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ROLE OF LAPLACE TRANSFORMS IN INTEGRAL CALCULUS

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ABSTRACT

In the first part of this survey paper we present one relation of Beta & Gamma function by using double integral and in the second part we present proof of this relation by using Laplace transforms theory.

Key words: *Laplace transforms Beta function, Gamma function,*

INTRODUCTION

We know that Laplace transform is an integral transform which is widely used in solving linear ordinary differential equations. [2] We can solve some complicated definite integrals by using Laplace transforms theory [1,2] In the first part of this survey paper we present one relation of Beta & Gamma function by using double integral [3] In the second part present proof of this relation by using Laplace transforms theory[5].

The main aim of this paper is to show that how Laplace transforms are useful in advanced integral calculus which is one of the important branches of Mathematics. The proof presented in second part is dependent on probability theory [6] which may be used to solve some more problems.

In this paper we present the application of Laplace transform in showing one equality of two Euler's special functions Beta and Gamma in advanced integral calculus.

2. DEFINITIONS AND STANDARD RESULTS

2.1. The Laplace transform: If f(t) is a function defied for all positive value of t then the Laplace transform of F(t) is defined as

$$L\,\{F(t)\} = f(p) = \int_0^\infty e^{-pt} f(t)\,dt \qquad (2.1)$$

Provided that the integral exist.

Where p is a parameter which is a real or a complex number.

If f(p) is a Laplace transform of a function f(t) then f(t) is called the inverse transform of the function f(p) and is written as

$$F(t) = L^{-1}\{f(t)\}$$

2.2. The Laplace transform of

$$t^n (n > -1)\, i\, \frac{\Gamma n+1}{p^{n+1}}, sp > 0. \qquad (2.2)$$

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2.3. Definition of convolution of Laplace transforms

Let F(t) and G(t) be two function of class A then the convolution of two function f(t) and G(t) denoted by F*G is defined by the relation

$$F*G=\int_0^t F(x)G(t-x)\,dx \tag{2.3}$$

2.4 Convolution theorem (statement)

Let F(t) and G(t) be two function of class A and let

$L^{-1}\{f(p)\}=F(t)$ and $L^{-1}\{g(p)\}=G(t)$

Then

$$L^{-1}\{f(p)g(p)\}=\int_0^t F(t)G(t-x)\,dx=F*G \tag{2.4}$$

2.5 Beta function: The Beta function is denoted by B(m,n)

For m>0, n>0 is defined by the relation

$$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx \tag{2.5}$$

2.6 Gamma function: The gamma function $\Gamma(x)$ for x > 0

Is defined by the relation

$$\Gamma(x)=\int_0^\infty e^{-y}y^{x-1}dy$$

This integration is convergent for t > 0

First part of the paper:

First we will prove one relation between Gamma and Beta function using double integral method.

By definition (2.6) we have

$$\Gamma(m)=\int_0^\infty t^{m-1}e^{-t}dt,(m>0)$$

Putting t = x^2 we get

$$\Gamma(m)=\int_0^\infty x^{2m-2}e^{-x^2}\,2xdx$$

$$\Gamma(m)=2\int_0^\infty x^{2m-1}e^{-x^2}\,dx \tag{2.6}$$

Similarly

$$\Gamma(n)=2\int_0^\infty y^{2n-1}e^{-y^2}\,dy \tag{2.7}$$

Multiplying (2.7) & (2.8) we have

$$\Gamma(m)\Gamma(n)=2\int_0^\infty x^{2m-2}e^{-x^2}dx*2\int_0^\infty y^{2n-1}e^{-y^2}dy$$

$$=4\int_0^\infty\int_0^\infty x^{2m-1}y^{2n-1}e^{-\left(x^2+y^2\right)}dxdy$$

Transforming to polar co-ordinates we have

$$\Gamma(m)\Gamma(n)=4\int_0^{\frac{\pi}{2}}\int_0^\infty(\mathrm{r}\cos\theta)^{2m-1}(\mathrm{r}\sin\theta)^{2n-1}e^{-r^2}\ \mathrm{r}d\theta dr$$

$$=4\int_0^{\frac{\pi}{2}}\int_0^\infty\cos^{2m-1}\theta\sin^{2n-1}\theta r^{2m+2n-1}e^{-r^2}d\theta dr$$

$$= 4\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta \cos^{2m-1}\theta d\theta \int_0^{\infty} r^{2m+2n-1} e^{-r^2} dr$$

$$\Gamma(m)\Gamma(n) = 2\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta \cos^{2m-1}\theta d\theta \quad \Gamma(m+n), \text{ From (2.7)}$$

Hence $$\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta \cos^{2m-1}\theta d\theta = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)} \tag{2.8}$$

The result (2.8) is true for all positive values of m and n

Substituting x=Sin$^2\theta$ in def (2.5) we have

$$B(m,n) = \int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-2} (\cos\theta)^{2n-2}\, 2\sin\theta\cos\theta dx$$

$$= \int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1} d\theta$$

$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \tag{2.9}$$

This is the relation between gamma and Beta function.

Second part of the paper (Required Result)

Now we will derive the above result (2.9) by using the theory of Laplace transforms. We will use the convolution theorem (2.4) of Laplace Transform to derive this result.

Consider the function,

$$F(t) = \int_0^t x^{m-1}(1-x)^{n-1} dx$$

We have

$$F(t) = \int_0^t F(x)G(t-x)dx, \text{ where } F_1(t) = t^{m-1} \,\&\, F_2(t) = t^{n-1}$$

$$= F_1 * F_2$$

Taking Laplace transform of both sides we have

$$L\{F(t)\} = L\{F_1 * F_2\}$$

$$= L\{F_1(t)\} * L\{F_2(t)\}$$

$$= L\{t^{m-1}\} L\{t^{n-1}\}$$

$$= \frac{\Gamma m}{p^m}\frac{\Gamma n}{p^n} = \frac{\Gamma m \Gamma n}{p^{m+n}} \tag{2.10}$$

Taking inverse Laplace transform of both sides of (2.10) we obtain

$$L^{-1}\{L(F(t))\} = F(t) = \int_0^t x^{m-1}(t-x)^{n-1} d$$

$$= L^{-1}\left\{\frac{\Gamma m \Gamma n}{p^{m+n}}\right\}$$

$$= \Gamma m \Gamma n L^{-1}\left\{\frac{1}{p^{m+n}}\right\}$$

$$= \frac{\Gamma m \Gamma n}{\Gamma(m+n)} t^{m+n-1} \tag{2.11}$$

Now putting t=1 in (2.11) we get

$$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx=\frac{\Gamma m\Gamma n}{\Gamma(m+n)}$$

CONCLUDING REMARKS:

By deriving relation in the second part of this paper we conclude that the Laplace transform can be used to derive such other relations. Thus finally we came to conclusion that Laplace transforms have a major role in integral calculus.

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