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FUZZY TRANSLATIONS AND FUZZY MULTIPLICATIONS OF INTERVAL-VALUED FUZZY *BG*-ALGEBRAS

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ABSTRACT

In this paper, we introduced the concept of fuzzy translations, fuzzy multiplications and fuzzy extensions of interval-valued fuzzy subalgebras of BG *-algebras and investigated some of their basic properties.*

Keywords: BG *-algebra, Interval-valued fuzzy set, Interval-valued fuzzy subalgebras, Fuzzy translation, Fuzzy multiplication, Fuzzy extension.*

AMS Subject Classification(2010): *06F35, 03E72, 03G25, 08A72.*

1. INTRODUCTION

In 1966, Imai and Iseki [4] introduced the two classes of logical algebras viz. BCK-algebras and BCI-algebras. It is known that the notion of BCI-algebra is a generalization of notion of BCK-algebras. In 2002, Neggers and Kim [9] introduced a new notion, called B-algebras which are related to wide classes of algebras such as BCI/BCK-algebras[5]. Kim and Kim [6] introduced the notion of BG-algebra which is a generalisation of B-algebra. The concept of a fuzzy set, was introduced by Zadeh[12]. In [1] Ahn and Lee applied the fuzzy notions to BG-algebras and introduced the notion of fuzzy BG subalgebras. In [13] Zadeh made an extension of the concept of fuzzy set by an interval-valued fuzzy set. Atanassov and Gargov [2] introduced the concept of interval-valued intuitionistic fuzzy set. In [3] different operators over interval-valued intuitionistic fuzzy sets are defined. A.B.Saeid [10] defined interval-valued fuzzy BG-algebras. The concept of fuzzy translation, fuzzy multiplication and fuzzy extension in fuzzy subalgebras and fuzzy ideals in BCK/BCI -algebras and fuzzy groups has been discussed in [7, 8, 11] . Motivated by this, In this paper we introduced fuzzy translation, fuzzy multiplication and fuzzy extension in interval-valued fuzzy BG-subalgebras.

2.PRELIMINARIES

Definition 2.1: A BG-algebra is a non-empty set X with a constant ‘0 ‘and a binary operation ‘*’ satisfying following axioms:

$(i)\ x*x=0,$

$(ii)\ x*0=x,$

$(iii)\ (x*y)*(0*y)=x,\ \forall\ x,y\in X.$

For brevity we also call X a BG-algebra.

Example 2.2: Let X={ 0, 1, 2, 3, 4} with the following caley table

*	0	1	2	3	4
0	0	4	3	2	1
1	1	0	4	3	2
2	2	1	0	4	3
3	3	2	1	0	4
4	4	3	2	1	0

Then (X, *, 0) is a BG-algebra.

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3. INTERVAL-VALUED FUZZY SETS

The notion of interval-valued fuzzy set was introduced by L.A.Zadeh[13]. To consider the notion of interval-valued fuzzy sets, we need the following definitions. An interval number on $[0,1]$, denoted by $\hat{a}$, is defined as the closed sub interval of $[0,1]$, where $\hat{a} = [\underline{a}, \overline{a}]$,satisfying $0 \le \underline{a} \le \overline{a} \le 1$. Let $D[0,1]$ denote the set of all such interval numbers on $[0,1]$ and also denote the interval numbers $[0,0]$ and $[1,1]$ by $\hat{0}$ and $\hat{1}$ respectively.

Let $\hat{a}_1 = [\underline{a_1}, \overline{a_1}]$ and $\hat{a}_2 = [\underline{a_2}, \overline{a_2}] \in D[0,1]$. Define on D[0, 1] the relations $\le, =, <, +, .$ by

1. $\hat{a}_1 \le \hat{a}_2 \Leftrightarrow \underline{a_1} \le \underline{a_2}$ and $\overline{a_1} \le \overline{a_2}$
2. $\hat{a}_1 = \hat{a}_2 \Leftrightarrow \underline{a_1} = \underline{a_2}$ and $\overline{a_1} = \overline{a_2}$
3. $\hat{a}_1 < \hat{a}_2 \Leftrightarrow \underline{a_1} < \underline{a_2}$ and $\overline{a_1} < \overline{a_2}$
4. $\hat{a}_1 + \hat{a}_2 \Leftrightarrow [\underline{a_1} + \underline{a_2}, \overline{a_1} + \overline{a_2}]$
5. $\hat{a}_1.\hat{a}_2 \Leftrightarrow [min(\underline{a_1}\underline{a_2}, \underline{a_1}\overline{a_2}, \overline{a_1}\underline{a_2}, \overline{a_1}\overline{a_2}), max(\underline{a_1}\underline{a_2}, \underline{a_1}\overline{a_2}, \overline{a_1}\underline{a_2}, \overline{a_1}\overline{a_2})] = [\underline{a_1}\underline{a_2}, \overline{a_1}\overline{a_2}]$
6. $k\hat{a} = [k\underline{a}, k\overline{a}]$ where $0 \le k \le 1$

