International Journal of Mathematical Archive-6(6), 2015, 162-167
Available online through www.ijma.info ISSN 2229 – 5046

FUZZY FILTERS ON β-ALGEBRAS

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(Received On: 17-06-15; Revised & Accepted On: 30-06-15)

ABSTRACT

In this paper, we define the notion on fuzzy filter on β-algebras and investigate some their properties and results.

Keywords: *β-algebra, β-filter, fuzzy filters.*

AMS Classification: *08A72, 03E72.*

1. INTRODUCTION

In 2002, J.Neggers and H.S.Kim [2], introduced a new class of algebras called β-algebras. Fuzzy set theory was developed by Lofti.A.Zadeh [5]. Since then its applications have been growing rapidly in many disciplines. The study of fuzzy algebraic structures was initiated with the introduction of the concept of the fuzzy subgroup by A.Rosenfeld [4]. Then many researchers have been engaged in extending the concepts and results of abstract algebra. The notion of filters was introduecd by Henri Cartan in 1937. In 1991, C.S. Hoo [1], introduced the concept of the filters in BCI-algebras. In 2013, A.Rezeai and A.Bourmand [3], introduced the notion of generalized fuzzy filters (ideals) of BE-algebras. In our earlier paper, we introduced the notion filters in β-algebras. In this paper, we introduce the concept of fuzzy filters in β-algebras and prove some of their properties and theorems.

2. PRELIMINARES

In this section we recall some basic definitions that are required in the sequel.

Definition 2.1: A β-algebra is a nonempty set X with a constant 0 and two binary operations + and –satisfying the following axioms:

1) $x - 0 = x$
2) $(0 - x) + x = 0$
3) $(x - y) - z = x - (z + y)$ for all $x, y, z \in X$.

Definition 2.2: A filter of X, is a non empty subset S, such that $x \in S$ and $y \in S \Rightarrow x \Delta y \in S$, where $x \Delta y = x*(x*y)$ and $0 \notin S$.

Definition 2.3: Let X and Y be two β-algebras. A mapping $f: X \to Y$ is said to be a β- homomorphism, if $f(x+y) = f(x)+f(y)$ and $f(x-y) = f(x)-f(y)$ for all $x, y \in X$.

Definition 2.4: Let X be a β-algebra and A be β-subalgebra. A is said to be a β- filter on X, if for all $x, y \in A$, $x \Delta y = x + (x + y)$ and $x \nabla y = x - (x - y) \in A$.

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3. FUZZY β – FILITER

In this section, we introduce the notion of fuzzy β-filter on a β-subalgebra. We begin with the definition.

Definition 3.1: Let X be a β-algebra and A be fuzzy β-subalgebra. A is said to be fuzzy β- filter on X, if it satisfies the following conditions. For all x, y ∈ A,

1) $\mu_A(x \, \Delta \, y) \geq \min\{\mu_A(x),\ \mu_A(x+y)\}$ and $\mu_A(x \, \nabla \, y) \geq \min\{\mu_A(x),\ \mu_A(x-y)\}$
2) $\mu_A(y) \geq \mu_A(x)$, if $x \leq y$.

Example 3.2: Let X= {0, 1, 2, 3} be a β-algebra with constant 0 and two binary operations + and – defined on X with the cayley's table

+	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	0	0	2	3
3	3	3	3	3

-	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3

Now, A = {2, 3} is β-filter on X.

A is fuzzy β-subalgebra, defined by the membership function, $\mu_A(x) = \begin{cases} 0.4, if\ x = 2 \\ 0.5, if\ x = 3 \end{cases}$

Then we can observe that, A is a fuzzy β-filter on X.

Example 3.3: Let X= {0, 1, 2, 3} be a β-algebra with constant 0 and two binary operations + and – defined on X with the cayley's table

+	0	1	2	3
0	0	0	0	0
1	1	1	1	2
2	0	3	2	3
3	3	1	2	3

-	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3

Now, A = {1, 2, 3} is β-filter on X.

A is fuzzy β-subalgebra defined by the membership function, $\mu_A(x) = \begin{cases} 0.7, if\ x = 1 \\ 0.5, if\ x = 2 \\ 0.6, if\ x = 3 \end{cases}$

Then we can observe that, A is not a fuzzy β-filter on X.

