v_{;\,ij} + m^2 v = 0$$

where 'v' is the real scalar field and m is the rest-mass of scalar meson field. Here (;) semicolon followed by an index denotes covariant differentiation. When m = 0, the scalar field 'v' is known as massless scalar field or micro matter field.

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To best of our knowledge no author has studied Bianchi type-1 space time in the context of general theory of relativity when the gravitational field in presence of cosmological constant is a mixture of massless scalar field and perfect fluid. So in the present paper, we are interested to study the role of Λ (cosmological constant) in deriving mesonic perfect fluid solutions for the spatially homogeneous and anisotropic Bianchi type-1 space time in general theory of relativity. The advantage of introducing Λ here is to reveal the cosmological characters of the nature. In section 2 of this article, we have derived the field equations and solved the field equations corresponding to three different cases in section 3. In addition to it some physical and geometrical properties of the solutions are studied. In section 4, the conclusion part of the article is given.

2. FIELD EQUATIONS

We have considered Bianchi type-I metric in the form

$$ds^2 = -dt^2 + A^2\, dx^2 + B^2\, dy^2 + C^2\, dz^2 \tag{1}$$

where A, B and C are functions of cosmic time 't' which ensures that the space-time is spatially homogeneous.

The Einstein field equations for gravitating mesonic perfect fluid with cosmological term Λg_{ij} may be written as

$$G_{ij} \equiv R_{ij} - \frac{1}{2} g_{ij} R + \Lambda g_{ij} = -8\pi T_{ij} \tag{2}$$

where the units are chosen such that G= 1 = C and R_{ij} is the Ricci tensor, R is the Ricci scalar, g_{ij} is the metric tensor and $T_{ij} = \left(T_{ij}^{p} + T_{ij}^{v}\right)$ is the Energy momentum tensor of the matter.

The energy momentum tensor T_{ij}^{p} for perfect fluid distribution is given by

$$T_{ij}^{p} = (\rho + p) u_i u_j + p g_{ij} \tag{3}$$

Together with

$$g_{ij}\, u^i u^j = -1 \tag{4}$$

Here ρ, p and u_i are respectively the mass energy density, isotropic pressure and four-velocity vector of the perfect fluid.

The energy momentum tensor T_{ij}^{v} for a micro matter field representing massless scalar field distribution is taken as

$$T_{ij}^{v} = v_i v_j - ½\, g_{ij}\, v_k v^k \tag{5}$$

together with

$$g^{ij}\, v_{;\, ij} = 0\ . \tag{6}$$

Here the scalar field v is a function of cosmic time.

Using co-moving coordinate system, the field equation (2) and the Klein-Gorden equation (6) for the metric (1) are obtained as

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4 C_4}{BC} + \Lambda = -8\pi\left(p + \frac{v_4^2}{2}\right) \tag{7}$$

$$\frac{C_{44}}{C} + \frac{A_{44}}{A} + \frac{C_4 A_4}{CA} + \Lambda = -8\pi\left(p + \frac{v_4^2}{2}\right), \tag{8}$$

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4 B_4}{AB} + \Lambda = -8\pi\left(p + \frac{v_4^2}{2}\right), \tag{9}$$

$$\frac{A_4 B_4}{AB} + \frac{B_4 C_4}{BC} + \frac{C_4 A_4}{CA} + \Lambda = 8\pi\left(\rho + \frac{v_4^2}{2}\right) \tag{10}$$

and $$v_{44} + [\log(ABC)]_4\, v_4 = 0 \tag{11}$$

where the subscript '4' denotes the ordinary differentiation with respect to time.

3. COSMOLOGICAL MODEL

Field equations (7) to (11) is an underdetermined system having six equations in seven unknowns viz., A, B, C, p, ρ, Λ and v. So, to make the system consistent, two additional conditions are to be considered.

Consider the bartropic equation of state

$$p = \gamma\rho, \quad 0 \le \gamma \le 1 \tag{12}$$

as first additional condition.