Now consider two intervals $\hat{a}_1 = [\underline{a_1}, \overline{a_1}], \hat{a}_2 = [\underline{a_2}, \overline{a_2}] \in D[0,1]$ then we define refine minimum rmin as $rmin(\hat{a}_1, \hat{a}_2) = [min(\underline{a_1}, \underline{a_2}), min(\overline{a_1}, \overline{a_2})]$ and refine maximum rmax as $rmax(\hat{a}_1, \hat{a}_2) = [max(\underline{a_1}, \underline{a_2}), max(\overline{a_1}, \overline{a_2})]$ generally if $\hat{a}_i = [\underline{a_1}, \overline{a_i}], \hat{b}_i = [\underline{b_1}, \overline{b_i}] \in D[0,1]$ for i= 1,2,3,...then we define $rmax(\hat{a}_i, \hat{b}_i) = [max(\underline{a_i}, \underline{b_i}), max(\overline{a_i}, \overline{b_i})]$ and $rmin(\hat{a}_i, \hat{b}_i) = [min(\underline{a_i}, \underline{b_i}), min(\overline{a_i}, \overline{b_i})]$ and $rinf_i(\hat{a}_i) = [\wedge_i \underline{a_i}, \wedge_i \overline{a_i}]$ and $rsup_i(\hat{a}_i) = [\vee_i \underline{a_i}, \vee_i \overline{a_i}]$

$(D[0,1], \le)$ is a complete lattice with $\wedge = rmin, \vee = rmax, \hat{0} = [0,0]$ and $\hat{1} = [1, 1]$ being the least and the greatest element respectively.

Definition 3.1 *An interval-valued fuzzy set defined on a non empty set X as an objects having the form* $\hat{\mu} = \{x, [\underline{\mu}(x), \overline{\mu}(x)]\}, \forall x \in X$ *where* $\underline{\mu}$ *and* $\overline{\mu}$ *are two fuzzy sets in X such that* $\underline{\mu}(x) \le \overline{\mu}(x)$ *for all* $x \in X$. *Let* $\hat{\mu}(x) = [\underline{\mu}(x), \overline{\mu}(x)], \forall x \in X$ *, then* $\hat{\mu}(x) \in D[0\,1], \forall x \in X$.

If $\hat{\mu}$ and $\hat{\nu}$ be two interval-valued fuzzy sets in X, then we define

- $\hat{\mu} \subset \hat{\nu} \Leftrightarrow$ for all $\underline{\mu}(x) \le \underline{\nu}(x)$ and $\overline{\mu}(x) \le \overline{\nu}(x)$.
- $\hat{\mu} = \hat{\nu} \Leftrightarrow$ for all $\underline{\mu}(x) = \underline{\nu}(x)$ and $\overline{\mu}(x) = \overline{\nu}(x)$.
- $(\hat{\mu} \cup \hat{\nu})(x) = \hat{\mu}(x) \vee \hat{\nu}(x) = [max\{\underline{\mu}(x), \underline{\nu}(x)\}, max\{\overline{\mu}(x), \overline{\nu}(x)\}]$.
- $(\hat{\mu} \cap \hat{\nu})(x) = \hat{\mu}(x) \wedge \hat{\nu}(x) = [min\{\underline{\mu}(x), \underline{\nu}(x)\}, min\{\overline{\mu}(x), \overline{\nu}(x)\}]$.
- $(\hat{\mu} \times \hat{\nu})(x, y) = \hat{\mu}(x) \wedge \hat{\nu}(y) = [min\{\underline{\mu}(x), \underline{\nu}(y)\}, min\{\overline{\mu}(x), \overline{\nu}(y)\}]$.
- $\hat{\mu}^c(x) = [1 - \overline{\mu}(x), 1 - \underline{\mu}(x)]$.

Definition 3.2 *Let* $\hat{\mu}$ *be an interval-valued fuzzy set in X. Then for every* $[0,0] < \hat{t} \le [1,1]$ *, the crisp set* $\hat{\mu}_t = \{x \in X \mid \hat{\mu}(x) \ge \hat{t}\}$ *is called the level subset of* $\hat{\mu}$.

Definition 3.3 *([10]) An interval-valued fuzzy set* $\hat{\mu}$ *in BG-algebra X is called an interval-valued fuzzy BG-subalgebra of X if* $\hat{\mu}(x * y) \ge rmin\{\hat{\mu}(x), \hat{\mu}(y)\}$ *for all* $x, y \in X$.

Example 3.4 *Consider* BG *-algebra* $X = \{0,1,2,3\}$ *with the following cayley table.*

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Define $\hat{\mu} : X \to D[0,1]$ by $\hat{\mu}(0) = \hat{\mu}(2) = [0.4,0.9], \hat{\mu}(1) = \hat{\mu}(3) = [0.3,0.5]$ then it is easy to verify that $\hat{\mu}$ is an interval-valued fuzzy BG-subalgebra of X.