Lemma 3.4: If A and B be any two fuzzy β-filters on X, then A ∩ B is also a fuzzy β-filter of X.

Proof:

$$\begin{aligned} (A \cap B)(x \, \Delta \, y) &= \min\{A(x \, \Delta \, y), B(x \, \Delta \, y)\} \\ &\geq \min\{\min\{A(x), A(x+y)\}, \min\{B(x), B(x+y)\}\} \\ &\geq \min\{\min\{A(x), B(x)\}, \min\{A(x+y), B(x+y)\}\} \\ &= \min\{(A \cap B)(x), (A \cap B)(x+y)\} \end{aligned}$$

Similarly, we can prove that

$(A \cap B)(x \, \nabla \, y) \geq \min\{(A \cap B)(x), (A \cap B)(x-y)\}$

Hence A ∩ B is also a fuzzy β-filter of X.

Theorem 3.5: Every fuzzy β-filter is also a fuzzy β-subalgebra.

This proof is obvious, directly follows from our definition of fuzzy β- filter.

However every fuzzy β subalgebra need not be a fuzzy β-filter shown by the following example.

Example 3.6: Let X= {0, 1, 2, 3} be a β-algebra with constant 0 and two binary operations + and – defined on X with the cayley's table

+	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	1	1	2	0
3	3	3	1	1

-	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3

Now, A = {1, 2} is β-filter on X.

A is fuzzy β-subalgebra defined by the membership function, $\mu_A(x) = \begin{cases} 0.4, if\ x = 1 \\ 0.3, if\ x = 2 \end{cases}$

Then we can observe that, A is not a fuzzy β-filter on X.

Since $\mu_A(2) \geq \mu_A(1) \Longrightarrow 0.3 \ngeq 0.4$

Theorem 3.7: If μ is a fuzzy β- filter of X, then $\mu_A(x\ \Delta\ y) \geq \mu_A(x)$ and $\mu_A(x\ \nabla\ y) \geq \mu_A(x)$, where $x \leq y$.

Proof: Assume that μ is fuzzy filter of X.

Let x, y ∈ X. Then we get,

$$\begin{aligned}
\mu_A(x\ \Delta\ y) &= \mu_A(x+(x+y)) \\
&\geq \min\{\mu_A(x), \mu_A(x+y)\} \\
&\geq \min\{\mu_A(x), \min\{\mu_A(x), \mu_A(y)\}\} \quad \textit{Since every } \text{fuzzy } \beta - \text{filter is also a fuzzy } \beta - \text{subalgebra.} \\
&= \min\{\mu_A(x), \mu_A(x)\} \quad \textit{since } x \leq y \Longrightarrow \mu_A(y) \geq \mu_A(x) \\
&= \mu_A(x)
\end{aligned}$$

Similarly, we can prove that, $\mu_A(x\ \nabla\ y) \geq \mu_A(x)$.

Definition 3.8: Let μ be a fuzzy β-filter in a β-subalgebra X. For s ∈ [0, 1], the set $\mu_s = \{x \in X / \mu(x) \geq s\}$ is called a level set of filter μ in X.

Theorem 3.9: A fuzzy subset A of β-algebra X is a fuzzy β-filter iff for any t ∈ [0, 1] the t - level subset $A_t = \{x \in X | A(x) \geq t\}$ is either a β-filter or $A_t \neq \emptyset$.

Proof: Assume that the level subset of A in X, $A_t \neq \emptyset$.

Then x, y ∈ A_t, A(x) ≥ t, A(y) ≥ t

Now,
$$\begin{aligned}
A(x\ \Delta\ y) &= A(x+(x+y)) \\
&\geq \min\{A(x), A(x+y)\} \\
&\geq \min\{A(x), \min\{A(x), A(y)\} \\
&= \min\{t, \min\{t, t\}\} \\
&= t
\end{aligned}$$

which implies $x\ \Delta\ y \in A_{t.}$

Similarly, we can prove that $x\ \nabla\ y \in A_t$.

Hence A_t is a β-filter of X.

Conversely, assume that A_t is a β-filter of X.

