FUZZY TRANSLATIONS AND FUZZY MULTIPLICATIONS OF INTERVAL-VALUED FUZZY BG-ALGEBRAS

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ABSTRACT
In this paper, we introduced the concept of fuzzy translations, fuzzy multiplications and fuzzy extensions of interval-valued fuzzy subalgebras of BG-algebras and investigated some of their basic properties.

Keywords: BG-algebra, Interval-valued fuzzy set, Interval-valued fuzzy subalgebras, Fuzzy translation, Fuzzy multiplication, Fuzzy extension.

AMS Subject Classification(2010): 06F35, 03E72, 03G25, 08A72.

1. INTRODUCTION

2. PRELIMINARIES

Definition 2.1: A BG-algebra is a non-empty set X with a constant ‘0’ and a binary operation ‘*’ satisfying following axioms:
(i) \( x \ast x = 0 \),
(ii) \( x \ast 0 = x \),
(iii) \( (x \ast y) \ast (0 \ast y) = x, \forall \ x, y \in X \).

For brevity we also call X a BG-algebra.

Example 2.2: Let \( X = \{ 0, 1, 2, 3, 4 \} \) with the following caley table

\[
\begin{array}{c|ccccc}
* & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 0 & 4 & 3 & 2 & 1 \\
1 & 1 & 0 & 4 & 3 & 2 \\
2 & 2 & 1 & 0 & 4 & 3 \\
3 & 3 & 2 & 1 & 0 & 4 \\
4 & 4 & 3 & 2 & 1 & 0 \\
\end{array}
\]

Then \(( X, \ast, 0)\) is a BG-algebra.
3. INTERVAL-VALUED FUZZY SETS

The notion of interval-valued fuzzy set was introduced by L.A.Zadeh[13]. To consider the notion of interval-valued fuzzy sets, we need the following definitions. An interval number on $[0,1]$, denoted by $[a,\bar{a}]$, is defined as the closed subinterval of $[0,1]$, where $a \leq \bar{a}$, satisfying $0 \leq a \leq \bar{a} \leq 1$. Let $D[0,1]$ denote the set of all such interval numbers on $[0,1]$ and also denote the interval numbers $[0,0]$ and $[1,1]$ by $\hat{0}$ and $\hat{1}$ respectively.

Let $\hat{a}_1 = [a_1,\bar{a}_1]$ and $\hat{a}_2 = [a_2,\bar{a}_2] \in D[0,1]$. Define on $D[0,1]$ the relations $\leq, =, >, \forall \hat{a}_1, \hat{a}_2 \in D[0,1]$ by

1. $\hat{a}_1 \leq \hat{a}_2 \iff a_1 \leq a_2 \text{ and } \bar{a}_1 \leq \bar{a}_2$
2. $\hat{a}_1 = \hat{a}_2 \iff a_1 = a_2 \text{ and } \bar{a}_1 = \bar{a}_2$
3. $\hat{a}_1 > \hat{a}_2 \iff a_1 > a_2 \text{ and } \bar{a}_1 > \bar{a}_2$
4. $\hat{a}_1 + \hat{a}_2 \iff [a_1 + a_2, \bar{a}_1 + \bar{a}_2]$
5. $\hat{a}_1 \hat{a}_2 \iff [\min(a_1, a_2, a_1, a_2, a_1, a_2), \max(a_1, a_2, a_1, a_2, a_1, a_2)] = [a_1, a_2, a_1, a_2]$
6. $k\hat{a} = [ka, k\bar{a}]$ where $0 \leq k \leq 1$

Now consider two intervals $\hat{a}_1 = [a_1,\bar{a}_1], \hat{a}_2 = [a_2,\bar{a}_2] \in D[0,1]$ then we define refine minimum $\text{rmin}$ as $\text{rmin}(\hat{a}_1, \hat{a}_2) = [\min(a_1, a_2, \min(a_1, a_2))$ and refine maximum $\text{rmax}$ as $\text{rmax}(\hat{a}_1, \hat{a}_2) = [\max(a_1, a_2, \max(a_1, a_2)]$ generally if $\hat{a}_i = [a_i,\bar{a}_i], \hat{b}_i = [b_i,\bar{b}_i] \in D[0,1]$ for $i = 1,2,3,...$ then we define $\text{rmax}(\hat{a}_i, \hat{b}_i) = [\max(a_i, b_i), \max(a_i, b_i)]$ and $\text{rmin}(\hat{a}_i, \hat{b}_i) = [\min(a_i, b_i), \min(a_i, b_i)]$ and $\text{rinf}(\hat{a}_i) = [\wedge, a_i, \wedge, a_i]$ and $\text{rsup}(\hat{a}_i) = [\vee, a_i, \vee, a_i]$

$D[0,1]$ is a complete lattice with $\wedge = \text{rmin}, \vee = \text{rmax}, \hat{0} = [0,0]$ and $\hat{1} = [1,1]$ being the least and the greatest element respectively.