Remark 3.5: *([10]) If* $\hat{\mu}_1$ *and* $\hat{\mu}_2$ *be any two interval-valued fuzzy BG-subalgebras of X. Then their intersection* $(\hat{\mu}_1 \cap \hat{\mu}_2)$ *is also an interval-valued fuzzy BG-subalgebra of X.*

4. FUZZY TRANSLATION AND FUZZY MULTIPLICATION OF INTERVAL-VALUED FUZZY BG-ALGEBRAS

In what follows X denotes a BG-algebra, and for any interval-valued fuzzy set $\hat{\mu} = [\underline{\mu}, \overline{\mu}]$ of X, we denote $\overline{T} = 1 - sup\{\overline{\mu}(x) \mid x \in X\}$ where $\hat{T} = (\underline{T}, \overline{T})$ such that $\underline{T} \le \overline{T}$.

Definition 4.1: *Let* $\hat{\mu}$ *an interval-valued fuzzy subset of X and* $0 \le \overline{\alpha} \le \overline{T}$ *where* $\hat{\alpha} = [\underline{\alpha}, \overline{\alpha}]$. *A mapping* $\hat{\mu}^T_{\hat{\alpha}} : X \to D[0,1]$ *is said to be an fuzzy* $\hat{\alpha}$ *translation of* $\hat{\mu}$ *if it satisfies* $\hat{\mu}^T_{\hat{\alpha}}(x) = \hat{\mu}(x) + \hat{\alpha}$ *for all* $x \in X$.

Example 4.2: *Consider the interval-valued fuzzy set* $\hat{\mu}$ *as in Example 3.4,* $\overline{T} = 1 - 0.9 = 0.1$ *let* $\hat{\alpha} = [0.02, 0.09] \in [\hat{0}, \hat{T}]$ *Then the* $\hat{\alpha}$ *translation of interval-valued fuzzy set* $\hat{\mu}$ *is given by*

$$\hat{\mu}^T_{\hat{\alpha}}(0) = \hat{\mu}^T_{\hat{\alpha}}(2) = [0.42,0.99], \hat{\mu}^T_{\hat{\alpha}}(1) = \hat{\mu}^T_{\hat{\alpha}}(3) = [0.32,0.59].$$

Definition 4.3: *Let* $\hat{\mu}$ *an interval-valued fuzzy subset of X and* $\hat{\beta} \in D[0,1]$ *.A mapping* $\hat{\mu}^M_{\hat{\beta}} : X \to D[0,1]$ *is said to be an fuzzy* $\hat{\beta}$ *multiplication of* $\hat{\mu}$ *if it satisfies* $\hat{\mu}^M_{\hat{\beta}}(x) = \hat{\beta}.\hat{\mu}(x)$ *for all* $x \in X$.

Example 4.4: *Consider the interval-valued fuzzy set* $\hat{\mu}$ *as in Example 3.4, let* $\hat{\beta} = [0.3,0.6]$ *Then the* $\hat{\beta}$ *multiplication of interval-valued fuzzy set* $\hat{\mu}$ *is given by*

$$\hat{\mu}^M_{\hat{\beta}}(0) = \hat{\mu}^M_{\hat{\beta}}(2) = [0.12,0.54], \hat{\mu}^M_{\hat{\beta}}(1) = \hat{\mu}^M_{\hat{\beta}}(3) = [0.09,0.30].$$

Definition 4.5: *Let* $\hat{\mu}$ *an interval-valued fuzzy subset of X and* $\hat{\alpha} \in [\hat{0}, \hat{T}]$ *and* $\hat{\beta} \in D[0,1]$. *A mapping* $\hat{\mu}^{MT}_{\hat{\beta}\hat{\alpha}} : X \to D[0,1]$ *is said to be a fuzzy magnified* $(\hat{\beta}\hat{\alpha})$ *translation of* $\hat{\mu}$ *if it satisfies* $\hat{\mu}^{MT}_{\hat{\beta}\hat{\alpha}}(x) = \hat{\beta}.\hat{\mu}(x) + \hat{\alpha}$ *for all* $x \in X$.

Theorem 4.6: *Let* $\hat{\mu}$ *be an interval-valued fuzzy subset of a BG-algebra X. Let* $\hat{\alpha} \in [\hat{0}, \hat{T}]$ *and* $\hat{\beta} \in D[0,1]$, *then* $\hat{\mu}$ *is interval-valued fuzzy BG-subalgebra X iff* $\hat{\mu}^{MT}_{\hat{\beta}\hat{\alpha}}$ *is an interval-valued fuzzy BG-subalgebra X.*

Proof: Let $\hat{\mu}$ be an interval-valued fuzzy BG-subalgebra X, then for $x, y \in X$, we have

$$\begin{aligned}\hat{\mu}^{MT}_{\hat{\beta}\hat{\alpha}}(x * y) &= \hat{\beta}.\hat{\mu}(x * y) + \hat{\alpha} \ge \hat{\beta}.rmin\{\hat{\mu}(x), \hat{\mu}(y)\} + \hat{\alpha} \\ &= rmin\{\hat{\beta}.\hat{\mu}(x) + \hat{\alpha}, \hat{\beta}.\hat{\mu}(y) + \hat{\alpha}\} \\ &= rmin\{\hat{\mu}^{MT}_{\hat{\beta}\hat{\alpha}}(x), \hat{\mu}^{MT}_{\hat{\beta}\hat{\alpha}}(y)\}.\end{aligned}$$

Hence $\hat{\mu}^{MT}_{\hat{\beta}\hat{\alpha}}$ is an interval-valued fuzzy BG-subalgebra X.

