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ASYMPTOTIC GENERALIZATION OF A FIXED POINT THEOREM IN PARTIAL METRIC SPACES

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ABSTRACT

This communication ventilates an asymptotic generalization of a fixed point theorem for generalized contractions with constants, recently achieved by Kikkawa and Suzuki in "M. Kikkawa, T. Suzuki, Three fixed point theorems for generalized contractions with constants in complete metric spaces, Nonlinear Analysis, 69, (2008), 2942-2949." and extended by Popescu in "O. Popescu, Two fixed point theorems for generalized contractions with constants in complete metric space, Cent.Eur . J. Math.7 (3), (2009), 529-538." in the light of Partial metric spaces.

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1. INTRODUCTION

Classical metric assumes zero identically on the diagonal of its domain. There are some mappings which are not metrics only for assuming nonzero values at some points of the diagonals of their domains. One such wetness is m: $X \times X \rightarrow [0, \infty)$, where X = [0,∞), defined by m(x, y) = max{x, y} for all x, y ϵ X. S.G.Matthews [6] introduced the concept of partial metric space using such mapping to measure distance between two points of a non empty set and observed that how this replacement of the tools of measurement of distance propagate differences in the theory of metric spaces. Now a day it is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation and in the domain theory of computer science. In this paper we show that a fixed point theorem achieved by Ovidiu Popescu [8] can be rediscovered as a limiting case of a fixed point theorem but in the language of partial metric spaces. For better understanding of motivation and source of such fixed point theorem we refer [1, 2, 3, 4, 5, 9, 10]. Let's begin with some basic definitions. As in [6, 7], a mapping p: X × X → [0,∞), where X is a non empty set is said to be partial metric if whenever x, y, z ϵ X the following conditions hold:

(i) x = y ⇔ p(x, x) = p(x, y) = p(y, y);
(ii) p(x, y) = p(y, x);
(iii) P(x, y) ≥ p(x, x);
(iv) p(x, y)≤ p(x, z) + p(z, y) - p(z, z).

and the ordered pair (X, p) is called partial metric space. The aforementioned mapping 'm' ensures the existence of such partial metric. According to [7] a sequence $\{x_n\}$ in a partial metric space (X, p) is said converges to x ∈ X iff $\lim_{n\to\infty} p(x_n, x) = p(x, x) = \lim_{n\to\infty} p(x_n, x_n)$. A sequence $\{x_n\}$ in (X, p) is a Cauchy sequence iff $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and a partial metric space is said to be complete iff every Cauchy sequence in (X, p) is convergent.

2. A LEMMA FOLLOWED BY THE MAIN THEOREM

Lemma: 2.1: In a partial metric space (X, p) if $\{x_n\}$ converges to x and $\{y_n\}$ converges to y then

$$\lim_{n\to\infty} p(x_n, y_n) = p(x, y).$$

Proof: $p(x_n, y) \le p(x_n, x) + p(x, y) - p(x, x) \Longrightarrow \lim_{n\to\infty} [p(x_n, y) - p(x, y)] \le 0.$ (2.1)

Also $p(x, y) \le p(x, x_n) + p(x_n, y) - p(x_n, x_n) \Longrightarrow \lim_{n\to\infty} [p(x, y) - p(x_n, y)] \le 0.$ (2.2)

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From (2.1) and (2.2) we have $\lim_{n\to\infty} p(x_n, y) = p(x, y)$.

Similarly we can show that $\lim_{n\to\infty} p(x, y_n) = p(x, y)$ (2.3)

Now $p(x_n, y_n) \le p(x_n, x) + p(x, y_n) - p(x, x)$

Taking limit $n\to\infty$ and using (2.3) we have $\lim_{n\to\infty} p(x_n, y_n) \le p(x, y)$ (2.4)

and
$$\begin{aligned} p(x, y) &= \lim_{n\to\infty} p(x, y_n) \\ &\le \lim_{n\to\infty} [\, p(x, x_n) + p(x_n, y_n) - p(x_n, x_n)] \\ &= \lim_{n\to\infty} p(x_n, y_n) \end{aligned} \tag{2.5}$$

From (2.4) and (2.5) we have $\lim_{n\to\infty} p(x_n, y_n) = p(x, y)$.