For all x, y ∈ X, x Δ y and x ∇ y ∈ A_t
⇒ A(x Δ y) ≥ t and A(x ∇ y) ≥ t.
A(x Δ y) = A(x+(x+y)) ≥ t = min{A(x), A(x+y)}.

Similarly, we can prove that, A(x ∇ y) ≥ t.

Thus proving that A is a fuzzy β-filter.

Corollary 3.10: Any β-filter of a β-algebra can be realized as a level β-filter of some fuzzy β-filter of X.

Theorem 3.11: Let f be an onto β-algebra homomorphism from X to Y. If B is a fuzzy β-filter of Y, then f^{-1}(B) is also a fuzzy β-filter on X.

Proof: Let B be a fuzzy β-filter of Y.
For x, y ∈ X, then

$$\begin{aligned} f^{-1}(\mu_B(x\,\Delta\,y)) &= f^{-1}(\mu_B(x+(x+y))) \\ &= \mu_B(f(x+(x+y))) \\ &= \mu_B(f(x)+f(x+y)) \\ &\geq \min\{\mu_B(f(x)), \mu_B(f(x+y))\} \\ &= \min\{f^{-1}(\mu_B(x)) + f^{-1}(\mu_B(x+y))\} \end{aligned}$$

Similarly, we can prove that, $f^{-1}(\mu_B(x\,\nabla\,y)) \geq \min\{f^{-1}(\mu_B(x)) - f^{-1}(\mu_B(x-y))\}$

Let x, y ∈ X, such that x ≥ y. Since B is fuzzy β-filter, we have $\mu_B(f(y)) \geq \mu_B(f(x)) = f^{-1}(\mu_B(x))$ such that $f^{-1}(\mu_B(y)) \geq f^{-1}(\mu_B(x))$.

Definition 3.12: Let X and Y be two β-algebras. Let A and B be fuzzy β-filters in $X \times Y$. Then the cartesian product of A and B defined as, A × B = $\{(\mu_A \times \mu_B)(x\,\Delta\,y)$ *and* $(\mu_A \times \mu_B)(x\,\nabla\,y)\}$
where
$\mu_A \times \mu_B(x\,\Delta\,y) = \min\{(\mu_A \times \mu_B)(x), (\mu_A \times \mu_B)(x+y)\}$
$\mu_A \times \mu_B(x\,\nabla\,y) = \min\{(\mu_A \times \mu_B)(x), (\mu_A \times \mu_B)(x-y)\}$

Theorem 3.13: Cartesian product of any two fuzzy β-filters is again a fuzzy β-filter.

Proof: Take x = (x_1, x_2), y = (y_1, y_2) ∈ X × Y and μ = $\mu_A \times \mu_B$.

$$\begin{aligned} \mu_{(A\times B)}(x\,\Delta\,y) &= \mu((x_1,x_2)\,\Delta\,(y_1,y_2)) \\ &= (\mu_A \times \mu_B)\{(x_1,x_2) + ((x_1,x_2)+(y_1,y_2))\} \\ &= \min\{\mu_A(x_1+(x_1+y_1)), \mu_B(x_2+(x_2+y_2)) \\ &\geq \min\{\min(\mu_A(x_1)+\mu_A(x_1+y_1)), \min(\mu_B(x_2)+\mu_B(x_2+y_2))\} \\ &= \min\{\min(\mu_A(x_1), \mu_B(x_2)), \min(\mu_A(x_1+y_1), \mu_B(x_2+y_2))\} \\ &= \min\{(\mu_A \times \mu_B)(x_1,x_2), (\mu_A \times \mu_B)((x_1+y_1),(x_2+y_2))\} \\ &= \min\{(\mu_A \times \mu_B)(x), (\mu_A \times \mu_B)(x+y)\} \end{aligned}$$

Similarly, we can prove that, $\mu_{(A\times B)}(x\,\nabla\,y) \geq \min\{(\mu_A \times \mu_B)(x), (\mu_A \times \mu_B)(x+y)\}$.

Thus proving that A× B is again a fuzzy β-filter.

Let X and Y be two β-algebras. Let A and B be fuzzy β-filters of X and Y respectively, By the above theorem A× B is a fuzzy β-filter of X ×Y. Now A_s and B_{s_1} are level β-filters corresponding to A and B respectively.

