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NOTES ON BIPOLAR-VALUED MULTI FUZZY SUBGROUPS OF A GROUP

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ABSTRACT

In this paper, we study some of the properties of bipolar-valued multi fuzzy subgroup and prove some results on these.

Key Words: *Bipolar-valued fuzzy subset, bipolar-valued multi fuzzy subset, bipolar-valued multi fuzzy subgroup.*

INTRODUCTION

In 1965, Zadeh [12] introduced the notion of a fuzzy subset of a set, fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. Since then it has become a vigorous area of research in different domains, there have been a number of generalizations of this fundamental concept such as intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, soft sets etc [5]. Lee [7] introduced the notion of bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0, 1] to [–1, 1]. In a bipolar-valued fuzzy subset, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degree (0, 1] indicates that elements somewhat satisfy the property and the membership degree [–1, 0) indicates that elements somewhat satisfy the implicit counter property. Bipolar-valued fuzzy sets and intuitionistic fuzzy sets look similar each other. However, they are different each other [7, 8]. We introduce the concept of bipolar-valued multi fuzzy subgroup and established some results.

1. PRELIMINARIES

1.1 Definition: A bipolar-valued fuzzy set (BVFS) A in X is defined as an object of the form $A = \{< x, A^+(x), A^-(x) >/ x\in X\}$, where $A^+ : X\to [0, 1]$ and $A^- : X\to [-1, 0]$. The positive membership degree $A^+(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set A and the negative membership degree $A^-(x)$ denotes the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar-valued fuzzy set A. If $A^+(x) \neq 0$ and $A^-(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for A and if $A^+(x) = 0$ and $A^-(x) \neq 0$, it is the situation that x does not satisfy the property of A, but somewhat satisfies the counter property of A. It is possible for an element x to be such that $A^+(x) \neq 0$ and $A^-(x) \neq 0$ when the membership function of the property overlaps that of its counter property over some portion of X.

1.2 Example: A = {< a, 0.5, –0.3 >, < b, 0.1, –0.7 >, < c, 0.5, –0.4 >} is a bipolar-valued fuzzy subset of X= {a, b, c}.

1.3 Definition: A bipolar-valued multi fuzzy set (BVMFS) A in X is defined as an object of the form $A = \{< x, A_i^+(x), A_i^-(x) >/ x\in X\}$, where $A_i^+ : X\to [0, 1]$ and $A_i^- : X\to [-1, 0]$. The positive membership degrees $A_i^+(x)$ denote the satisfaction degree of an element x to the property corresponding to a bipolar-valued multi fuzzy set A and the negative membership degrees $A_i^-(x)$ denote the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar-valued multi fuzzy set A. If $A_i^+(x) \neq 0$ and $A_i^-(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for A and if $A_i^+(x) = 0$ and $A_i^-(x) \neq 0$, it is the situation that x does not satisfy the property of A, but somewhat satisfies the counter property of A. It is possible for an element x to be such that $A_i^+(x) \neq 0$ and $A_i^-(x) \neq 0$ when the membership function of the property overlaps that of its counter property over some portion of X, where i = 1 to n.

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1.4 Example: A = {< a, 0.5, 0, 6, 0.3, –0.3, –0.6, –0.5 >, < b, 0.1, 0.4, 0.7, –0.7, –0.3, –0.6 >, < c, 0.5, 0.3, 0.8, –0.4, –0.5, –0.3 >} is a bipolar-valued multi fuzzy subset of X = {a, b, c}.

1.5 Definition: Let G be a group. A bipolar-valued multi fuzzy subset A of G is said to be a bipolar-valued multi fuzzy subgroup of G (BVMFSG) if the following conditions are satisfied

(i) $A_i^+(xy) \geq \min\{A_i^+(x), A_i^+(y)\}$
(ii) $A_i^+(x^{-1}) \geq A_i^+(x)$
(iii) $A_i^-(xy) \leq \max\{A_i^-(x), A_i^-(y)\}$
(iv) $A_i^-(x^{-1}) \leq A_i^-(x)$ for all x and y in G.