3.1 Mesonic Stiff Fluid Model

Taking $\gamma = 1$, equation (12) reduces to

$$p = \rho \tag{13}$$

which represents stiff fluid (Zel'dovich, 1962). In this case adding equations (7), (8) and (9) with three times of equation (10), we obtained

$$\frac{(ABC)_{44}}{ABC} + 3\Lambda = 12\pi\left(\rho - p\right). \tag{14}$$

On integration, eqn. (11) yields

$$v_4 = \frac{\alpha}{ABC}, \tag{15}$$

where $\alpha \neq 0$ is the constant of integration.

Now use of eqn. (13) in eqn. (14), we get

$$(ABC)_{44} + 3\Lambda ABC = 0. \tag{16}$$

In order to get exact and explicit solutions, following three cases i.e.,

(i) $\Lambda = 0$ (ii) $\Lambda > 0$ (iii) $\Lambda < 0$ considered as second additional condition as discussed earlier.

Case-i: For $\Lambda = 0$, eqn.(16) after integration yields

$$ABC = (\alpha_1 t + \alpha_2) \tag{17}$$

where $\alpha_1\ (\neq 0)$ and α_2 are constants of integration.

Use of (17), eqn. (15) after integration reduces

$$v = \frac{\alpha}{\alpha_1} \ln(\alpha_1 t + \alpha_2) + \beta \tag{18}$$

where β is a constant of integration.

Equation (17) can be written as

$$A = (\alpha_1 t + \alpha_2)^{n_1}, \quad B = (\alpha_1 t + \alpha_2)^{n_2}, \quad C = (\alpha_1 t + \alpha_2)^{n_3} \tag{19a, b, c}$$

where n_i, i = 1,2,3 are real constants such that $\sum_{i=1}^{3} n_i = 1$.

Using equations (19a, b, c) in equation (10), we obtained

$$\frac{\alpha_1^2 \sum_{\substack{i,j=1 \\ i \neq j}}^{3} n_i\, n_j}{(a_1 t + a_2)^2} = 8\pi\left(\rho + \frac{v_4^2}{2}\right). \tag{20}$$

Equation (20) can be written as

$$\left(\rho + \frac{v_4^2}{2}\right) = \frac{q^2}{(a_1 t + a_2)^2} \tag{21}$$

where $$q^2 = \frac{\alpha_1^2 \sum\limits_{\substack{i,j=1 \\ i \neq j}}^{3} n_i n_j}{8\pi}.$$

With the help of equations (13), (15) and (19a, b, c), equation (21) yields

$$\rho = p = \frac{2q^2 - \alpha^2}{2\,(a_1 t + a_2)^2}. \tag{22}$$

Thus the anisotropic homogeneous cosmological micro-stiff fluid model is given by

$$ds^2 = -dt^2 + (a_1 t + a_2)^{2n_1} dx^2 + (a_1 t + a_2)^{2n_2} dy^2 + (a_1 t + a_2)^{2n_3} dz^2. \tag{23}$$

This model exhibits singularities at infinite past and future as well.

Some properties of the model (23):

The physical parameters involved in the model behaves as follows:

(a) as $t \to 0$, the meson field $v \to$ a constant and the energy density cum pressure $\rho(=p) \to$ a constant, subject to the condition $\alpha_1^2 \sum\limits_{\substack{i,j=1 \\ i \neq j}}^{3} n_i n_j > 4\pi\alpha^2$. In this case the space-time reduces to a flat space time in Einstein's theory.

(b) as $t \to \infty$, $v \to \infty$ and $\rho(=p) \to 0$, So in this case the micro-stiff fluid model of the universe reduces to vacuum model of the universe. The same case also arises when $q = \frac{\alpha}{2}$ or $\alpha_1^2 \sum\limits_{\substack{i,j=1 \\ i \neq j}}^{3} n_i n_j = 4\pi\alpha^2$.

(c) when $\alpha = 0$ then the meson field v becomes constant and $\rho = p = \dfrac{q^2}{(a_1 t + a_2)^2}$.

(d) the scalar expansion for the model $\theta = \dfrac{\alpha_1}{(a_1 t + a_2)}$ shows that the universe is expanding with increase of time but the rate of expansion is slow with increase of time.