3. MAIN RESULTS

Theorem: 3.1: Let (X, p) be a complete partial metric space. Define a non increasing function $\Theta: [0, \frac{k}{k+4}) \longrightarrow (\frac{1}{2}, 1]$ where $k\in\mathbb{N}$ and $k \ge 4$ by

$$\Theta(r) = \begin{cases} 1 & if\ 0 \le r \le \frac{\sqrt{5}k-k}{2(k+4)}, \\ \frac{k(k-kr-4r)}{r^2(k+4)^2} & if\ \frac{\sqrt{5}k-k}{2(k+4)} \le r \le \frac{k}{(4+k)\sqrt{2}}, \\ \frac{k}{k+kr+4r} & if\ \frac{k}{(4+k)\sqrt{2}} \le r < \frac{k}{k+4}. \end{cases}$$

Let S, T be self mappings on X satisfying the following:

(a) S is continuous;
(b) T(X) subset of S(X);
(c) S and T commute.

Suppose that there exists $r \in [0, \frac{k}{k+4})$ such that
$\Theta(r)\, p(Sx, Tx) \le p(Sx, Sy) \Rightarrow p(Tx, Ty) \le r\, G_{S,T}(x, y)$ for all $x, y \in X$,

where
$$G_{S,T}(x,y) = \max\{p(Sx, Sy), p(Sx, Tx), p(Sy, Ty), \frac{p(Sx,Ty) + p(Sy,Tx)}{2} + \frac{1}{k}p(Tx, Tx) + \frac{1}{k}p(Sy, Sy)\}$$

Then there exists a unique common fixed point of S and T.

Proof: By condition (b), we can define a mapping I on X satisfying SIx = Tx for all $x \in X$ and I commutes with S.

Since $\Theta(r) \le 1$, we have $\Theta(r)\, p(Sx, Tx) = \Theta(r)\, p(Sx, SIx) \le p(Sx, SIx)$. So by the hypothesis

$$\begin{aligned} p(SIx, SIIx) &= p(Tx, TIx) \\ &\le r\, G_{S,T}(x, Ix) \\ &= r \max\{p(Sx, SIx), p(Sx, Tx), p(SIx, TIx), \frac{p(Sx,TIx) + p(Tx,SIx)}{2} + \frac{1}{k}p(Tx, Tx) + \frac{1}{k}p(SIx, SIx)\} \\ &\le r \max\{p(Sx, SIx), p(Sx, SIx), p(SIx, SIIx), \frac{p(Sx,SIIx) + p(SIx,SIx)}{2} + \frac{2}{k}p(SIx, SIIx)\} \\ &\le r \max\{p(Sx, SIx), p(Sx, SIx), p(SIx, SIIx), \\ &\qquad \frac{p(Sx,SIx) + p(SIx,SIIx) - p(SIx,SIx) + p(SIx,SIx)}{2} + \frac{2}{k}p(SIx, SIIx)\} \end{aligned}$$

Therefore we have $p(SIx, SIIx) \le r\, p(Sx, SIx)$ (3.1)

Let us fix $u_0 \in X$. Put $u_{n+1} = Iu_n$ and $Su_{n+1} = Tu_n$.

Now $p(Su_n, Su_{n+1}) = p(SIu_{n-1}, SIIu_{n-1}) \le r\, p(Su_{n-1}, SIu_{n-1}) \le \ldots\ldots\ldots\ldots \le r^n\, p(Su_0, Su_1)$.

$\Rightarrow \lim_{n,m\to\infty} p(Su_n, Su_m) = 0$. Thus $\{Su_n\}$ is a Cauchy sequence in (X, p).

Since (X, p) is complete $\{Su_n\}$ as well as $\{Tu_n\}$ converges to a point z ϵ X such that $\lim_{n\to\infty} p(Su_n, z) = p(z, z) = \lim_{n\to\infty} p(Su_n, Su_n) = 0$. In a similar way we can show that $\lim_{n,m\to\infty} p(SSu_n, SSu_m) = 0$. So for $\varepsilon = \frac{p(Sx,\ z)}{3}$ there exists a natural number M such that $p(Su_n, z) \le \varepsilon \quad \forall n \ge M$.