Theorem 3.14: Let X and Y be two β-algebras. Let A_s and B_{s_1} be two fuzzy β-filters on $X \times Y$. Then $(A_s \times B_{s_1})$ is also a β-filter if s ≥ s_1.

Proof: Take x = (x_1, x_2), y = (y_1, y_2) ∈ X × Y and μ = $\mu_A \times \mu_B$. If s ≥ s_1,

Using above theorem, we get $\mu(A_s \times B_{s_1})(x\,\Delta\,y) \geq s_1$,

Similarly, we can prove that, $\mu(A_s \times B_{s_1})(x\,\nabla\,y) \geq s_1$.

4. FUZZY STRONG β-FILTER

In this section, we introduce the notion of fuzzy strong β-filter on a β-subalgebra. We begin with the definition and examples as follows.

Definition 4.1: Let X be a β-algebra and A be fuzzy β-subalgebra. A is said to be fuzzy strong β- filter on X, if it satisfies for all x, y ∈ A

1) $\mu_A(x \, \Delta \, y) = \mu_A(x \, \nabla \, y)$
2) $\mu_A(y) \geq \mu_A(x)$, if $x \leq y$.

Example 4.2: Let X= {0, 1, 2, 3} be a β-algebra with constant 0 and two binary operations + and – defined on X with the cayley's table

+	0	1	2	3
0	0	0	0	0
1	1	2	1	3
2	0	3	2	2
3	3	1	3	3

-	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3

Now, A = {2, 3} is β-filter on X.

A is fuzzy β-subalgebra, defined as, $\mu_A(x) = \begin{cases} 0.4, if\ x = 2 \\ 0.5, if\ x = 3 \end{cases}$

Then we can observe that, A is a fuzzy strong β-filter on X.

Theorem 4.3: Every Fuzzy strong β-filter is also a fuzzy β-subalgebra.

Converse part of the theorem need not be true.

Example 4.4: Let X= {0, 1, 2, 3} be a β-algebra with constant 0 and two binary operations + and – defined on X with the cayley's table

+	0	1	2	3
0	0	0	0	0
1	1	2	3	0
2	1	0	2	3
3	3	1	3	3

-	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3

Now, A = {2,3} is β-filter on X.

A is fuzzy β-subalgebra defined the membership function, $\mu_A(x) = \begin{cases} 0.4, if\ x = 3 \\ 0.3, if\ x = 2 \end{cases}$

Then we can observe that, A is not a fuzzy strong β-filter on X.

Since $\mu_A(2 \Delta \, 3) \neq \mu_A(2 \, \nabla \, 3) \Rightarrow \mu_A(3) \neq \mu_A(2) \Rightarrow 0.4 \neq 0.3$.

Theorem 4.5: Every Fuzzy strong β-filter is also a fuzzy β-filter.

From the example 4.4., the converse part of above theorem need not be true, in general.

Lemma 4.6: If A and B be two fuzzy strong β-filters on X, then A ∩ B is also a fuzzy strong β-filter of X.

Theorem 4.7: If μ is a fuzzy β- filter of X, then $\mu_A(x \, \Delta \, y) \geq \mu_A(y)$ where $y \leq x$.

Theorem 4.8: Let f be onto β-algebra homomorphism X to Y respectively. If B is a fuzzy strong β-filter of Y, then f^{-1}(B) is also a fuzzy strong β-filter on X.

5. REFERENCES

1. Hoo C.S., Filters and Ideals in BCI-algebras, Math. Japan, 36(1991), no.5, pp-987-997.
2. Neggers J. and Kim Hee Sik, On β-algebras, Math. Solvaca, (2002), 52 (5), pp-517-530.
3. Rezeai A. and Bourmand Saied, Generalized Fuzzy Filters (Ideals) of BE-Algebras, Journal of Uncertain systems, 7(2013), no. 2, pp-152-160.
4. Rosenfeld A., Fuzzy Groups, Journal Math. Anal. Appl., (1971) 35, pp-512-517.
5. Zadeh L.A., Fuzzy sets, inform. Control (1965), 8(3), pp-338-353.

Source of support: Nil, Conflict of interest: None Declared