**Definition 3.1** An interval-valued fuzzy set defined on a non empty set $X$ as an objects having the form $\hat{\mu} = \{x, [\mu(x), \bar{\mu}(x)] \}, \forall x \in X$ where $\mu$ and $\bar{\mu}$ are two fuzzy sets in $X$ such that $\mu(x) \leq \bar{\mu}(x)$ for all $x \in X$. Let $\hat{\mu}(x) = [\mu(x), \bar{\mu}(x)], \forall x \in X$, then $\hat{\mu}(x) \in D[0,1], \forall x \in X$.

If $\hat{\mu}$ and $\hat{\nu}$ be two interval-valued fuzzy sets in $X$, then we define

- $\hat{\mu} \subset \hat{\nu} \iff \forall \mu(x) \leq \nu(x)$ and $\bar{\mu}(x) \leq \bar{\nu}(x)$.
- $\hat{\mu} = \hat{\nu} \iff \forall \mu(x) = \nu(x)$ and $\bar{\mu}(x) = \bar{\nu}(x)$.
- $(\hat{\mu} \cup \hat{\nu})(x) = \hat{\mu}(x) \vee \hat{\nu}(x) = [\max(\mu(x), \nu(x)), \max(\bar{\mu}(x), \bar{\nu}(x))]$.
- $(\hat{\mu} \cap \hat{\nu})(x) = \hat{\mu}(x) \wedge \hat{\nu}(x) = [\min(\mu(x), \nu(x)), \min(\bar{\mu}(x), \bar{\nu}(x))]$.
- $(\hat{\mu} \times \hat{\nu})(x,y) = \hat{\mu}(x) \wedge \hat{\nu}(y) = [\min(\mu(x), \nu(y)), \min(\bar{\mu}(x), \bar{\nu}(y))]$.
- $\hat{\mu}^c(x) = [1 - \mu(x), 1 - \bar{\mu}(x)]$.

**Definition 3.2** Let $\hat{\mu}$ be an interval-valued fuzzy set in $X$. Then for every $0,0 < \hat{\iota} \leq [1,1]$, the crisp set $\hat{\mu}_i = \{x \in X | \hat{\mu}(x) \geq \hat{\iota}\}$ is called the level subset of $\hat{\mu}$.

**Definition 3.3** ([10]) An interval-valued fuzzy set $\hat{\mu}$ in BG-algebra $X$ is called an interval-valued fuzzy BG-subalgebra of $X$ if $\hat{\mu}(x \times y) \geq \text{rmin}(\hat{\mu}(x), \hat{\mu}(y))$ for all $x, y \in X$.  

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Example 3.4 Consider BG-algebra $X = \{0, 1, 2, 3\}$ with the following Cayley table.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
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<td>3</td>
<td>2</td>
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<td>2</td>
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<td>3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Define $\hat{\mu} : X \to D[0,1]$ by $\hat{\mu}(0) = \hat{\mu}(2) = [0.4, 0.9], \hat{\mu}(1) = \hat{\mu}(3) = [0.3, 0.5]$; then it is easy to verify that $\hat{\mu}$ is an interval-valued fuzzy BG-subalgebra of $X$.

Remark 3.5: ([10]) If $\hat{\mu}_1$ and $\hat{\mu}_2$ be any two interval-valued fuzzy BG-subalgebras of $X$. Then their intersection $(\hat{\mu}_1 \cap \hat{\mu}_2)$ is also an interval-valued fuzzy BG-subalgebra of $X$.

4. FUZZY TRANSLATION AND FUZZY MULTIPLICATION OF INTERVAL-VALUED FUZZY BG-ALGEBRAS

In what follows $X$ denotes a BG-algebra, and for any interval-valued fuzzy set $\hat{\mu} = [\mu, \mu]$ of $X$, we denote $\overline{T} = 1 - \sup \{\mu(x) | x \in X\}$ such that $T \leq \overline{T}$.