1.6 Example: Let G = {1, –1, i, –i} be a group with respect to the ordinary multiplication. Then A = {< 1, 0.5, 0.6, 0.4, –0.6, –0.5, –0.3 >, < –1, 0.4, 0.5, 0.3, –0.5, –0.4, –0.2 >, < i, 0.2, 0.3, 0.2, –0.4, –0.3, –0.1 >, < –i, 0.2, 0.3, 0.2, –0.4, –0.3, –0.1 >} is a bipolar-valued multi fuzzy subgroup of G.

1.7 Definition: Let $A = \langle A_i^+, A_i^- \rangle$ and $B = \langle B_i^+, B_i^- \rangle$ be any two bipolar-valued multi fuzzy subsets of sets G and H, respectively. The product of A and B, denoted by A×B, is defined as $A\times B = \{\langle (x, y), (A_i\times B_i)^+(x, y), (A_i\times B_i)^-(x, y)\rangle /$ for all x in G and y in H} where $(A_i\times B_i)^+(x, y) = \min\{A_i^+(x), B_i^+(y)\}$ and $(A_i\times B_i)^-(x, y) = \max\{A_i^-(x), B_i^-(y)\}$ for all x in G and y in H.

1.8 Definition: Let $A = \langle A_i^+, A_i^- \rangle$ be a bipolar-valued multi fuzzy subset in a set S, the strongest bipolar-valued multi fuzzy relation on S, that is a bipolar-valued multi fuzzy relation on A is $V = \{\langle (x, y), V_i^+(x, y), V_i^-(x, y)\rangle$ / x and y in S} given by $V_i^+(x, y) = \min\{A_i^+(x), A_i^+(y)\}$ and $V_i^-(x, y) = \max\{A_i^-(x), A_i^-(y)\}$ for all x and y in S.

2. PROPERTIES

2.1 Theorem: Let $A = \langle A_i^+, A_i^- \rangle$ be a bipolar-valued multi fuzzy subgroup of G. Then $A_i^+(x^{-1}) = A_i^+(x)$ and $A_i^-(x^{-1}) = A_i^-(x)$, $A_i^+(x) \leq A_i^+(e)$ and $A_i^-(x) \geq A_i^-(e)$ for all x in G and the identity element e in G.

Proof: Let x be in G. Now $A_i^+(x) = A_i^+((x^{-1})^{-1}) \geq A_i^+(x^{-1}) \geq A_i^+(x)$. Therefore $A_i^+(x) = A_i^+(x^{-1})$ for all x in G. And $A_i^-(x) = A_i^-((x^{-1})^{-1}) \leq A_i^-(x^{-1}) \leq A_i^-(x)$. Therefore $A_i^-(x^{-1}) = A_i^-(x)$ for all x in G.

Now $A_i^+(e) = A_i^+(xx^{-1}) \geq \min\{A_i^+(x), A_i^+(x^{-1})\} = A_i^+(x)$. Therefore $A_i^+(e) \geq A_i^+(x)$ for all x in G. And $A_i^-(e) = A_i^-(xx^{-1}) \leq \max\{A_i^-(x), A_i^-(x^{-1})\} = A_i^-(x)$. Therefore $A_i^-(e) \leq A_i^-(x)$ for all x in G.

2.2 Theorem: Let $A = \langle A_i^+, A_i^- \rangle$ be a bipolar-valued multi fuzzy subgroup of G. Then

(i) $A_i^+(xy^{-1}) = A_i^+(e)$ implies that $A_i^+(x) = A_i^+(y)$ for x and y in G.
(ii) $A_i^-(xy^{-1}) = A_i^-(e)$ implies that $A_i^-(x) = A_i^-(y)$ for x and y in G.

Proof: Now $A_i^+(x) = A_i^+(xy^{-1}y) \geq \min\{A_i^+(xy^{-1}), A_i^+(y)\} = \min\{A_i^+(e), A_i^+(y)\} = A_i^+(y) = A_i^+(yx^{-1}x) \geq \min\{A_i^+(yx^{-1}), A_i^+(x)\} = \min\{A_i^+(e), A_i^+(x)\} = A_i^+(x)$. Therefore $A_i^+(x) = A_i^+(y)$ for x and y in G. And $A_i^-(x) = A_i^-(xy^{-1}y) \leq \max\{A_i^-(xy^{-1}), A_i^-(y)\} = \max\{A_i^-(e), A_i^-(y)\} = A_i^-(y) = A_i^-(yx^{-1}x) \leq \max\{A_i^-(yx^{-1}), A_i^-(x)\} = \max\{A_i^-(e), A_i^-(x)\} = A_i^-(x)$. Therefore $A_i^-(x) = A_i^-(y)$ for x and y in G.