(e) the shear scalar $\boldsymbol{\sigma}^2$ for this model is $\boldsymbol{\sigma}^2 = \dfrac{2}{3}\left(\dfrac{\alpha_1}{(a_1 t + a_2)}\right)^2 \left(1 - 3\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{3} n_i n_j\right)$ As $t \to \infty$, $\boldsymbol{\sigma}^2 \to 0$ and as $t \to 0$, $\boldsymbol{\sigma}^2 \to$ a constant. Thus it shows that the shape of the universe changes uniformly in x, y and z directions depending on the parameters n_i, i = 1, 2, 3. However the rate of change of the shape of the universe becomes slow with increase of time. Further we obtained

$$\lim_{T \to \infty} \left(\frac{\boldsymbol{\sigma}}{\theta}\right) = \lim_{T \to \infty} \sqrt{\frac{2}{3}\left(1 - 3 \sum_{\substack{i,j=1 \\ i \neq j}}^{3} n_i n_j\right)} = \frac{\sqrt{2}}{\sqrt{3}} \sqrt{1 - 3 \sum_{\substack{i,j=1 \\ i \neq j}}^{3} n_i n_j}$$

which indicates that the universe remains anisotropic throughout the evolution. This can also be seen by considering the present day observational limits in the temperature anisotropy.

(f) the velocity field from geodesic motion is given by acceleration $\dot{u}_i$. Here we found that there is no acceleration i.e. $\dot{u}_i = \overline{0}$. Hence the mesonic fluid flow is geodesic in nature. Also the vorticity tensor w_{ij} becomes zero and hence the rotation 'ω' turns out to be zero. Hence the model is non-rotating in nature.

(g) the spatial volume V = $(\alpha_1 t + \alpha_2)$ clearly shows the anisotropic expansion of the universe with time and the universe starts expanding with a constant volume and blows up at infinite future.

Case-ii: For $\Lambda > 0$, equation (16) after integration yields

$$ABC = \alpha_3 \cos\sqrt{3\Lambda}t + \alpha_4 \sin\sqrt{3\Lambda}t \tag{24}$$

where α_3 and α_4 are constants of integration. From eqn. (24), we can write the explicit form for A, B and C as

$$\left.\begin{aligned} A &= \left(\alpha_3 \cos\sqrt{3\Lambda}t + \alpha_4 \sin\sqrt{3\Lambda}t\right)^{n_1}, \\ B &= \left(\alpha_3 \cos\sqrt{3\Lambda}t + \alpha_4 \sin\sqrt{3\Lambda}t\right)^{n_2} \\ &and \\ C &= \left(\alpha_3 \cos\sqrt{3\Lambda}t + \alpha_4 \sin\sqrt{3\Lambda}t\right)^{n_3} \end{aligned}\right\} \tag{25}$$

where n_i, i = 1, 2, 3 are real constants and satisfies the relation

$$\sum_{i=1}^{3} n_i = 1. \tag{26}$$

Here the over determinacy for determining three unknowns A, B and C from four field eqns. (7)-(10) can be settled by actual substitution of the values of A, B and C from eqn. (25) in eqn.(10).

Thus we obtain

$$\sum_{\substack{i,j=1 \\ i\neq j}}^{3} n_i n_j = \frac{\left[8\pi(\rho + \frac{v_4^2}{2}) - \Lambda\right]}{3\Lambda} \cdot \left[\frac{\alpha_3 \cos\sqrt{3\Lambda}t + \alpha_4 \sin\sqrt{3\Lambda}t}{\alpha_4 \cos\sqrt{3\Lambda}t - \alpha_3 \sin\sqrt{3\Lambda}t}\right]^2. \tag{27}$$

As n_i, $i = 1, 2, 3$ are real constants, so also $\sum_{\substack{i,j=1 \\ i\neq j}}^{3} n_i n_j$ is a real constant. But this relation cannot be hold good in eqn. (27) as its L.H.S part is constant but R.H.S part is a function of 't'.

Thus for $\Lambda > 0$ and $p = \rho$, it is not possible to determine real physical model of the universe.