Definition 4.1: Let $\hat{\mu}$ an interval-valued fuzzy subset of $X$ and $0 \leq \overline{\alpha} \leq \overline{T}$ where $\overline{\alpha} = [\alpha, \overline{\alpha}]$. A mapping $\hat{\mu}_T : X \to D[0,1]$ is said to be an fuzzy $\overline{\alpha}$ translation of $\hat{\mu}$ if it satisfies $\hat{\mu}_T(x) = \hat{\mu}(x) + \overline{\alpha}$ for all $x \in X$.

Example 4.2: Consider the interval-valued fuzzy set $\hat{\mu}$ as in Example 3.4, let $\overline{\alpha} = [0.02, 0.09] \in [0, \overline{T}]$ Then the $\overline{\alpha}$ translation of interval-valued fuzzy set $\hat{\mu}$ is given by

$$\hat{\mu}_T(0) = \hat{\mu}_T(2) = [0.42, 0.99], \hat{\mu}_T(1) = \hat{\mu}_T(3) = [0.32, 0.59].$$

Definition 4.3: Let $\hat{\mu}$ an interval-valued fuzzy subset of $X$ and $\hat{\beta} \in D[0,1]$. A mapping $\hat{\mu}^\beta_M : X \to D[0,1]$ is said to be a fuzzy $\hat{\beta}$ multiplication of $\hat{\mu}$ if it satisfies $\hat{\mu}^\beta_M(x) = \hat{\beta} \cdot \hat{\mu}(x)$ for all $x \in X$.

Example 4.4: Consider the interval-valued fuzzy set $\hat{\mu}$ as in Example 3.4, let $\hat{\beta} = [0.3, 0.6]$ Then the $\hat{\beta}$ multiplication of interval-valued fuzzy set $\hat{\mu}$ is given by

$$\hat{\mu}_\beta^\beta(0) = \hat{\mu}_\beta^\beta(2) = [0.12, 0.54], \hat{\mu}_\beta^\beta(1) = \hat{\mu}_\beta^\beta(3) = [0.09, 0.30].$$

Definition 4.5: Let $\hat{\mu}$ an interval-valued fuzzy subset of $X$ and $\overline{\alpha} \in [0, \overline{T}]$ and $\hat{\beta} \in D[0,1]$. A mapping $\hat{\mu}^\beta_{\overline{\alpha}} : X \to D[0,1]$ is said to be a fuzzy magnified $\hat{\beta} \overline{\alpha}$ translation of $\hat{\mu}$ if it satisfies $\hat{\mu}^\beta_{\overline{\alpha}}(x) = \hat{\beta} \cdot \hat{\mu}(x) + \overline{\alpha}$ for all $x \in X$.

Theorem 4.6: Let $\hat{\mu}$ be an interval-valued fuzzy subset of a BG-algebra $X$. Let $\overline{\alpha} \in [0, \overline{T}]$ and $\hat{\beta} \in D[0,1]$, then $\hat{\mu}$ is interval-valued fuzzy BG-subalgebra $X$ iff $\hat{\mu}^\beta_{\overline{\alpha}}$ is an interval-valued fuzzy BG-subalgebra $X$.

Proof: Let $\hat{\mu}$ be an interval-valued fuzzy BG-subalgebra $X$, then for $x, y \in X$, we have

$$\hat{\mu}^\beta_{\overline{\alpha}}(x \cdot y) = \hat{\beta} \cdot \hat{\mu}(x \cdot y) + \overline{\alpha} \geq \hat{\beta} \cdot \min\{\hat{\mu}(x), \hat{\mu}(y)\} + \overline{\alpha}$$

$$= \min\{\hat{\beta} \cdot \hat{\mu}(x) + \overline{\alpha}, \hat{\beta} \cdot \hat{\mu}(y) + \overline{\alpha}\}$$

$$= \min\{\hat{\mu}^\beta_{\overline{\alpha}}(x), \hat{\mu}^\beta_{\overline{\alpha}}(y)\}.$$

Hence $\hat{\mu}^\beta_{\overline{\alpha}}$ is an interval-valued fuzzy BG-subalgebra $X$.  