Conversely, assume $\hat{\mu}^{MT}_{\hat{\beta}\hat{\alpha}}$ is an interval-valued fuzzy BG-subalgebra X, then for $x, y \in X$, we have

$$\begin{aligned}\hat{\beta}.\hat{\mu}(x*y)+\hat{\alpha} &= \hat{\mu}^{MT}_{\hat{\beta}\hat{\alpha}}(x*y) \geq rmin\{\hat{\mu}^{MT}_{\hat{\beta}\hat{\alpha}}(x), \hat{\mu}^{MT}_{\hat{\beta}\hat{\alpha}}(y)\} \\ &= rmin\ \{\hat{\beta}.\ \hat{\mu}(x)+\hat{\alpha}, \hat{\beta}.\hat{\mu}(y)+\hat{\alpha}\} \\ &= \hat{\beta}.rmin\{\hat{\mu}(x), \hat{\mu}(y)\}+\hat{\alpha}\end{aligned}$$

$$\Rightarrow \hat{\beta}.\hat{\mu}(x*y)+\hat{\alpha} \geq \hat{\beta}.rmin\{\hat{\mu}(x), \hat{\mu}(y)\}+\hat{\alpha}.$$

which implies $\hat{\mu}(x*y) \geq rmin\{\hat{\mu}(x), \hat{\mu}(y)\}$ for all $x, y \in X$.

Hence $\hat{\mu}^{MT}_{\hat{\beta}\hat{\alpha}}$ is an interval-valued fuzzy BG-subalgebra X.

Corollary 4.7: *Let $\hat{\mu}$ be an interval-valued fuzzy subset of a BG-algebra X and $\hat{\alpha} \in [\hat{0}, \hat{T}]$ then $\hat{\mu}$ is an interval-valued fuzzy BG-subalgebra X iff $\hat{\mu}^{T}_{\hat{\alpha}}$ is an interval-valued fuzzy BG-subalgebra X.*

Proof: Put $\hat{\beta} = \hat{1}$ in Theorem 4.6.

Corollary 4.8: *Let $\hat{\mu}$ be an interval-valued fuzzy subset of a BG-algebra X and $\hat{\beta} \in D[0,1]$ then $\hat{\mu}$ is an interval-valued fuzzy BG-subalgebra X iff $\hat{\mu}^{M}_{\hat{\beta}}$ is an interval-valued fuzzy BG-subalgebra X.*

Proof: Put $\hat{\alpha} = \hat{0}$ in Theorem 4.6.

Theorem 4.9: *Let $\hat{\mu}$ be an interval-valued fuzzy BG-subalgebra X, then $\hat{\mu}^{MT}_{\hat{\beta}\hat{\alpha}}(0) \geq \hat{\mu}^{MT}_{\hat{\beta}\hat{\alpha}}(x)$ for all $x \in X$.*

Proof: Here $\hat{\mu}$ be an interval-valued fuzzy BG-subalgebra X therefore $\hat{\mu}^{MT}_{\hat{\beta}\hat{\alpha}}$ is an interval-valued fuzzy BG-subalgebra X.

$$\begin{aligned}\hat{\mu}^{MT}_{\hat{\beta}\hat{\alpha}}(0) = \hat{\mu}^{MT}_{\hat{\beta}\hat{\alpha}}(x*x) &= \hat{\beta}.\hat{\mu}(x*x)+\hat{\alpha} \\ &\geq \hat{\beta}.rmin\{\hat{\mu}(x), \hat{\mu}(x)\}+\hat{\alpha} \\ &= rmin\ \{\hat{\beta}.\hat{\mu}(x)+\hat{\alpha}, \hat{\beta}.\hat{\mu}(x)+\hat{\alpha}\} \\ &= rmin\{[\underline{\hat{\beta}}.\underline{\hat{\mu}}(x)+\underline{\hat{\alpha}}, \overline{\hat{\beta}}.\overline{\hat{\mu}}(x)+\overline{\hat{\alpha}}], [\underline{\hat{\beta}}.\underline{\hat{\mu}}(x)+\underline{\hat{\alpha}}, \overline{\hat{\beta}}.\overline{\hat{\mu}}(x)+\overline{\hat{\alpha}}]\} \\ &= [min\ \{\underline{\hat{\beta}}.\underline{\hat{\mu}}(x)+\underline{\hat{\alpha}}, \underline{\hat{\beta}}.\underline{\hat{\mu}}(x)+\underline{\hat{\alpha}}\}, min\{\overline{\hat{\beta}}.\overline{\hat{\mu}}(x)+\overline{\hat{\alpha}}, \overline{\hat{\beta}}.\overline{\hat{\mu}}(x)+\overline{\hat{\alpha}}\}] \\ &= \hat{\mu}^{MT}_{\hat{\beta}\hat{\alpha}}(x)\end{aligned}$$

$$\Rightarrow \hat{\mu}^{MT}_{\hat{\beta}\hat{\alpha}}(0) \geq \hat{\mu}^{MT}_{\hat{\beta}\hat{\alpha}}(x).$$