We now show that $p(Tx, z) \le r \max\{p(z, Sx), p(Sx, Tx)\} \quad \forall x \in X - \{z\}$ (3.2)

$$\begin{aligned}\Theta(r)\, p(Su_n, Tu_n) &\le p(Su_n, Su_{n+1}) \le P(Su_n, z) + p(z, Su_{n+1}) - p(z, z)\\ &\le \frac{2p(Sx,z)}{3} \qquad \forall n \ge M\\ &= p(Sx, z) - \frac{p(Sx,z)}{3}\\ &\le p(Sx, z) - p(Su_n, z)\\ &\le p(Sx, Su_n) + p(Su_n, z) - p(Su_n, Su_n) - p(Su_n, z)\\ &\le p(Sx, Su_n)\end{aligned}$$

Then by our hypothesis $p(Tu_n, Tx) \le r G_{S,T}(x, u_n) \quad \forall n \ge M$. This implies (3.2).

Let us now prove that z is a fixed point of point of S.

In the case when $\{n: p(Su_n, Tu_n) > p(Su_n, SSu_n)\}$ is infinite there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $p(Su_{n_j}, Tu_{n_j}) > p(Su_{n_j}, SSu_{n_j})$

$$\begin{aligned}P(Sz, z) &= \lim_{j\to\infty} p(SSu_{n_j}, z)\\ &\le \lim_{j\to\infty} [\, p(SSu_{n_j}, Su_{n_j}) + p\left(Su_{n_j}, z\right) - p(Su_{n_j}, Su_{n_j})\,]\\ &\le \lim_{j\to\infty} \left[p\left(Su_{n_j}, Tu_{n_j}\right) + p\left(Su_{n_j}, z\right) - p\left(Su_{n_j}, Su_{n_j}\right)\right]\\ &= 0.\end{aligned}$$

In the other case when $\{n: p(Su_n, Tu_n) > p(Su_n, SSu_n)\}$ is finite, there exists $M_1 \in \mathbb{N}$ such that $p(Su_n, Tu_n) \le p(Su_n, SSu_n)\ \forall\, n \ge M_1$.

So by the assumption
$\Theta(r)\, p(Su_n, Tu_n) \le p(Su_n, Tu_n) \le p(Su_n, SSu_n) \quad \forall\, n \ge M_1$.

$\Rightarrow p(Tu_n, TSu_n) \le r\, G_{S,T}(u_n, Su_n) \quad \forall\, n \ge M_1$.

Since $\{STu_n\}$ converges to Sz, we obtain

$$\begin{aligned}p(Sz, z) &= \lim_{n\to\infty} p(STu_n, Tu_n) \qquad \text{[by \textbf{lemma (2.1)}]}\\ &= \lim_{n\to\infty} p(Tu_n, TSu_n)\\ &\le \lim_{n\to\infty} r \max\{p(Su_n, SSu_n), p(Su_n, Tu_n), p(SSu_n, TSu_n), \frac{p(Su_n, TSu_n) + p(SSu_n, Tu_n)}{2}\\ &\qquad + \frac{1}{k}\, p(Tu_n, Tu_n) + \frac{1}{k}\, p(SSu_n, SSu_n)\} \qquad (3.3)\end{aligned}$$

Hence using **lemma (2.1)** in (3.3) we have $p(Sz, z) \le r\, p(Sz, z) < p(Sz, z) \Longrightarrow p(Sz, z) = 0 \Longrightarrow Sz = z$.

Thus in both the cases z is a fixed point of S.

Now we will show that
$P(T^n z, T^{n+1} z) \le r\, p(T^{n-1}z, T^n z)$ for all n ϵℕ, where $T^0 z = z$.

Since $\Theta(r)\, p(ST^{n-1} z, T^n z) \le p(ST^{n-1}z, T^n Sz) = p(ST^{n-1}z, ST^n z)$

$$\begin{aligned}\Longrightarrow p(T^n z, T^{n+1} z) &\le r\, G_{S,T}(T^{n-1}z, T^n z)\\ &= r \max\{p(ST^{n-1}z, ST^n z), p(ST^{n-1}z, T^n z), p(ST^n z, T^{n+1}z), \frac{p(ST^{n-1}z, T^{n+1}z) + p(ST^n z, T^n z)}{2}\\ &\qquad + \frac{1}{k}\, p(T^n z, T^n z) + \frac{1}{k}\, p(T^n z, T^n z)\}\\ &\le r \max\{p(T^{n-1}z, T^n z), p(T^{n-1}z, T^n z), p(T^n z, T^{n+1}z), \frac{p(T^{n-1}z, T^{n+1}z) + p(T^n z, T^n z)}{2} + \frac{2}{k}\, p(T^n z, T^{n+1} z)\}\\ &\le r \max\{p(T^{n-1}z, T^n z), p(T^{n-1}z, T^n z), p(T^n z, T^{n+1}z),\\ &\qquad \frac{p(T^{n-1}z, T^n z) + p(T^n z, T^{n+1}z) - p(T^n z, T^n z) + p(T^n z, T^n z)}{2} + \frac{2}{k}\, p(T^n z, T^{n+1} z)\}\end{aligned}$$

Therefore p(T^nz, T^{n+1}z) ≤ r p(T^{n-1}z, T^nz).