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Conversely, assume \( \hat{\mu}_{\beta\alpha}^{MT} \) is an interval-valued fuzzy BG-subalgebra X, then for \( x, y \in X \), we have
\[
\hat{\beta}.\hat{\mu}(x * y) + \hat{\alpha} = \hat{\mu}_{\beta\alpha}^{MT}(x * y) \geq rmin\{\hat{\mu}_{\beta\alpha}^{MT}(x), \hat{\mu}_{\beta\alpha}^{MT}(y)\} \\
= rmin \{ \hat{\beta}.\hat{\mu}(x) + \hat{\alpha}, \hat{\beta}.\hat{\mu}(y) + \hat{\alpha} \} \\
= \hat{\beta}.rmin\{\hat{\mu}(x), \hat{\mu}(y)\} + \hat{\alpha}
\]
\[
\Rightarrow \hat{\beta}.\hat{\mu}(x * y) + \hat{\alpha} \geq \hat{\beta}.rmin\{\hat{\mu}(x), \hat{\mu}(y)\} + \hat{\alpha}.
\]
which implies \( \hat{\mu}(x * y) \geq rmin\{\hat{\mu}(x), \hat{\mu}(y)\} \) for all \( x, y \in X \).

Hence \( \hat{\mu}_{\beta\alpha}^{MT} \) is an interval-valued fuzzy BG-subalgebra X.

**Corollary 4.7:** Let \( \hat{\mu} \) be an interval-valued fuzzy subset of a BG-algebra X and \( \hat{\alpha} \in [0, \hat{T}] \) then \( \hat{\mu} \) is an interval-valued fuzzy BG-subalgebra X iff \( \hat{\mu}_{T}^{\hat{\alpha}} \) is an interval-valued fuzzy BG-subalgebra X.

**Proof:** Put \( \hat{\beta} = \hat{1} \) in Theorem 4.6.

**Corollary 4.8:** Let \( \hat{\mu} \) be an interval-valued fuzzy subset of a BG-algebra X and \( \hat{\beta} \in D[0,1] \) then \( \hat{\mu} \) is an interval-valued fuzzy BG-subalgebra X iff \( \hat{\mu}_{\beta}^{MT} \) is an interval-valued fuzzy BG-subalgebra X.

**Proof:** Put \( \hat{\alpha} = \hat{0} \) in Theorem 4.6.

**Theorem 4.9:** Let \( \hat{\mu} \) be an interval-valued fuzzy BG-subalgebra X, then \( \hat{\mu}_{\beta\alpha}^{MT}(0) \geq \hat{\mu}_{\beta\alpha}^{MT}(x) \) for all \( x \in X \).

**Proof:** Here \( \hat{\mu} \) be an interval-valued fuzzy BG-subalgebra X therefore \( \hat{\mu}_{\beta\alpha}^{MT} \) is an interval-valued fuzzy BG-subalgebra X.

\[
\hat{\mu}_{\beta\alpha}^{MT}(0) = \hat{\mu}_{\beta\alpha}^{MT}(x * x) = \hat{\beta}.\hat{\mu}(x * x) + \hat{\alpha} \\
\geq \hat{\beta}.rmin\{\hat{\mu}(x), \hat{\mu}(x)\} + \hat{\alpha} \\
= rmin \{ \hat{\beta}.\hat{\mu}(x) + \hat{\alpha}, \hat{\beta}.\hat{\mu}(x) + \hat{\alpha} \} \\
= rmin\{\hat{\beta}.\hat{\mu}(x) + \hat{\alpha}, \hat{\beta}.\hat{\mu}(x) + \hat{\alpha}, \hat{\beta}.\hat{\mu}(x) + \hat{\alpha}\} \\
= \hat{\mu}_{\beta\alpha}^{MT}(x)
\]
\[
\Rightarrow \hat{\mu}_{\beta\alpha}^{MT}(0) \geq \hat{\mu}_{\beta\alpha}^{MT}(x).
\]

**Theorem 4.10:** Let \( \hat{\mu}_{\beta\alpha}^{MT} \) be an interval-valued fuzzy BG-subalgebra X where \( \hat{\alpha} \in [0, \hat{T}] \) and \( \hat{\beta}, \hat{i} \in D[0,1] \) with \( \hat{i} \geq \hat{\alpha} \), then the level subset \( U_{\beta,a}^{MT}(\hat{\mu}, \hat{i}) = \{x \in X \mid \hat{\beta}.\hat{\mu}(x) \geq \hat{i} - \hat{\alpha}, \forall \hat{i} \in Im(\hat{\mu}) \} \) is a subalgebra of X.