Case-iii: Suppose $\Lambda < 0$ i.e., $\Lambda = \frac{-\omega^2}{3}$. In this case eqn. (16) reduces to

$$(ABC)_{44} - \omega^2 ABC = 0. \tag{28}$$

On integration, eqn. (28) yields

$$ABC = \beta_1 e^{\omega t} + \beta_2 e^{-\omega t} \tag{29}$$

where β_1 and β_2 are arbitrary constants of integration. From eqn. (29), we can write the explicit form for A, B and C as

$$\left.\begin{aligned} A &= \left(\beta_1 e^{\omega t} + \beta_2 e^{-\omega t}\right)^{r_1}, \\ B &= \left(\beta_1 e^{\omega t} + \beta_2 e^{-\omega t}\right)^{r_2} \\ &and \\ C &= \left(\beta_1 e^{\omega t} + \beta_2 e^{-\omega t}\right)^{r_3} \end{aligned}\right\} \tag{30}$$

where r_i, i = 1, 2, 3 are real constants satisfying the condition

$$\sum_{i=1}^{3} r_i = 1. \tag{31}$$

Here also the over determinacy for finding three unknowns A,B and C from four eqns. (7) to (10) can be settled by putting the values of A,B and C from eqn. (30) in eqn.(10).Thus we get

$$\sum_{\substack{i,j=1 \\ i\neq j}}^{3} r_i r_j = \frac{1}{3} \cdot \left[\frac{\beta_1 e^{\omega t} + \beta_2 e^{-\omega t}}{\beta_1 e^{\omega t} - \beta_2 e^{-\omega t}}\right]^2 \cdot \left[\frac{8\pi\left(\rho + \frac{v_4^2}{2}\right) - \Lambda}{-\Lambda}\right]. \tag{32}$$

Since r_i, i = 1, 2, 3 are real constants, so $\sum_{\substack{i,j=1 \\ i\neq j}}^{3} r_i r_j$ is also a real constant. But this relation holds good in eqn. (32) only when

$$\beta_2 = 0 \ and \ \frac{8\pi\left(\rho + \frac{v_4^2}{2}\right) - \Lambda}{-\Lambda} = 1. \tag{33}$$

Thus from eqn. (32) and (33), we obtain

$$\sum_{\substack{i,j=1 \\ i\neq j}}^{3} r_i r_j = \frac{1}{3} \tag{34}$$

and

$$8\pi\left(\rho + \frac{v_4^2}{2}\right) = 0. \tag{35}$$

From eqn. (35), we get

$$\rho = -\frac{v_4^2}{2}. \tag{36}$$

Thus from (36) and (13), we have

$$\rho = p = -\frac{v_4^2}{2} \quad . \tag{37}$$

Now putting the value of $\beta_2 = 0$ from eqn. (33) in eqn. (29), we obtain

$$ABC = \beta_1 e^{\omega t} \quad . \tag{38}$$

Solving (34) and (31), one can get

$$r_1 = r_2 = r_3 = 1/3. \tag{39}$$

The explicit expressions of A, B and C in (38) can be expressed as

$$A = B = C = (\beta_1)^{1/3}.e^{\frac{\omega t}{3}} \quad . \tag{40}$$

Using eqn. (40) in eqn. (15), we obtain

$$v = \frac{-\alpha}{\beta_1 \omega}\cdot\frac{1}{e^{\omega t}} + \beta_3 \tag{41}$$

where β_3 is the constant of integration.

With the help of eqn. (41), eqn. (37) yields

$$\rho = p = -\frac{\alpha^2}{2\beta_1^2}\cdot\frac{1}{e^{2\omega t}}\ . \tag{42}$$

Thus corresponding to our solution (40) the Bianchi type-1 metric (1) can be expressed as

$$ds^2 = -dt^2 + \left(\beta_1 e^{\omega t}\right)^{2/3}\left(dx^2 + dy^2 + dz^2\right). \tag{43}$$

The model obtained as above is the spatially-homogeneous isotropic Bianchi type-1stiff fluid micro model. This model does not lead to well known Kasner, (1921) model for $\Lambda < 0$.