Theorem 4.10: *Let $\hat{\mu}^{MT}_{\hat{\beta}\hat{\alpha}}$ be an interval-valued fuzzy BG-subalgebra X where $\hat{\alpha} \in [\hat{0}, \hat{T}]$ and $\hat{\beta}, \hat{t} \in D[0,1]$ with $\hat{t} \geq \hat{\alpha}$, then the level subset $U^{MT}_{\beta,\alpha}(\hat{\mu}, \hat{t}) = \{x \in X \mid \hat{\beta}.\hat{\mu}(x) \geq \hat{t} - \hat{\alpha}\}, \forall\ \hat{t} \in Im(\hat{\mu})$ is a subalgebra of X.*

Proof: Let $x, y \in U^{MT}_{\beta,\alpha}(\hat{\mu}, \hat{t}) \Rightarrow \hat{\beta}.\hat{\mu}(x) \geq \hat{t} - \hat{\alpha}$ and $\hat{\beta}.\hat{\mu}(y) \geq \hat{t} - \hat{\alpha}$

$\Rightarrow \hat{\beta}.\hat{\mu}(x)+\hat{\alpha} \geq \hat{t}$ and $\hat{\beta}.\hat{\mu}(y)+\hat{\alpha} \geq \hat{t}$

$\Rightarrow \hat{\mu}^{MT}_{\hat{\beta}\hat{\alpha}}(x) \geq \hat{t}$ and $\hat{\mu}^{MT}_{\hat{\beta}\hat{\alpha}}(y) \geq \hat{t}$, Now

$$\begin{aligned}\hat{\mu}^{MT}_{\beta\hat{\alpha}}(x*y) &\geq rmin\ \{\hat{\mu}^{MT}_{\beta\hat{\alpha}}(x), \hat{\mu}^{MT}_{\beta\hat{\alpha}}(y)\}\\ &\geq rmin\,\{\hat{t},\hat{t}\}\\ &= rmin\{[\underline{\hat{t}},\overline{\hat{t}}],[\underline{\hat{t}},\overline{\hat{t}}]\}\\ &= [min\{\underline{\hat{t}},\underline{\hat{t}}\}, min\{\overline{\hat{t}},\overline{\hat{t}}\}]\\ &= [\underline{\hat{t}},\overline{\hat{t}}] = \hat{t}\end{aligned}$$

$$\Rightarrow \hat{\beta}.\hat{\mu}(x*y)+\hat{\alpha} \geq \hat{t}$$

$$\Rightarrow (x*y) \in U^{MT}_{\hat{\beta},\alpha}(\hat{\mu},\hat{t}).$$

Theorem 4.11: *Intersection and union of any two fuzzy translation of an interval-valued fuzzy BG-subalgebra of X is also an interval-valued fuzzy BG-subalgebra of X.*

Proof: Let $\hat{\mu}^T_{\hat{\alpha}}$ and $\hat{\mu}^T_{\hat{\delta}}$ be two fuzzy translatons of an interval-valued fuzzy BG-subalgebra $\hat{\mu}$, where $\hat{\alpha},\hat{\delta} \in [\hat{0},\hat{T}]$. Assume that $\hat{\alpha} \leq \hat{\delta}$ By Corollary 4.7 $\hat{\mu}^T_{\hat{\alpha}}$ and $\hat{\mu}^T_{\hat{\delta}}$ are interval-valued fuzzy BG-subalgebras of X.

Now

$$\begin{aligned}(\hat{\mu}^T_{\hat{\alpha}} \cup \hat{\mu}^T_{\hat{\delta}})(x) &= rmax\{\hat{\mu}^T_{\hat{\alpha}}(x), \hat{\mu}^T_{\hat{\delta}}(x)\}\\ &= rmax\,\{\hat{\mu}(x)+\hat{\alpha}, \hat{\mu}(x)+\hat{\delta}\}\\ &= rmax\{[\underline{\hat{\mu}}(x)+\underline{\hat{\alpha}}, \overline{\hat{\mu}}(x)+\overline{\hat{\alpha}}],[\underline{\hat{\mu}}(x)+\underline{\hat{\delta}}, \overline{\hat{\mu}}(x)+\overline{\hat{\delta}}]\}\\ &= [max\{\underline{\hat{\mu}}(x)+\underline{\hat{\alpha}}, \underline{\hat{\mu}}(x)+\underline{\hat{\delta}}\}, max\{\overline{\hat{\mu}}(x)+\overline{\hat{\alpha}}, \overline{\hat{\mu}}(x)+\overline{\hat{\delta}}\}]\\ &= [\underline{\hat{\mu}}(x)+\underline{\hat{\delta}}, \overline{\hat{\mu}}(x)+\overline{\hat{\delta}}]\\ &= \hat{\mu}(x)+\hat{\delta} = \hat{\mu}^T_{\hat{\delta}}(x).\end{aligned}$$