2.3 Theorem: Let $A = \langle A_i^+, A_i^- \rangle$ be a bipolar-valued multi fuzzy subgroup of a group G.

(i) If $A_i^+(xy^{-1}) = 1$, then $A_i^+(x) = A_i^+(y)$ for x and y in G.
(ii) If $A_i^-(xy^{-1}) = -1$, then $A_i^-(x) = A_i^-(y)$ for x and y in G.

Proof: Now $A_i^+(x) = A_i^+(xy^{-1}y) \geq \min\{A_i^+(xy^{-1}), A_i^+(y)\} = \min\{1, A_i^+(y)\} = A_i^+(y) = A_i^+(y^{-1}) = A_i^+(x^{-1}xy^{-1}) \geq \min\{A_i^+(x^{-1}), A_i^+(xy^{-1})\} = \min\{A_i^+(x^{-1}), 1\} = A_i^+(x^{-1}) = A_i^+(x)$. Therefore $A_i^+(x) = A_i^+(y)$ for x and y in G. Hence (i) is proved. Also $A_i^-(x) = A_i^-(xy^{-1}y) \leq \max\{A_i^-(xy^{-1}), A_i^-(y)\} = \max\{-1, A_i^-(y)\} = A_i^-(y) = A_i^-(y^{-1}) = A^-(x^{-1}xy^{-1}) \leq \max\{A_i^-(x^{-1}), A_i^-(xy^{-1})\} = \max\{A_i^-(x^{-1}), -1\} = A^-(x^{-1}) = A_i^-(x)$. Therefore $A_i^-(x) = A_i^-(y)$ for x and y in G. Hence (ii) is proved.

2.4 Theorem: Let $A = \langle A_i^+, A_i^- \rangle$ be a bipolar-valued multi fuzzy subgroup of a group G.
(i) If $A_i^+(xy^{-1}) = 0$, then either $A_i^+(x) = 0$ or $A_i^+(y) = 0$ for x and y in G.
(ii) If $A_i^-(xy^{-1}) = 0$, then either $A_i^-(x) = 0$ or $A_i^-(y) = 0$ for x and y in G.

Proof: Let x and y in G.

(i) By the definition $A_i^+(xy^{-1}) \geq \min\{A_i^+(x), A_i^+(y)\}$ which implies that $0 \geq \min\{A_i^+(x), A_i^+(y)\}$. Therefore either $A_i^+(x) = 0$ or $A_i^+(y) = 0$.

(ii) By the definition $A_i^-(xy^{-1}) \le \max\{A_i^-(x), A_i^-(y)\}$ which implies that $0 \le \max\{A_i^-(x), A_i^-(y)\}$. Therefore either $A_i^-(x) = 0$ or $A_i^-(y) = 0$.

2.5 Theorem: If $A = \langle A_i^+, A_i^-\rangle$ be a bipolar-valued multi fuzzy subgroup of G, then

(i) $A_i^+(xy) = A_i^+(yx)$ if and only if $A_i^+(x) = A_i^+(y^{-1}xy)$ for x and y in G.

(ii) $A_i^-(xy) = A_i^-(yx)$ if and only if $A_i^-(x) = A_i^-(y^{-1}xy)$ for x and y in G.