Some physical and geometrical features of the model (43):

i. The spatial volume in the model is given by $V = (-g)^{1/2} = ABC = \beta_1 e^{\omega t}$.

Thus $V \to \beta_1$ as $t \to 0$ and $V \to \infty$ as $t \to \infty$. Thus the universe starts expanding from a constant volume and becomes infinite large volume as $t \to \infty$ .Hence we concluded that the universe may blows up at infinite future.

ii. The scalar expansion θ and the anisotropy $|\sigma|$ are defined by (Roychoudhury, 1955) as $\theta = u^i_{;i} = \frac{V_4}{V}$

where V the volume element and

$$\sigma^2 = \frac{1}{12}\left[\left(\frac{g_{11,4}}{g_{11}} - \frac{g_{22,4}}{g_{22}}\right)^2 + \left(\frac{g_{22,4}}{g_{22}} - \frac{g_{33,4}}{g_{33}}\right)^2 + \left(\frac{g_{33,4}}{g_{33}} - \frac{g_{11,4}}{g_{11}}\right)^2\right].$$

Now the scalar expansion θ and anisotropy $|\sigma|$ in the model we found

$\theta = \omega > 0$ and $\sigma^2 = 0$. Thus $\lim_{t\to\infty} \frac{\sigma}{\theta} = 0$.

The above result shows that the universe is expanding in nature with constant expansion. Also the universe is non-shearing and isotropic throughout the evolution. Hence the spatially homogeneous anisotropic cosmological model in Barber's second theory reduces to homogeneous isotropic model.

i. The Kretschmann curvature invariant defined by $L = R_{hijk} R^{hijk}$\, where R_{hijk} is the Riemann curvature tensor. Here L is found to be

$$L = \frac{2}{3}\left(\Lambda^2\right) > 0$$
.

As L is a + ve constant, the result confirms that the model has no geometrical singularity.

ii. The massless scalar field v in this model is found to be

$$v = \frac{-\alpha}{\beta_1 \omega} \cdot \frac{1}{e^{\omega t}} + \beta_3.$$

Thus $v \to$ a constant as $t \to 0$ and $v \to \beta_3$ as $t \to \infty$. If $\alpha = 0$, then $v = \beta_3$.

In this case the massless scalar field v does not exist.

iii. The energy density and proper pressure in the model are given by

$$\rho = p = -\frac{\alpha^2}{2\beta_1^2} \cdot \frac{1}{e^{2\omega t}}.$$

Here ρ (= p)$\to$ a –ve constant as $t \to 0$ and ρ (= p) $\to 0$ as $t \to \infty$.

Thus it is evident from the result that at initial time the result leads to unphysical situation but at infinite future the model shows singularity. If $\alpha = 0$, then ρ (= p) will be constant. In this case the space time reduces to flat space time.

iv. The Hubble's parameter H in the model is found as $H = \frac{\omega}{3}$. Thus H is not a function of time and hence we concluded that the model is of steady-state.

v. The scale factor S^3 in the model can be determined as $S^3 = ABC = \beta_1 e^{\omega t}$.

So, S increases as time increase.

vi. The deceleration parameter 'q' in the model is found to be $q = -\frac{VV_{44}}{V_4^2} = -1$. As q = -1, so the model of the universe corresponds to an inflationary model.

4. CONCLUSION

In this paper, we studied the role of cosmological constant Λ for deriving mesonic perfect fluid solutions in view of the spatially homogeneous anisotropic Bianchi type-1 space time in general relativity corresponding to three distinct cases. In first case i.e. when $\Lambda = 0$, it is seen that the anisotropic homogeneous cosmological micro-stiff fluid model of the universe exist. The model which exist is uniformly expanding, non-rotating and geodesic. In second case, i.e when $\Lambda > 0$, it is shown that the real physical model of the universe does not exist. However, in third case i.e when $\Lambda < 0$, the perfect fluid characterized by the equation of state $p = \rho$ degenerates spatially homogeneous isotropic model of the universe in Einstein theory. It is also observed that the model found is inflationary, uniformly expanding, non-shearing and has no geometrical singularity.

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