Also

$$\begin{aligned}(\hat{\mu}^T_{\hat{\alpha}} \cap \hat{\mu}^T_{\hat{\delta}})(x) &= rmin\{\hat{\mu}^T_{\hat{\alpha}}(x), \hat{\mu}^T_{\hat{\delta}}(x)\}\\ &= rmin\,\{\hat{\mu}(x)+\hat{\alpha}, \hat{\mu}(x)+\hat{\delta}\}\\ &= rmin\{[\underline{\hat{\mu}}(x)+\underline{\hat{\alpha}}, \overline{\hat{\mu}}(x)+\overline{\hat{\alpha}}],[\underline{\hat{\mu}}(x)+\underline{\hat{\delta}}, \overline{\hat{\mu}}(x)+\overline{\hat{\delta}}]\}\\ &= [min\{\underline{\hat{\mu}}(x)+\underline{\hat{\alpha}}, \underline{\hat{\mu}}(x)+\underline{\hat{\delta}}\}, min\{\overline{\hat{\mu}}(x)+\overline{\hat{\alpha}}, \overline{\hat{\mu}}(x)+\overline{\hat{\delta}}\}]\\ &= [\underline{\hat{\mu}}(x)+\underline{\hat{\alpha}}, \overline{\hat{\mu}}(x)+\overline{\hat{\alpha}}]\\ &= \hat{\mu}(x)+\hat{\alpha} = \hat{\mu}^T_{\hat{\alpha}}(x).\end{aligned}$$

5. FUZZY TRANSLATIONS AND FUZZY MULTIPLICATIONS OF INTERVA-VALUED FUZZY BG -ALGEBRAS UNDER HOMOMORPHISM

Definition 5.1: *Let X and Y be two BG-algebras. Then a mapping $f: X \to Y$ is said to be homomorphism if* $f(x*y) = f(x)*f(y), \forall x,y \in X$.

Theorem 5.2: *Let $f: X \to Y$ be a homomorphism of BG-algebras. If $\hat{\mu}$ be an interval-valued fuzzy subalgebras of Y, then $f^{-1}(\hat{\mu}^T_{\hat{\alpha}})$ an interval-valued fuzzy subalgebra of X.*

Proof: Let $\hat{\mu}$ be an interval-valued fuzzy subalgebras of Y, therefore by Corollary 4.7 $\hat{\mu}^T_{\hat{\alpha}}$ is also an interval-valued fuzzy subalgebras of Y. Now $f^{-1}(\hat{\mu})^T_{\hat{\alpha}}$ is defined by $f^{-1}(\hat{\mu}^T_{\hat{\alpha}})(x) = \hat{\mu}^T_{\hat{\alpha}}(f(x))\ \forall x \in X$. Let $x, y, \in X$

Now

$$\begin{aligned} f^{-1}(\hat{\mu}^T_{\hat{\alpha}})(x*y) &= \hat{\mu}^T_{\hat{\alpha}}\{f(x*y)\} \\ &= \hat{\mu}^T_{\hat{\alpha}}\{f(x)*f(y)\} \geq rmin\{\hat{\mu}^T_{\hat{\alpha}}(f(x)), \hat{\mu}^T_{\hat{\alpha}}(f(y))\} \\ &= rmin\{f^{-1}(\hat{\mu}^T_{\hat{\alpha}})(x), f^{-1}(\hat{\mu}^T_{\hat{\alpha}})(y)\}. \end{aligned}$$

Hence $f^{-1}(\hat{\mu}^T_{\hat{\alpha}})$ an interval-valued fuzzy subalgebras of X.

Theorem 5.3: *Let $f: X \rightarrow Y$ be a homomorphism of BG-algebras. If $\hat{\mu}$ be an interval- valued fuzzy subalgebras of Y, then $f^{-1}(\hat{\mu}^M_{\hat{\beta}})$ an interval-valued fuzzy subalgebra of X.*

Proof. Same as Theorem 5.2.

Theorem 5.4: *Let $f: X \rightarrow Y$ be an onto homomorphism of BG-algebras. If $\hat{\mu}$ be an interval-valued fuzzy subset of Y such that $f^{-1}(\hat{\mu}^T_{\hat{\alpha}})$ is an interval-valued fuzzy subalgebra of X. Then $\hat{\mu}$ is also an interval-valued fuzzy subalgebra of Y.*