Proof: Let x and y be in G. Assume that $A_i^+(xy) = A_i^+(yx)$. So, $A_i^+(y^{-1}xy) = A_i^+(y^{-1}yx) = A_i^+(ex) = A_i^+(x)$. Therefore $A_i^+(x) = A_i^+(y^{-1}xy)$ for x and y in G. Conversely assume that $A_i^+(x) = A_i^+(y^{-1}xy)$. We get $A_i^+(xy) = A_i^+(xyxx^{-1}) = A_i^+(yx)$. Therefore $A_i^+(xy) = A_i^+(xy)$ for x and y in G. Hence $A_i^+(xy) = A_i^+(yx)$ if and only if $A_i^+(x) = A_i^+(y^{-1}xy)$ for x and y in G. Also assume that $A_i^-(xy) = A_i^-(yx)$. We get $A_i^-(y^{-1}xy) = A_i^-(y^{-1}yx) = A_i^-(ex) = A_i^-(x)$. Therefore $A_i^-(x) = A_i^-(y^{-1}xy)$ for x and y in G. Conversely assume that $A_i^-(x) = A_i^-(y^{-1}xy)$. So $A_i^-(xy) = A_i^-(xyxx^{-1}) = A_i^-(yx)$. Therefore $A_i^-(xy) = A_i^-(xy)$ for x and y in G. Hence $A_i^-(xy) = A_i^-(yx)$ if and only if $A_i^-(x) = A_i^-(y^{-1}xy)$ for x and y in G.

2.6 Theorem: If $A = \langle A_i^+, A_i^-\rangle$ is a bipolar-valued multi fuzzy subgroup of a group G, then $H = \{x\in G \mid A_i^+(x) = 1, A_i^-(x) = -1\}$ is either empty or a subgroup of G.

Proof: If no element satisfies this condition, then H is empty. If x and y in H, then $A_i^+(xy^{-1}) \ge \min\{A_i^+(x), A_i^+(y)\}= \min\{1, 1\}= 1$. Therefore $A_i^+(xy^{-1}) = 1$. And $A_i^-(xy^{-1}) \le \max\{A_i^-(x), A_i^-(y)\} = \max\{-1, -1\}= -1$. Therefore $A_i^-(xy^{-1}) = -1$. That is $xy^{-1}\in H$. Hence H is a subgroup of G. Hence H is either empty or a subgroup of G.

2.7 Theorem: If $A = \langle A_i^+, A_i^-\rangle$ is a bipolar-valued multi fuzzy subgroup of G, then $H = \{x\in G \mid A_i^+(x) = A_i^+(e)$ and $A_i^-(x) = A_i^-(e)\}$ is a subgroup of G.

Proof: Here $H = \{x\in G \mid A_i^+(x) = A_i^+(e)$ and $A_i^-(x) = A_i^-(e)\}$ by Theorem 2.1, $A_i^+(x^{-1}) = A_i^+(x) = A_i^+(e)$ and $A_i^-(x^{-1}) = A_i^-(x) = A_i^-(e)$. Therefore $x^{-1}\in H$. Now $A_i^+(xy^{-1}) \ge \min\{A_i^+(x), A_i^+(y)\} = \min\{A_i^+(e), A_i^+(e)\} = A_i^+(e)$ and $A_i^+(e) = A_i^+((xy^{-1})(xy^{-1})^{-1}) \ge \min\{A_i^+(xy^{-1}), A_i^+(xy^{-1})\} = A_i^+(xy^{-1})$. Hence $A_i^+(e) = A_i^+(xy^{-1})$. Also $A_i^-(xy^{-1}) \le \max\{A_i^-(x), A_i^-(y)\} = \max\{A_i^-(e), A_i^-(e)\} = A_i^-(e)$ and $A_i^-(e) = A_i^-((xy^{-1})(xy^{-1})^{-1}) \le \max\{A_i^-(xy^{-1}), A_i^-(xy^{-1})\} = A_i^-(xy^{-1})$. Therefore $A_i^-(e) = A_i^-(xy^{-1})$. Hence $A_i^+(e) = A_i^+(xy^{-1})$ and $A_i^-(e) = A_i^-(xy^{-1})$. Therefore $xy^{-1}\in H$. Hence H is a subgroup of G.

2.8 Theorem: Let G be a group. If $A = \langle A_i^+, A_i^-\rangle$ is a bipolar-valued multi fuzzy subgroup of G, then $A_i^+(xy) = \min\{A_i^+(x), A_i^+(y)\}$ and $A_i^+(xy) = \max\{A_i^-(x), A_i^-(y)\}$ for each x andy in G with $A_i^+(x) \ne A_i^+(y)$ and $A_i^-(x) \ne A_i^-(y)$.