Using induction we get
P (T^nz, T^{n+1}z) ≤ r^np(Tz, z) ∀ n ϵ ℕ (3.4)

We next show that p(T^nz, z) ≤ p(Tz, z) (3.5)

For n = 1 this holds.

Let us suppose p(T^nz, z) ≤ p(Tz, z) holds for some n ϵ ℕ with n≥2.

If T^nz = z then T^{n+1}z = Tz and p(T^{n+1}z, z) = p(Tz, z) ≤ p(Tz, z).

If T^nz ≠ z.

$$\begin{aligned} p(T^{n+1}z, z) &\le r \max\{p(z, ST^n z), p(ST^n z, T^{n+1}z)\} \quad \text{[by (3.2)]} \\ &= r \max\{p(z, T^n z), p(T^n z, T^{n+1}z)\} \\ &\le r \max\{p(Tz, z), r^n p(Tz, z)\} \\ &= r\, p(Tz, z) < p(Tz, z). \end{aligned}$$

Thus p(T^nz, z) ≤ p(Tz, z) for all n ϵ ℕ. Now we will show that z is a fixed point of T.

Case-1: In the first case r ϵ $[0, \frac{k}{(4+k)\sqrt{2}})$ and (r) ≤ $\frac{k(k-kr-4r)}{r^2(k+4)^2}$.

We first prove that p(T^nz, Tz) ≤ r p(Tz, z) ∀ n ≥ 2. (3.6)

For n = 2, (3.6) holds by (3.4).

Suppose that p(T^nz, Tz) ≤ r p(Tz, z) holds for some n > 2.

$$\begin{aligned} \text{Since } p(z, Tz) &\le p(z, T^n z) + p(T^n z, Tz) - p(T^n z, T^n z) \\ &\le p(z, T^n z) + r\, p(z,Tz) \end{aligned}$$

$\Longrightarrow$ (1-r) p(z, Tz) ≤ p(z, T^nz)

$$\begin{aligned} \text{Hence } \Theta(r)\, p(ST^n z, TT^n z) &= \Theta(r) p(T^n z, T^{n+1}z) \\ &\le \frac{k(k-kr-4r)}{r^2(k+4)^2} p(T^n z, T^{n+1}z) \\ &= \frac{1-\frac{(k+4)}{k}r}{\{\frac{(k+4)}{k}\}^2 r^2}\, p(T^n z, T^{n+1}z) \\ &\le \frac{1-\frac{(k+4)}{k}r}{\{\frac{(k+4)}{k}\}^2 r^n} r^n\, p(Tz, z) \qquad \forall\, n > 2 \quad \text{[by (3.4)]} \\ &\le (1-r)\, p(Tz, z) \\ &\le p(z, T^n z) = p(Sz, ST^n z). \end{aligned}$$

$$\begin{aligned} \text{Hence } p(T^{n+1}z, Tz) &\le r\, G_{S,T}(T^n z, z) \\ &= r \max\{p(ST^n z, Sz), p(ST^n z, T^{n+1} z), p(Sz, Tz), \frac{p(ST^n z,Tz)+ p(Sz, T^{n+1}z)}{2} \\ &\qquad + \frac{1}{k} p(T^{n+1}z, T^{n+1}z) + \frac{1}{k} p(Sz,Sz)\} \\ &\le r \max\{p(T^n z, z), p(T^n z, T^{n+1} z), p(z, Tz), \frac{p(T^n z,Tz)+ p(z,T^{n+1}z)}{2} + \frac{1}{k} p(T^{n+1}z, z)\} \\ &= r\, p(Tz, z). \end{aligned}$$

$\Longrightarrow$p(T^nz, Tz) ≤ r p(Tz, z).

Thus we have proved p(T^nz, Tz) ≤ r p(Tz, z) ∀ n ≥ 2.