**Proof:** Let \( x, y \in U_{\beta,a}^{MT}(\hat{\mu}, \hat{i}) \Rightarrow \hat{\beta}.\hat{\mu}(x) \geq \hat{i} - \hat{\alpha} \) and \( \hat{\beta}.\hat{\mu}(y) \geq \hat{i} - \hat{\alpha} \)
\[
\Rightarrow \hat{\beta}.\hat{\mu}(x) + \hat{\alpha} \geq \hat{i} \quad \text{and} \quad \hat{\beta}.\hat{\mu}(y) + \hat{\alpha} \geq \hat{i}
\]
\[
\Rightarrow \hat{\mu}_{\beta\alpha}^{MT}(x) \geq \hat{i} \quad \text{and} \quad \hat{\mu}_{\beta\alpha}^{MT}(y) \geq \hat{i},
\]

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Now
\[ \hat{\mu}^{MT}_{\hat{\alpha}}(x \ast y) \geq rmin \{ \hat{\mu}^{MT}_{\hat{\beta}}(x), \hat{\mu}^{MT}_{\hat{\alpha}}(y) \} \]
\[ \geq rmin \{ \hat{t}, \hat{t} \} \]
\[ = rmin\{[\hat{l}, \hat{t}], [\hat{l}, \hat{t}]\} \]
\[ = [\min\{\hat{l}, \hat{t}\}, \min\{\hat{t}, \hat{t}\}] \]
\[ = [\hat{l}, \hat{t}] = \hat{t} \]

\[ \Rightarrow \hat{\beta}, \hat{\mu}(x \ast y) + \hat{\alpha} \geq \hat{t} \]

\[ \Rightarrow (x \ast y) \in U^{MT}_{\hat{\beta}, \hat{\alpha}}(\hat{\mu}, \hat{t}). \]

**Theorem 4.11:** Intersection and union of any two fuzzy translation of an interval-valued fuzzy BG-subalgebra of X is also an interval-valued fuzzy BG-subalgebra of X.

**Proof:** Let \( \hat{\mu}^{T}_{\hat{\alpha}} \) and \( \hat{\mu}^{T}_{\hat{\delta}} \) be two fuzzy translations of an interval-valued fuzzy BG-subalgebra \( \hat{\mu} \), where \( \hat{\alpha}, \hat{\delta} \in [0, \hat{T}] \). Assume that \( \hat{\alpha} \leq \hat{\delta} \) By Corollary 4.7 \( \hat{\mu}^{T}_{\hat{\alpha}} \) and \( \hat{\mu}^{T}_{\hat{\delta}} \) are interval-valued fuzzy BG-subalgebras of X.

Now
\[ (\hat{\mu}^{T}_{\hat{\alpha}} \cup \hat{\mu}^{T}_{\hat{\delta}})(x) = \text{rmax}\{\hat{\mu}^{T}_{\hat{\alpha}}(x), \hat{\mu}^{T}_{\hat{\delta}}(x)\} \]
\[ = \text{rmax}\{\hat{\mu}(x) + \hat{\alpha}, \hat{\mu}(x) + \hat{\delta}\} \]
\[ = \text{rmax}\{[\underline{\mu}(x) + \hat{\alpha}, \overline{\mu}(x) + \hat{\alpha}], [\underline{\mu}(x) + \hat{\delta}, \overline{\mu}(x) + \hat{\delta}]\} \]
\[ = [\max\{\underline{\mu}(x) + \hat{\alpha}, \overline{\mu}(x) + \hat{\alpha}\}, \max\{\underline{\mu}(x) + \hat{\delta}, \overline{\mu}(x) + \hat{\delta}\}] \]
\[ = [\hat{\mu}(x) + \hat{\alpha}, \hat{\mu}(x) + \hat{\delta}] \]
\[ = \hat{\mu}(x) + \hat{\delta} = \hat{\mu}^{T}_{\hat{\delta}}(x). \]

Also
\[ (\hat{\mu}^{T}_{\hat{\alpha}} \cap \hat{\mu}^{T}_{\hat{\delta}})(x) = \text{rmin}\{\hat{\mu}^{T}_{\hat{\alpha}}(x), \hat{\mu}^{T}_{\hat{\delta}}(x)\} \]
\[ = \text{rmin}\{\hat{\mu}(x) + \hat{\alpha}, \hat{\mu}(x) + \hat{\delta}\} \]
\[ = \text{rmin}\{[\underline{\mu}(x) + \hat{\alpha}, \overline{\mu}(x) + \hat{\alpha}], [\underline{\mu}(x) + \hat{\delta}, \overline{\mu}(x) + \hat{\delta}]\} \]
\[ = [\min\{\underline{\mu}(x) + \hat{\alpha}, \overline{\mu}(x) + \hat{\alpha}\}, \min\{\underline{\mu}(x) + \hat{\delta}, \overline{\mu}(x) + \hat{\delta}\}] \]
\[ = [\underline{\mu}(x) + \hat{\alpha}, \overline{\mu}(x) + \hat{\alpha}] \]
\[ = \hat{\mu}(x) + \hat{\alpha} = \hat{\mu}^{T}_{\hat{\alpha}}(x). \]

