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ON THE DEGREE OF APPROXIMATION OF FUNCTION BELONGING TO THE LIPSCHITZ CLASS BY (E, q) (C, δ) PRODUCT MEANS OF ITS CONJUGATE FOURIER SERIES

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ABSTRACT

In this paper a theorem on the degree of Approximation of function belonging to the Lipschitz Class by (E, q) (C, δ) Product Means of its Conjugate Fourier Series have been established.

Keywords: *Cesâro matrix, Euler matrix, Degree of Approximation.*

1. DEFINITION AND NOTATIONS

Let f be 2π - periodic and L-integrable over $[-\pi,\pi]$. The Fourier series of f at a point is given by

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \tag{1.1}$$

The conjugate series of the Fourier series (1.1) is given by

$$\sum_{n=1}^{\infty}(b_n\cos nx-a_n\sin nx) \tag{1.2}$$

A function $f\in \mathrm{Lip}\,\alpha$ $(0<\alpha\le 1)$ if

$$f(x+t)-f(x)=\left(|t|^{\alpha}\right). \tag{1.3}$$

The degree of Approximation of a function $f:R\to R$ by a trigonometric polynomial t_n of order n is defined by zygmund [1, p-114],

$$\|t_n-f\|=\sup\{|t_n(x)-f(x)|:x\in R\} \tag{1.4}$$

Let $\sum_{n=0}^{\infty}a_n$ be given infinite series with the sequence (s_n) of partial sums of its first (n+1) terms. The Euler means of the sequence (s_n) are defined by

$$(E,q)=E_n^q=(q+1)^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}S_k,(q\ge 0),$$

Where E_n^0 is defined to be s_n. If $t_n\to s$: as $n\to\infty$, we say that (S_n) or $\sum_{n=0}^{\infty}a_n$ is summable $(E,q)(q>0)$ to S or symbolically we write $(S_n)\in S(E,q)$,for q>0 see Hardy [2,p-180] and for real and complex values of $q\neq -1$,see Chandra [4].

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The sequence (S_n) is said to be summable (C, δ) $(\delta > -1)$ to limit S if, $\left(A_n^{\delta}\right)^{-1}\sum_{k=0}^{n} A_{n-k}^{\delta-1}S_k \to S \; as \, n \to \infty$

Where A_n^{δ} are the binomial coefficients. See Zygmund [1, p-76]

The (E, q) transform of the (C, δ) transform defines the (E, q) (C, δ) transform of the partial sums S_n of the series $\sum_{n=0}^{\infty} a_n$.

The Transform (E, q) (C, δ) reduces to (E, q) and (C, δ) respectively for δ = 0 and q = 0. Thus if

$$(E_q c_\delta)_n = (1+q)^{-n}\sum_{v=0}^{n}\binom{n}{v} q^{n-v}(A_v^{\delta})^{-1}\sum_{k=0}^{n} A_{v-k}^{\delta-1}S_k \to S \text{ as } n \to \infty .$$

Then the series $\sum_{n=0}^{\infty} a_n$ is said to be summable by (E, q) (C, δ) means or simply summable (E, d) (C, δ) to S.

Let S_n (f; x) be the n^{th} partial sum of the series (1.1).Then (E, q) (C, δ) means of (S_n (f; x)), Where q > 0 and δ >-1, is given by

$$(E_q c_\delta)_n(f;x) = (1+q)^{-n}\sum_{v=0}^{n}\binom{n}{v} q^{n-v}(A_v^{\delta})^{-1}\sum_{k=0}^{n} A_{v-k}^{\delta-1}S_k(f;x) \qquad (1.5)$$

We shall use the following notations for each $x \in R$.

$$\psi(t) = f(x+t) - f(x-t) \qquad (1.6)$$

$$\overline{K}_n(t) = \frac{1}{2\pi(1+q)^n}\sum_{v=0}^{n}\binom{n}{v} q^{n-v}(A_v^{\delta})^{-1}\sum_{k=0}^{n} A_{v-k}^{\delta-1}\frac{\operatorname{Sin}(2k+1)(t/2)}{2\sin(t/2)} \qquad (1.7)$$

$$\widetilde{f}(x) = -\frac{1}{4\pi}\int_0^{\pi}\psi(t) \qquad (1.8)$$

MAIN THEOREM

Theorem 2.1: If $f : R \to R$ is 2π periodic and Lebesgue integrable on $[-\pi,\pi]$ and $f \in Lip\,\alpha$ class then the degree of approximation of function f by (E, q) (C, δ) product means of its conjugate Fourier Series (1.2) of f satisfies, for n = 0, 1, 2......

$$\left\|(E_q C_\delta)_n(\widetilde{f} : x) - \widetilde{f}(x)\right\|_\infty = \begin{cases} o\left(\dfrac{1}{(n+1)\alpha}\right) & ;(0<\alpha<\delta\le 1) \\ & ;(0<\alpha\le 1, \delta>1) \\ o\left(\dfrac{\log(n+1)}{n+1,\alpha}\right) & ;(0<\alpha\le\delta\le 1)\end{cases} \qquad (2.1)$$

3. For the proof of our theorem, we need the following lemmas:

Lemma 1: [1, p-94]: For $(0 < \delta \le 1), n = 1, 2, 3....\; 0 < t \le \pi$

$$\left|\widetilde{k}_v^{\delta}(t)\right| \le A_\delta y^{-\delta} t^{-(\delta+1)} \qquad (3.1)$$

Where A_δ depending on δ only

Lemma 2: [5] For q > 0

$$\sum_{v=0}^{n}\binom{n}{v} q^{n-v}(v+1)^{-1} = o\{(1+q)^{n+1}/(n+1)\} \qquad (3.2)$$