Proof: Assume that $A_i^+(x) > A_i^+(y)$ and $A_i^-(x) < A_i^-(y)$. Then $A_i^+(y) = A_i^+(x^{-1}xy) \ge \min\{A_i^+(x^{-1}), A_i^+(xy)\}= \min\{A_i^+(x), A_i^+(xy)\} = A_i^+(xy) \ge \min\{A_i^+(x), A_i^+(y)\} = A_i^+(y)$. Therefore $A_i^+(xy) = A_i^+(y) = \min\{A_i^+(x), A_i^+(y)\}$. And $A_i^-(y) = A_i^-(x^{-1}xy) \le \max\{A_i^-(x^{-1}), A_i^-(xy)\}= \max\{A_i^-(x), A_i^-(xy)\} = A_i^-(xy) \le \max\{A_i^-(x), A_i^-(y)\} = A_i^-(y)$. Therefore $A_i^-(xy) = A_i^-(y) = \max\{A_i^-(x), A_i^-(y)\}$.

2.9 Theorem: If $A = \langle A_i^+, A_i^-\rangle$ and $B = \langle B_i^+, B_i^-\rangle$ are two bipolar-valued multi fuzzy subgroups of a group G, then their intersection $A\cap B$ is a bipolar-valued multi fuzzy subgroup of G.

Proof: Let $A = \{< x, A_i^+(x), A_i^-(x) > / x\in G\}$, $B = \{< x, B_i^+(x), B_i^-(x) > / x\in G\}$. Let $C = A\cap B$ and $C = \{< x, C_i^+(x), C_i^-(x) > / x\in G\}$.Now $C_i^+(xy^{-1}) = \min\{A_i^+(xy^{-1}), B_i^+(xy^{-1})\}\ge \min\{\min\{A_i^+(x), A_i^+(y)\}, \min\{B_i^+(x), B_i^+(y)\}\} \ge \min\{\min\{A_i^+(x), B_i^+(x)\}, \min\{A_i^+(y), B_i^+(y)\}\}= \min\{C_i^+(x), C_i^+(y)\}$. Therefore $C_i^+(xy^{-1}) \ge \min\{C_i^+(x), C_i^+(y)\}$. Also $C_i^-(xy^{-1}) = \max\{A_i^-(xy^{-1}), B_i^-(xy^{-1})\}\le \max\{\max\{A_i^-(x), A_i^-(y)\}, \max\{B_i^-(x), B_i^-(y)\}\} \le \max\{\max\{A_i^-(x), B_i^-(x)\}, \max\{A_i^-(y), B_i^-(y)\}\} = \max\{C_i^-(x), C_i^-(y)\}$. Therefore $C_i^-(xy^{-1}) \le \max\{C_i^-(x), C_i^-(y)\}$. Hence $A\cap B$ is a bipolar-valued multi fuzzy subgroup of G.

2.10 Theorem: The intersection of a family of bipolar-valued multi fuzzy subgroups of a group G is a bipolar-valued multi fuzzy subgroup of G.

Proof: The Theorem is true by Theorem 2.9.

2.11 Theorem: If $A= \langle A_i^+, A_i^-\rangle$ and $B = \langle B_i^+, B_i^-\rangle$ are any two bipolar-valued multi fuzzy subgroups of the groups G_1 and G_2 respectively, then $A\times B = \langle (A_i\times B_i)^+, (A_i\times B_i)^-\rangle$ is a bipolar-valued multi fuzzy subgroup of $G_1\times G_2$.

Proof: Let A and B be two bipolar-valued multi fuzzy subgroups of the groups G_1 and G_2 respectively. Let x_1 and x_2 be in G_1, y_1 and y_2 be in G_2. Then (x_1, y_1) and (x_2, y_2) are in $G_1\times G_2$. Now, $(A_i\times B_i)^+[(x_1, y_1)(x_2, y_2)^{-1}] = (A_i\times B_i)^+(x_1x_2^{-1}, y_1y_2^{-1}) = \min\{A_i^+(x_1x_2^{-1}), B_i^+(y_1y_2^{-1})\} \ge \min\{\min\{A_i^+(x_1), A_i^+(x_2)\}, \min\{B_i^+(y_1), B_i^+(y_2)\}\} = \min\{\min\{A_i^+(x_1), B_i^+(y_1)\},$