Proof: Since f is onto homomorphism therefore to each $x', y' \in Y$, there exists $x, y \in X$ such that $f(x) = x'$, $f(y) = y'$ and $f(x*y) = f(x)*f(y) = x'*y'$ .Now since $f^{-1}(\hat{\mu}^T_{\hat{\alpha}})$ is an interval-valued fuzzy subalgebras of X.
Therefore

$$f^{-1}(\hat{\mu}^T_{\hat{\alpha}})(x*y) \geq rmin\{f^{-1}(\hat{\mu}^T_{\hat{\alpha}})(x), f^{-1}(\hat{\mu}^T_{\hat{\alpha}})(y)\}$$

$$\Rightarrow \hat{\mu}^T_{\hat{\alpha}} f(x*y) \geq rmin\{\hat{\mu}^T_{\hat{\alpha}} f(x), \hat{\mu}^T_{\hat{\alpha}} f(y)\}$$

$$\Rightarrow \hat{\mu}^T_{\hat{\alpha}}\{f(x)*f(y)\} \geq rmin\{\hat{\mu}^T_{\hat{\alpha}} f(x), \hat{\mu}^T_{\hat{\alpha}} f(y)\}$$

$$\Rightarrow \hat{\mu}^T_{\hat{\alpha}}\{(x'*y'\} \geq rmin\{\hat{\mu}^T_{\hat{\alpha}}(x'), \hat{\mu}^T_{\hat{\alpha}}(y')\}.$$

$\Rightarrow \hat{\mu}^T_{\hat{\alpha}}$ is an interval valued fuzzy subalgebra of Y. Hence By Corollary 4.7 $\hat{\mu}$ is an interval valued fuzzy subalgebra of Y.

Theorem 5.5: *Let $f: X \rightarrow Y$ be an onto homomorphism of BG-algebras. If $\hat{\mu}$ be an interval-valued fuzzy subset of Y such that $f^{-1}(\hat{\mu}^M_{\hat{\beta}})$ is an interval-valued fuzzy subalgebra of X. Then $\hat{\mu}$ is also an interval-valued fuzzy subalgebra of Y.*

Proof: Same as Theorem 5.4.

Theorem 5.6: *Let $f: X \rightarrow Y$ be an epimorphism, where X, Y are two two interval-valued fuzzy subalgebras and $\hat{\alpha} \in [\hat{0}, \hat{T}]$. Then the inverse image of $\hat{\alpha}$ translation of any fuzzy subalgebra $\hat{\mu}$ of Y is same as the α translation of the inverse image of fuzzy subalgebra $\hat{\mu}$.*

Proof: Let $f: X \rightarrow Y$ be an epimorphism, where X, Y are two two interval-valued fuzzy subalgebras and $\hat{\alpha} \in [\hat{0}, \hat{T}]$. Let $\hat{\mu}$ be a fuzzy subalgebra of Y. Therefore by Theorem 4.6 The $\hat{\alpha}$ translation $\hat{\mu}^T_{\hat{\alpha}}$ is a fuzzy subalgebra of Y. Therefore by Corollary 4.7 $f^{-1}(\hat{\mu}^T_{\hat{\alpha}})$ is a fuzzy subalgebra of X.

Also $f^{-1}(\hat{\mu}^T_{\hat{\alpha}})(x) = \hat{\mu}^T_{\hat{\alpha}}(f(x)) = \hat{\mu}(f(x)) + \hat{\alpha} = f^{-1}(\hat{\mu})(x) + \hat{\alpha} = (f^{-1}(\hat{\mu}))^T_{\hat{\alpha}}(x)$.

Hence $f^{-1}(\hat{\mu}^T_{\hat{\alpha}}) = (f^{-1}(\hat{\mu}))^T_{\hat{\alpha}}$

Theorem 5.7: *Let $f: X \rightarrow Y$ be an epimorphism, where X, Y are two two interval- valued fuzzy subalgebras and $\hat{\beta} \in D[0,1]$. Then the inverse image of $\hat{\beta}$ multiplication of any fuzzy subalgebra $\hat{\mu}$ of Y is same as the β multiplication of the inverse image of fuzzy subalgebra $\hat{\mu}$.*

Proof: Same as Theorem 5.6.

6. INTERVAL-VALUED FUZZY SUBALGEBRA EXTENSION

Definition 6.1: *Let $\hat{\mu}_1$ and $\hat{\mu}_2$ be two interval-valued fuzzy subsets of X. If $\hat{\mu}_1(x) \le \hat{\mu}_2(x)$ for all $x \in X$, then we say that $\hat{\mu}_2$ is a fuzzy (an interval-valued) S - extension of $\hat{\mu}_1$.*

Definition 6.2: *Let $\hat{\mu}_1$ and $\hat{\mu}_2$ be two interval-valued fuzzy subsets of X such that $\hat{\mu}_2$ is a fuzzy extension of $\hat{\mu}_1$. If $\hat{\mu}_1$ is an interval-valued fuzzy sub algebra of X implies that $\hat{\mu}_2$ is an interval-valued fuzzy sub algebra of X, then $\hat{\mu}_2$ is called fuzzy S-extension of $\hat{\mu}_1$.*

Theorem 6.3: *Let $\hat{\mu}$ be an interval-valued fuzzy subalgebra of X and $\hat{\alpha} \in [\hat{0}, \hat{T}]$, then the fuzzy $\hat{\alpha}$ translation $\hat{\mu}_{\hat{\alpha}}^T$ of $\hat{\mu}$ is a fuzzy S-extension of $\hat{\mu}$.*

Proof: Here $\hat{\mu}$ be an interval-valued fuzzy subalgebra of X, then by Corollary 4.7 the fuzzy $\hat{\alpha}$ translation $\hat{\mu}_{\hat{\alpha}}^T$ of $\hat{\mu}$ is also an interval-valued fuzzy subalgebra of X for all $\hat{\alpha} \in [\hat{0}, \hat{T}]$. Also $\hat{\mu}_{\hat{\alpha}}^T(x) = \hat{\mu}(x) + \hat{\alpha} \ge \hat{\mu}(x)\ \forall x \in X$. Therefore $\hat{\mu}_{\hat{\alpha}}^T$ is a fuzzy S-extension of $\hat{\mu}$.