Let if possible Tz ≠ z. Then by (3.6) T^nz ≠ z ∀ n ϵ ℕ.

$$\begin{aligned} \text{By (3.2) we have } p(T^{n+1}z, z) &\le r \max\{p(z, ST^n z), p(ST^n z, T^{n+1}z)\} \\ &= r \max\{p(z, T^n z), p(T^n z, T^{n+1}z)\} \\ &\le r \max\{p(T^n z, z), r^n p(Tz, z)\} \qquad \text{[by (3.4)]} \end{aligned}$$

Now p(z, Tz) ≤ p(z, T^nz) + p(T^nz, Tz) - p(T^nz, T^nz)

⟹ p(T^nz, z)≥ p(Tz, z) − p(T^nz,Tz) ≥ p(Tz, z) − r p(Tz, z) = (1− r) p(Tz, z).

So, there exists v∊ ℕ such that p(T^nz, z) ≥ r^np(Tz, z) ∀ n ≥ v.

Hence p(T^{n+1}z, z) ≤ r p(T^nz, z) ≤…..≤ r^{n-v+1} p($T^v z, z$) ∀ n ≥ v.

This implies $\lim_{n\to\infty}$ p(T^{n+1}z, z) = 0.

So from (3.4) we can say that {T^nz} converges to z. This contradicts (3.6). Thus we obtain Tz = z.

Case-2: In this case $\frac{k}{(4+k)\sqrt{2}}$ ≤ r < $\frac{k}{k+4}$, we will show that there exists a subsequence {n_j} of {n} such that
Θ(r) p(Su_{n_j}, $Su_{u_{n_j+1}}$) ≤ p(Su_{n_j}, z) ∀ j ∊ ℕ.

Now, p(Su_n, Su_{n+1}) = p(SIu_{n-1}, SIIu_{n-1}) ≤ r p(Su_{n-1}, SIu_{n-1}) = r p(Su_{n-1}, Su_n).

⇒ p(Su_n, Su_{n+1}) ≤ r p(Su_{n-1}, Su_n).

We assume $\frac{k}{(k+kr+4r)}$ p(Su_{n-1}, Su_n) > p(Su_{n-1}, z) and $\frac{k}{(k+kr+4r)}$ p(Su_n, Su_{n+1}) > p(Su_n, z).

Then we have

$$
\begin{aligned}
p(Su_{n-1}, Su_n) &\le p(Su_{n-1}, z) + p(Su_n, z) - p(z, z)\\
&< \frac{k}{(k+kr+4r)} p(Su_{n-1}, Su_n) + \frac{k}{(k+kr+4r)} p(Su_n, Su_{n+1})\\
&= \frac{k}{(k+kr+4r)} [p(Su_{n-1}, Su_n) + p(Su_n, Su_{n+1})]\\
&\le \frac{k}{(k+kr+4r)} (1+r)\, p(Su_{n-1}, Su_n) < p(Su_{n-1}, Su_n) \text{ which is a contradiction.}
\end{aligned}
$$

Therefore either

$$\frac{k}{(k+kr+4r)} p(Su_{n-1}, Su_n) \le p(Su_{n-1}, z)$$

Or $$\frac{k}{(k+kr+4r)} p(Su_n, Su_{n+1}) \le p(Su_n, z).$$

Thus either Θ(r) p(Su_{2n-1}, Su_{2n}) ≤ p(Su_{2n-1}, z) or Θ(r) p(Su_{2n}, Su_{2n+1}) ≤ p(Su_{2n}, z) ∀ n ∈ ℕ. Thus there exists a subsequence {n_j} of {n} such that Θ(r) p(Su_{n_j}, $Su_{u_{n_j+1}}$) ≤ p(Su_{n_j}, z) = p(Su_{n_j}, Sz) ∀ j∈ ℕ. Therefore we have

p(z, Tz) = $\lim_{n\to\infty}$ p$\left(Tu_{n_j}, \text{Tz}\right)$ ≤ $\lim_{n\to\infty}$ r $G_{S,T}$ (u_{n_j}, z) = r p(z, Tz) < p(Tz, z) which is a contradiction.

Thus Tz = z. Hence Tz = Sz = z. Let z_1 be another fixed point of S and T. Then by (3.2) we have
p(Tz_1, z) ≤ r max{p(z, Sz_1), p(Sz_1, Tz_1)} ⇒ $p(z_1, z)$< $p(z_1, z)$ a contradiction.

Hence $z_1 = z$.

Note: In the above theorem if we let k→ ∞, the **Theorem 2.1** in [8] is achieved in the language of partial metric spaces.

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