$\min\{A_i^+(x_2), B_i^+(y_2)\}\} = \min\{(A_i\times B_i)^+(x_1, y_1), (A_i\times B_i)^+(x_2, y_2)\}$. Therefore $(A_i\times B_i)^+[(x_1, y_1)(x_2, y_2)^{-1}] \geq \min\{(A_i\times B_i)^+(x_1, y_1), (A_i\times B_i)^+(x_2, y_2)\}$. Also $(A_i\times B_i)^-[(x_1, y_1)(x_2, y_2)^{-1}] = (A_i\times B_i)^-(x_1x_2^{-1}, y_1y_2^{-1}) = \max\{A_i^-(x_1x_2^{-1}), B_i^-(y_1y_2^{-1})\} \leq \max\{\max\{A_i^-(x_1), A_i^-(x_2)\}, \max\{B_i^-(y_1), B_i^-(y_2)\}\} = \max\{\max\{A_i^-(x_1), B_i^-(y_1)\}, \max\{A_i^-(x_2), B_i^-(y_2)\}\} = \max\{(A_i\times B_i)^-(x_1, y_1), (A_i\times B_i)^-(x_2, y_2)\}$. Therefore $(A_i\times B_i)^-[(x_1, y_1)(x_2, y_2)^{-1}] \leq \max\{(A_i\times B_i)^-(x_1, y_1), (A_i\times B_i)^-(x_2, y_2)\}$. Hence $A\times B$ is a bipolar-valued multi fuzzy subgroup of $G_1\times G_2$.

2.12 Theorem: Let $A = \langle A_i^+, A_i^-\rangle$ and $B = \langle B_i^+, B_i^-\rangle$ be any two bipolar-valued multi fuzzy subsets of the groups G and H respectively. Suppose that e and e^l are the identity elements of G and H respectively. If $A\times B$ is a bipolar-valued multi fuzzy subgroup of $G\times H$, then at least one of the following two statements must hold.

(i) $B_i^+(e^l) \geq A_i^+(x)$ for all x in G and $B_i^-(e^l) \leq A_i^-(x)$ for all x in G

(ii) $A_i^+(e) \geq B_i^+(y)$ for all y in H and $A_i^-(e) \leq B_i^-(y)$ for all y in H.

Proof: Let $A\times B$ is a bipolar-valued multi fuzzy subgroup of $G\times H$. By contraposition, suppose that none of the statements (i) and (ii) holds. Then we can find a in G and b in H such that $A_i^+(a) > B_i^+(e^l)$, $A_i^-(a) < B_i^-(e^l)$ and $B_i^+(b) > A_i^+(e)$, $B_i^-(b) < A_i^-(e)$. We have $(A_i\times B_i)^+(a, b) = \min\{A_i^+(a), B_i^+(b)\} > \min\{A_i^+(e), B_i^+(e^l)\} = (A_i\times B_i)^+(e, e^l)$. Also $(A_i\times B_i)^-(a, b) = \max\{A_i^-(a), B_i^-(b)\} < \max\{A_i^-(e), B_i^-(e^l)\} = (A_i\times B_i)^-(e, e^l)$. Thus $A\times B$ is not a bipolar-valued multi fuzzy subgroup of $G\times H$. Hence either $B_i^+(e^l) \geq A_i^+(x)$ for all x in G and $B_i^-(e^l) \leq A_i^-(x)$ for all x in G or $A_i^+(e) \geq B_i^+(y)$ for all y in H and $A_i^-(e) \leq B_i^-(y)$ for all y in H.

2.13 Theorem: Let $A= \langle A_i^+, A_i^-\rangle$ and $B = \langle B_i^+, B_i^-\rangle$ be any two bipolar-valued multi fuzzy subsets of the groups G and H, respectively and $A\times B$ is a bipolar-valued multi fuzzy subgroup of $G\times H$. Then the following are true:

(i) If $A_i^+(x) \leq B_i^+(e^l)$ for all x in G and $A_i^-(x) \geq B_i^-(e^l)$ for all x in G, then A is a bipolar-valued multi fuzzy subgroup of G where e^l is identity element of H.

(ii) If $B_i^+(x) \leq A_i^+(e)$ for all x in H and $B_i^-(x) \geq A_i^-(e)$ for all x in H, then B is a bipolar-valued multi fuzzy subgroup of H where e is identity element of G.