5. **FUZZY TRANSLATIONS AND FUZZY MULTIPLICATIONS OF INTERVAL-VALUED FUZZY BG-ALGEBRAS UNDER HOMOMORPHISM**

**Definition 5.1:** Let \( X \) and \( Y \) be two BG-algebras. Then a mapping \( f : X \rightarrow Y \) is said to be homomorphism if
\[ f(x \ast y) = f(x) \ast f(y), \forall x, y \in X. \]

**Theorem 5.2:** Let \( f : X \rightarrow Y \) be a homomorphism of BG-algebras. If \( \hat{\mu} \) be an interval-valued fuzzy subalgebras of \( Y \), then \( f^{-1}(\hat{\mu}^{T}_{\hat{\alpha}}) \) an interval-valued fuzzy subalgebra of \( X \).

**Proof:** Let \( \hat{\mu} \) be an interval-valued fuzzy subalgebras of \( Y \), therefore by Corollary 4.7 \( \hat{\mu}^{T}_{\hat{\alpha}} \) is also an interval-valued fuzzy subalgebras of \( Y \). Now \( f^{-1}(\hat{\mu}^{T}_{\hat{\alpha}}) \) is defined by \( f^{-1}(\hat{\mu}^{T}_{\hat{\alpha}})(x) = \hat{\mu}^{T}_{\hat{\alpha}}(f(x)) \forall x \in X. \) Let \( x, y, \in X \)
Now \( f^{-1}(\hat{\mu}_a^T) (x \ast y) = \hat{\mu}_a^T \{ f(x \ast y) \} \)
\[ = \hat{\mu}_a^T \{ f(x) \ast f(y) \} \geq rmin \{ \hat{\mu}_a^T(f(x)), \hat{\mu}_a^T(f(y)) \} \]
\[ = rmin \{ f^{-1}(\hat{\mu}_a^T(x)), f^{-1}(\hat{\mu}_a^T(y)) \}. \]

Hence \( f^{-1}(\hat{\mu}_a^T) \) an interval-valued fuzzy subalgebras of X.

**Theorem 5.3:** Let \( f : X \rightarrow Y \) be a homomorphism of BG-algebras. If \( \hat{\mu} \) be an interval-valued fuzzy subalgebra of Y, then \( f^{-1}(\hat{\mu}_a^T) \) an interval-valued fuzzy subalgebra of X.

**Proof.** Same as Theorem 5.2.

**Theorem 5.4:** Let \( f : X \rightarrow Y \) be an onto homomorphism of BG-algebras. If \( \hat{\mu} \) be an interval-valued fuzzy subset of Y such that \( f^{-1}(\hat{\mu}_a^T) \) is an interval-valued fuzzy subalgebra of X. Then \( \hat{\mu} \) is also an interval-valued fuzzy subalgebra of Y.

**Proof:** Since f is onto homomorphism therefore to each \( x', y' \in Y \), there exists \( x, y \in X \) such that \( f(x) = x', f(y) = y' \) and \( f(x \ast y) = f(x) \ast f(y) = x' \ast y' \). Now since \( f^{-1}(\hat{\mu}_a^T) \) is an interval-valued fuzzy subalgebras of X. Therefore
\[ f^{-1}(\hat{\mu}_a^T)(x \ast y) \geq rmin \{ f^{-1}(\hat{\mu}_a^T)(x), f^{-1}(\hat{\mu}_a^T)(y) \} \]
\[ = \hat{\mu}_a^T f(x \ast y) \geq rmin \{ \hat{\mu}_a^T f(x), \hat{\mu}_a^T f(y) \} \]
\[ = \hat{\mu}_a^T f(x) \ast f(y) \geq rmin \{ \hat{\mu}_a^T f(x), \hat{\mu}_a^T f(y) \} \]
\[ = \hat{\mu}_a^T (x' \ast y') \geq rmin \{ \hat{\mu}_a^T (x'), \hat{\mu}_a^T (y') \}. \]
\[ = \hat{\mu}_a^T \] is an interval valued fuzzy subalgebra of Y. Hence By Corollary 4.7 \( \hat{\mu} \) an interval valued fuzzy subalgebra of Y.