Theorem 6.4: *Let $\hat{\mu}$ be an interval-valued fuzzy subalgebra of X and $\hat{\alpha}, \hat{\beta} \in [\hat{0}, \hat{T}]$. If $\hat{\alpha} \ge \hat{\beta}$ then the fuzzy $\hat{\alpha}$ translation $\hat{\mu}_{\hat{\alpha}}^T$ of $\hat{\mu}$ is a fuzzy S-extension of the fuzzy $\hat{\beta}$ translation $\hat{\mu}_{\hat{\beta}}^T$ of $\hat{\mu}$.*

Proof: Here $\hat{\mu}$ be an interval-valued fuzzy subalgebra of X, then by Corollary 4.7 the fuzzy $\hat{\alpha}$ translation $\hat{\mu}_{\hat{\alpha}}^T$ and fuzzy $\hat{\beta}$ translation $\hat{\mu}_{\hat{\beta}}^T$ of $\hat{\mu}$ is also an interval-valued fuzzy subalgebra of X. Since Also $\hat{\alpha} \ge \hat{\beta}$ therefore $\hat{\mu}(x) + \hat{\alpha} \ge \hat{\mu}(x) + \hat{\beta}\ \forall x \in X$. Therefore $\hat{\mu}_{\hat{\alpha}}^T(x) \ge \hat{\mu}_{\hat{\beta}}^T(x)\ \forall x \in X$. Therefore $\hat{\mu}_{\hat{\alpha}}^T$ is a fuzzy S-extension of $\hat{\mu}_{\hat{\beta}}^T$.

Theorem 6.5: *Intersection of any two interval-valued fuzzy S-extensions of an interval-valued fuzzy subalgebra $\hat{\mu}$ is a fuzzy S-extensions of $\hat{\mu}$.*

Proof: Let $\hat{\mu}_1$ and $\hat{\mu}_2$ be two interval-valued fuzzy S-extensions of a fuzzy subalgebra $\hat{\mu}$ of X. Then $\hat{\mu}_1(x) \ge \hat{\mu}(x)$ and $\hat{\mu}_2(x) \ge \hat{\mu}(x)\ \forall x \in X$. Now By Remark 3.5 $\hat{\mu}_1 \cap \hat{\mu}_2$ is an interval-valued fuzzy subalgebra of X. Now

$$(\hat{\mu}_1 \cap \hat{\mu}_2)(x) = rmin\{\hat{\mu}_1(x), \hat{\mu}_2(x)\} \ge rmin\{\hat{\mu}(x), \hat{\mu}(x)\} = \hat{\mu}(x).$$

Hence $(\hat{\mu}_1 \cap \hat{\mu}_2)$ is fuzzy S-extensions of $\hat{\mu}$.

Theorem 6.6: *Let $\hat{\mu}$ be an interval-valued fuzzy set of X and $\hat{\alpha} \in [\hat{0}, \hat{T}]$ and $\hat{\beta} \in D[0\,1]$ .If $\hat{\mu}_{\hat{\beta}}^M$ is a fuzzy subalgebra of X then the fuzzy $\hat{\alpha}$ translation $\hat{\mu}_{\hat{\alpha}}^T$ of $\hat{\mu}$ is a fuzzy S-extension of $\hat{\mu}_{\hat{\beta}}^M$.*

Proof: Let $\hat{\alpha} \in [\hat{0}, \hat{T}]$ and $\hat{\beta} \in D[0\,1]$ and if fuzzy $\hat{\beta}$ multiplication $\hat{\mu}_{\hat{\beta}}^M$ is an interval-valued fuzzy subalgebra of X then by Corollary 4.7 the interval-valued fuzzy set $\hat{\mu}$ and the fuzzy $\hat{\alpha}$ translation $\hat{\mu}_{\hat{\alpha}}^T$ of $\hat{\mu}$ is an interval-valued fuzzy subalgebra of X. Now $\hat{\mu}_{\hat{\alpha}}^T(x) = \hat{\mu}(x) + \hat{\alpha} \ge \hat{\mu}(x) \ge \hat{\mu}(x).\hat{\beta} = \hat{\mu}_{\hat{\beta}}^M$. Hence $\hat{\mu}_{\hat{\alpha}}^T$ is a fuzzy S-extension of $\hat{\mu}_{\hat{\beta}}^M$.

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