(iii) either A is a bipolar-valued multi fuzzy subgroup of G or B is a bipolar-valued multi fuzzy subgroup of H where e and e^l are the identity elements of G and H respectively.

Proof: Let $A\times B$ be a bipolar-valued multi fuzzy subgroup of $G\times H$ and x and y in G. Then (x, e^l) and (y, e^l) are in $G\times H$. Now using the property if $A_i^+(x) \leq B_i^+(e^l)$ for all x in G and $A_i^-(x) \geq B_i^-(e^l)$ for all x in G where e^l is identity element of H we get, $A_i^+(xy^{-1}) = \min\{A_i^+(xy^{-1}), B_i^+(e^l e^l)\}= (A_i\times B_i)^+((xy^{-1}), (e^le^l)) = (A_i\times B_i)^+[(x, e^l)(y^{-1}, e^l)] \geq \min\{(A_i\times B_i)^+(x, e^l), (A_i\times B_i)^+(y^{-1}, e^l)\} = \min\{\min\{A_i^+(x), B_i^+(e^l)\}, \min\{A_i^+(y^{-1}), B_i^+(e^l)\}\}= \min\{A_i^+(x), A_i^+(y^{-1})\}\geq\min\{A_i^+(x), A_i^+(y)\}$. Therefore $A_i^+(xy^{-1}) \geq \min\{A_i^+(x), A_i^+(y)\}$ for all x and y in G. Also $A_i^-(xy^{-1}) = \max\{A_i^-(xy^{-1}), B_i^-(e^l e^l)\} = (A_i\times B_i)^-((xy^{-1}), (e^le^l)) = (A_i\times B_i)^-[(x, e^l)(y^{-1}, e^l)] \leq \max\{(A_i\times B_i)^-(x, e^l), (A_i\times B_i)^-(y^{-1}, e^l)\} = \max\{A_i^-(x), B_i^-(e^l)\}, \max\{A_i^-(y^{-1}), B_i^-(e^l)\}\}= \max\{A_i^-(x), A_i^-(y^{-1})\} \leq \max\{A_i^-(x), A_i^-(y)\}$. Therefore $A_i^-(xy^{-1}) \leq \max\{A_i^-(x), A_i^-(y)\}$ for all x and y in G. Hence A is a bipolar-valued multi fuzzy subgroup of G. Thus (i) is proved. Now using the property $B_i^+(x) \leq A_i^+(e)$ for all x in H and $B_i^-(x) \geq A_i^-(e)$ for all x in H we get, $B_i^+(xy^{-1}) = \min\{B_i^+(xy^{-1}), A_i^+(e.e)\}= (A_i\times B_i)^+((e.e), (xy^{-1})) = (A_i\times B_i)^+[(e, x)(e, y^{-1})] \geq \min\{(A_i\times B_i)^+(e, x), (A_i\times B_i)^+(e, y^{-1})\} = \min\{\min\{A_i^+(e), B_i^+(x)\}, \min\{A_i^+(e), B_i^+(y^{-1})\}\}= \min\{B_i^+(x), B_i^+(y^{-1})\} \geq \min\{B_i^+(x), B_i^+(y)\}$. Therefore $B_i^+(xy^{-1}) \geq \min\{B_i^+(x), B_i^+(y)\}$ for all x and y in H. Also $B_i^-(xy^{-1}) = \max\{B_i^-(xy^{-1}), A_i^-(ee)\} = (A_i\times B_i)^-((ee), (xy^{-1})) = (A_i\times B_i)^-[(e, x)(e, y^{-1})] \leq \max\{(A_i\times B_i)^-(e, x), (A_i\times B_i)^-(e, y^{-1})\} = \max\{\max\{A_i^-(e), B_i^-(x)\}, \max\{A_i^-(e), B_i^-(y^{-1})\}\} = \max\{B_i^-(x), B_i^-(y^{-1})\} \leq \max\{B_i^-(x), B_i^-(y)\}$. Therefore $B_i^-(xy^{-1}) \leq \max\{B_i^-(x), B_i^-(y)\}$ for all x and y in H. Hence B is a bipolar-valued multi fuzzy subgroup of H. Thus (ii) is proved. Hence (iii) is clear.