**Theorem 5.5:** Let \( f : X \rightarrow Y \) be an onto homomorphism of BG-algebras. If \( \hat{\mu} \) be an interval-valued fuzzy subset of Y such that \( f^{-1}(\hat{\mu}_a^T) \) is an interval-valued fuzzy subalgebra of X. Then \( \hat{\mu} \) is also an interval-valued fuzzy subalgebra of Y.

**Proof:** Same as Theorem 5.4.

**Theorem 5.6:** Let \( f : X \rightarrow Y \) be an epimorphism, where X, Y are two two interval-valued fuzzy subalgebras and \( \hat{\alpha} \in [\hat{0}, \hat{T}] \). Then the inverse image of \( \hat{\alpha} \) translation of any fuzzy subalgebra \( \hat{\mu} \) of Y is same as the \( \alpha \) translation of the inverse image of fuzzy subalgebra \( \hat{\mu} \).

**Proof:** Let \( f : X \rightarrow Y \) be an epimorphism, where X, Y are two two interval-valued fuzzy subalgebras and \( \hat{\alpha} \in [\hat{0}, \hat{T}] \). Let \( \hat{\mu} \) be a fuzzy subalgebra of Y. Therefore by Theorem 4.6 The \( \hat{\alpha} \) translation \( \hat{\mu}_a^T \) is a fuzzy subalgebra of Y. Therefore by Corollary 4.7 \( f^{-1}(\hat{\mu}_a^T) \) is a fuzzy subalgebra of X.

Also \( f^{-1}(\hat{\mu}_a^T)(x) = \hat{\mu}_a^T(f(x)) = \hat{\mu}(f(x)) + \hat{\alpha} = f^{-1}(\hat{\mu})(x) + \hat{\alpha} = (f^{-1}(\hat{\mu}))^\alpha_a(x) \).

Hence \( f^{-1}(\hat{\mu}_a^T) = (f^{-1}(\hat{\mu}))^\alpha_a \)

**Theorem 5.7:** Let \( f : X \rightarrow Y \) be an epimorphism, where X, Y are two two interval-valued fuzzy subalgebras and \( \hat{\beta} \in [0, 1] \). Then the inverse image of \( \hat{\beta} \) multiplication of any fuzzy subalgebra \( \hat{\mu} \) of Y is same as the \( \beta \) multiplication of the inverse image of fuzzy subalgebra \( \hat{\mu} \).
6. INTERVAL-VALUED FUZZY SUBALGEBRA EXTENSION

Definition 6.1: Let \( \hat{\mu}_1 \) and \( \hat{\mu}_2 \) be two interval-valued fuzzy subsets of \( X \). If \( \hat{\mu}_1(x) \leq \hat{\mu}_2(x) \) for all \( x \in X \), then we say that \( \hat{\mu}_2 \) is a fuzzy (an interval-valued) \( S \)-extension of \( \hat{\mu}_1 \).

Definition 6.2: Let \( \hat{\mu}_1 \) and \( \hat{\mu}_2 \) be two interval-valued fuzzy subsets of \( X \) such that \( \hat{\mu}_1 \) is a fuzzy extension of \( \hat{\mu}_2 \). If \( \hat{\mu}_1 \) is an interval-valued fuzzy subalgebra of \( X \) implies that \( \hat{\mu}_2 \) is an interval-valued fuzzy subalgebra of \( X \), then \( \hat{\mu}_2 \) is called fuzzy \( S \)-extension of \( \hat{\mu}_1 \).

Theorem 6.3: Let \( \hat{\mu} \) be an interval-valued fuzzy subalgebra of \( X \) and \( \dot{\alpha} \in [0, \hat{T}] \), then the fuzzy \( \dot{\alpha} \) translation \( \hat{\mu}^T_{\dot{\alpha}} \) of \( \hat{\mu} \) is a fuzzy \( S \)-extension of \( \hat{\mu} \).

Proof: Here \( \hat{\mu} \) be an interval-valued fuzzy subalgebra of \( X \), then by Corollary 4.7 the fuzzy \( \dot{\alpha} \) translation \( \hat{\mu}^T_{\dot{\alpha}} \) of \( \hat{\mu} \) is also an interval-valued fuzzy subalgebra of \( X \) for all \( \dot{\alpha} \in [0, \hat{T}] \). Also \( \hat{\mu}^T_{\dot{\alpha}}(x) = \hat{\mu}(x) + \dot{\alpha} \geq \hat{\mu}(x) \forall x \in X \). Therefore \( \hat{\mu}^T_{\dot{\alpha}} \) is a fuzzy \( S \)-extension of \( \hat{\mu} \).