Lemma 3: For δ >1,

$$\left|\tilde{k}_v^{\delta}(t)\right| = o(1)\left(\frac{\delta}{(v+1)t^2}\right) \tag{3.3}$$

4. PROOF OF THE THEOREM:

The n^{th} partial sum of the conjugate Fourier series [1, p-50] is $\overline{S}_n(f;x)$. Then

$$(E_q c_\delta)_n(\tilde{f};x)-\tilde{f}(x)=\frac{1}{\pi}\int_0^{\pi}\phi_x(t)(1+q)^{-n}\sum_{v=0}^{n}\binom{n}{v}q^{n-v}(A_v^{\delta})^{-1}\sum_{k=0}^{n}A_{v-k}^{\delta-1}D_k(t)d$$

$$\left|(E_q c_\delta)_n(\tilde{f};x)-\tilde{f}(x)\right|\le\frac{1}{\pi}\int_0^{\pi}\left|\phi_x(t)\right|\left|(1+q)^{-n}\sum_{v=0}^{n}\binom{n}{v}q^{n-v}(A_v^{\delta})^{-1}\sum_{k=0}^{n}A_{v-k}^{\delta-1}D_k(t)\right|d$$

$$\le \frac{1}{\pi}\left\{\int_0^{\frac{1}{(n+1)}}+\int_{\frac{1}{(n+1)}}^{\pi}\right\} \le |I_1|+|I_2|, say$$

Now, for $0 \le t \le 1/(n+1), \sin nt \le n \sin t$, see [1, p-91]

$$|I_1|\le\frac{1}{\pi}\int_0^{\frac{1}{(n+1)}}|\phi_x(t)|\left|(1+q)^{-n}\sum_{v=0}^{n}\binom{n}{v}q^{n-v}(A_v^{\delta})^{-1}\sum_{k=0}^{n}A_{v-k}^{\delta-1}D_k(t)\right|dt$$

$$\le \frac{1}{2\pi}\int_0^{\frac{1}{(n+1)}}|\phi_x(t)|\left|(1+q)^{-n}\sum_{v=0}^{n}\binom{n}{v}q^{n-v}(A_v^{\delta})^{-1}\sum_{k=0}^{n}A_{v-k}^{\delta-1}(2k+1)\right|dt$$

We have by Boos [3, p-104]

$$|I_1|\le\frac{1}{2\pi}\int_0^{\frac{1}{(n+1)}}|\phi_x(t)|(1+d)^{-n}\sum_{v=0}^{n}\binom{n}{v}q^{n-v}(2v+1)dt,$$

$$\le \frac{1}{2\pi}\int_0^{\frac{1}{(n+1)}}|\phi_x(t)|(2n+1)dt,$$

By (1.3)

$$0(n+1)\int_0^{\frac{1}{(n+1)}}t^{\alpha}dt = o(n+1)^{-\alpha} \tag{4.1}$$

By (1.8), we have

$$|I_2|\le\frac{1}{\pi}\int_{\frac{1}{(n+1)}}^{\pi}|\phi_x(t)|(1+q)^{-n}\sum_{v=0}^{n}\binom{n}{v}q^{n-v}\left|\tilde{k}_v^{\delta}(t)\right|dt$$

Condition –I

For $\delta \le 1$, by lemma 1, we have

$$|I_2|\le\frac{1}{\pi}\int_{\frac{1}{(n+1)}}^{\pi}|\phi_x(t)|(1+q)^{-n}\sum_{v=0}^{n}\binom{n}{v}q^{n-v}A_\delta v^{-\delta}t^{-(\delta+1)}dt$$

$$\le \frac{A_\delta}{\pi}\int_{\frac{1}{(n+1)}}^{\pi}|\phi_x(t)|t^{-(\delta+1)}(1+q)^{-n}\sum_{v=0}^{n}\binom{n}{v}q^{n-v}(v+1)^{-\delta}dt$$

By Lemma 2 and (1.2), we get

$$|I_2| = o((n+1)^{-\delta}) \int_{\frac{1}{(n+1)}}^{\pi} t^{\alpha-(\delta+1)} dt$$

Case-I: When $\alpha = \delta$, then

$$|I_2| = o((n+1)^{-\alpha}) \int_{\frac{1}{(n+1)}}^{\pi} t^{-1} dt$$

$$= o((n+1)^{-\alpha}) \log (n+1) \tag{4.2}$$

Case-II: When $\alpha < \delta$, then

$$|I_2| = o((n+1)^{-\delta})(t^{\alpha-\delta})^{\pi} \frac{1}{(n+1)}$$

$$= o((n+1)^{-\alpha}) \tag{4.3}$$

Combining (4.2) and (4.3) we have,

$$|I_2| = \begin{cases} o\{(n+1)^{-\alpha}\}, \ (0 < \alpha < \delta \le 1) \\ o\{(n+1)^{-\alpha} \log(n+1)\}, \ (0 < \alpha \le \delta \le 1) \end{cases}$$

Condition-II:

For $\delta > 1$, by lemma 3, we have

$$|I_2| \le \frac{1}{\pi} \int_{\frac{1}{(n+1)}}^{\pi} |\phi_x(t)| (1+q)^{-n} \sum_{v=0}^{n} \binom{n}{v} q^{n-v} \frac{\delta}{(v+1)t^2} dt \tag{4.4}$$

By lemma 3 and (1.2), we have

$$o((n+1)^{-1}) \int_{\frac{1}{(n+1)}}^{\pi} t^{\alpha-2} dt = o((n+1)^{-\alpha}) \tag{4.5}$$

Now combining the estimate (4.1), (4.4) and (4.5) we get required result.

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