2.14 Theorem: Let $A = \langle A_i^+, A_i^-\rangle$ be a bipolar-valued multi fuzzy subset of a group (G, .) and $V = \langle V_i^+, V_i^-\rangle$ be the strongest bipolar-valued multi fuzzy relation of G. Then A is a bipolar-valued multi fuzzy subgroup of G if and only if V is a bipolar-valued multi fuzzy subgroup of $G\times G$.

Proof: Suppose that A is a bipolar-valued multi fuzzy subgroup of G. Then for any $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in $G\times G$. We have $V_i^+(xy^{-1}) = V_i^+[(x_1, x_2)(y_1, y_2)^{-1}] = V_i^+(x_1y_1^{-1}, x_2y_2^{-1}) = \min\{A_i^+(x_1y_1^{-1}), A_i^+(x_2y_2^{-1})\} \geq \min\{\min\{A_i^+(x_1), A_i^+(y_1)\}, \min\{A_i^+(x_2), A_i^+(y_2)\}\} = \min\{\min\{A_i^+(x_1), A_i^+(x_2)\}, \min\{A_i^+(y_1), A_i^+(y_2)\}\}= \min\{V_i^+(x_1, x_2), V_i^+(y_1, y_2)\}= \min\{V_i^+(x), V_i^+(y)\}$. Therefore $V_i^+(xy^{-1}) \geq \min\{V_i^+(x), V_i^+(y)\}$ for all x and y in $G\times G$. Also we have $V_i^-(xy^{-1})= V_i^-[(x_1, x_2)(y_1, y_2)^{-1}] = V_i^-(x_1y_1^{-1}, x_2y_2^{-1}) = \max\{A_i^-(x_1y_1^{-1}), A_i^-(x_2y_2^{-1})\}\leq \max\{\max\{A_i^-(x_1), A_i^-(y_1)\}, \max\{A_i^-(x_2), A_i^-(y_2)\}\} = \max\{\max\{A_i^-(x_1), A_i^-(x_2)\}, \max\{A_i^-(y_1), A_i^-(y_2)\}\} = \max\{V_i^-(x_1, x_2), V_i^-(y_1, y_2)\} = \max\{V_i^-(x), V_i^-(y)\}$. Therefore $V_i^-(xy^{-1}) \leq \max\{V_i^-(x), V_i^-(y)\}$ for all x and y in $G\times G$. This proves that V is a bipolar-valued multi fuzzy subgroup of $G\times G$. Conversely, assume that V is a bipolar-valued multi fuzzy subgroup of $G\times G$, then for any $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in $G\times G$, we have $\min\{A_i^+(x_1y_1^{-1}), A_i^+(x_2y_2^{-1})\} = V_i^+(x_1y_1^{-1}, x_2y_2^{-1}) = V_i^+[(x_1, x_2)(y_1, y_2)^{-1}] = V_i^+(xy^{-1}) \geq \min\{V_i^+(x), V_i^+(y)\}= \min\{V_i^+(x_1, x_2), V_i^+(y_1, y_2)\}= \min\{\min\{A_i^+(x_1), A_i^+(x_2)\}, \min\{A_i^+(y_1), A_i^+(y_2)\}\}$. If we put $x_2 = y_2 = e$, we get, $A_i^+(x_1y_1^{-1}) \geq \min\{A_i^+(x_1), A_i^+(y_1)\}$ for all x_1 and y_1 in G. Also we have $\max\{A_i^-(x_1y_1^{-1}), A_i^-(x_2y_2^{-1})\} = V_i^-(x_1y_1^{-1}, x_2y_2^{-1}) = V_i^-[(x_1, x_2)(y_1, y_2)^{-1}] = V_i^-(xy^{-1}) \leq \max\{V_i^-(x), V_i^-(y)\}$

$= \max\{V_i^-(x_1, x_2), V_i^-(y_1, y_2)\} = \max\{\max\{A_i^-(x_1), A_i^-(x_2)\}, \max\{A_i^-(y_1), A_i^-(y_2)\}\}$. If we put $x_2 = y_2 = e$, we get $A_i^-(x_1y_1^{-1}) \leq \max\{A_i^-(x_1), A_i^-(y_1)\}$ for all x_1 and y_1 in G. Hence A is a bipolar-valued multi fuzzy subgroup of G.

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