Theorem 6.4: Let \( \hat{\mu} \) be an interval-valued fuzzy subalgebra of \( X \) and \( \dot{\alpha}, \dot{\beta} \in [0, \hat{T}] \). If \( \dot{\alpha} \geq \dot{\beta} \) then the fuzzy \( \dot{\alpha} \) translation \( \hat{\mu}^T_{\dot{\alpha}} \) of \( \hat{\mu} \) is a fuzzy \( S \)-extension of the fuzzy \( \dot{\beta} \) translation \( \hat{\mu}^T_{\dot{\beta}} \) of \( \hat{\mu} \).

Proof: Here \( \hat{\mu} \) be an interval-valued fuzzy subalgebra of \( X \), then by Corollary 4.7 the fuzzy \( \dot{\alpha} \) translation \( \hat{\mu}^T_{\dot{\alpha}} \) and fuzzy \( \dot{\beta} \) translation \( \hat{\mu}^T_{\dot{\beta}} \) of \( \hat{\mu} \) is also an interval-valued fuzzy subalgebra of \( X \). Since Also \( \dot{\alpha} \geq \dot{\beta} \) therefore \( \hat{\mu}(x) + \dot{\alpha} \geq \hat{\mu}(x) + \dot{\beta} \forall x \in X \). Therefore \( \hat{\mu}^T_{\dot{\alpha}}(x) \geq \hat{\mu}^T_{\dot{\beta}}(x) \forall x \in X \). Therefore \( \hat{\mu}^T_{\dot{\alpha}} \) is a fuzzy \( S \)-extension of \( \hat{\mu}^T_{\dot{\beta}} \).

Theorem 6.5: Intersection of any two interval-valued fuzzy \( S \)-extensions of an interval-valued fuzzy subalgebra \( \hat{\mu} \) is a fuzzy \( S \)-extensions of \( \hat{\mu} \).

Proof: Let \( \hat{\mu}_1 \) and \( \hat{\mu}_2 \) be two interval-valued fuzzy \( S \)-extensions of a fuzzy subalgebra \( \hat{\mu} \) of \( X \). Then \( \hat{\mu}_1(x) \geq \hat{\mu}(x) \) and \( \hat{\mu}_2(x) \geq \hat{\mu}(x) \forall x \in X \). Now By Remark 3.5 \( \hat{\mu}_1 \cap \hat{\mu}_2 \) is an interval-valued fuzzy subalgebra of \( X \). Now

\[
(\hat{\mu}_1 \cap \hat{\mu}_2)(x) = r\min\{\hat{\mu}_1(x), \hat{\mu}_2(x)\} \geq r\min\{\hat{\mu}(x), \hat{\mu}(x)\} = \hat{\mu}(x).
\]

Hence \( \hat{\mu}_1 \cap \hat{\mu}_2 \) is fuzzy \( S \)-extensions of \( \hat{\mu} \).

Theorem 6.6: Let \( \hat{\mu} \) be an interval-valued fuzzy set of \( X \) and \( \dot{\alpha} \in [0, \hat{T}] \) and \( \dot{\beta} \in D[0,1] \). If \( \hat{\mu}^M_{\dot{\beta}} \) is a fuzzy subalgebra of \( X \) then the fuzzy \( \dot{\alpha} \) translation \( \hat{\mu}^T_{\dot{\alpha}} \) of \( \hat{\mu} \) is a fuzzy \( S \)-extension of \( \hat{\mu}^M_{\dot{\beta}} \).

Proof: Let \( \dot{\alpha} \in [0, \hat{T}] \) and \( \dot{\beta} \in D[0,1] \) and if fuzzy \( \dot{\beta} \) multiplication \( \hat{\mu}^M_{\dot{\beta}} \) is an interval-valued fuzzy subalgebra of \( X \) then by Corollary 4.7 the interval-valued fuzzy set \( \hat{\mu} \) and the fuzzy \( \dot{\alpha} \) translation \( \hat{\mu}^T_{\dot{\alpha}} \) of \( \hat{\mu} \) is an interval-valued fuzzy subalgebra of \( X \). Now \( \hat{\mu}^T_{\dot{\alpha}}(x) = \hat{\mu}(x) + \dot{\alpha} \geq \hat{\mu}(x) \geq \hat{\mu}(x) \dot{\beta} = \hat{\mu}^M_{\dot{\beta}} \). Hence \( \hat{\mu}^T_{\dot{\alpha}} \) is a fuzzy \( S \)-extension of \( \hat{\mu}^M_{\dot{\beta}} \).
REFERENCES


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