WEAKLY $I_g^*$- CLOSED SETS AND WEAKLY $I_g^{*}$-CLOSED SETS
IN IDEAL TOPOLOGICAL SPACES

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ABSTRACT

In this paper, we define a new class of closed sets namely weakly closed sets in ideal topological spaces. Also, we study some characterizations and properties of weakly closed sets with respect to an ideal topological space.

Key Words: $wI_g^*$-closed set, $wI_g^*$-open set, $wI_g^{*}$-closed set, $wI_g^{*}$-open set.

1. INTRODUCTION

One of the important tools in General Topology is the ideals. Newcomb (1967) [9], Rancin(1972), Samuals(1975), Hamlet and JanKovic [4] (1990,1992,1990) motivated the research in general topology. A generalized closed set in topological space was introduced by Levine [7] (1967) in 1970. The notion of ideal topological space was studied by Kurotowski [6] (1933) and Vaidyanathaswamy [11] (1945). Jafari and Rajesh introduced $I_g^*$-closed set with respect to an ideal. In this paper, we introduced and study a new class of closed sets in ideal topological spaces called $wI_g^*$ and $wI_g^{*}$ closed sets with respect to an ideal which is the weaker form of $I_g^*$ and $I_g^{*}$ closed sets in ideal topological spaces.

2. PRELIMINARIES

Throughout the present paper (X, $\tau$) always means a topological space. Let A be a subset of topological space (X, $\tau$). The closure (resp. interior) of A are denoted by cl(A) (resp. Int(A)). An ideal (Kuratowski, 1933) [6] on a set X is a nonempty collection of subsets of X with heredity property and finite additivity property that is it satisfies the following two conditions:

1. $A \in I$ and $B \subseteq A$ then $B \in I$ (heredity)
2. $A \in I$ and $B \in I$ implies $A \cup B \in I$ (finite additivity)

Definition 2.1 [4]: A topological space $(X, \tau)$ with an ideal I on X and if $\emptyset(X)$ is the set of all subsets of X, a set operator $(\cdot)^*$: $\emptyset(X) \rightarrow \emptyset(X)$, called a local functionof A with respect to $\tau$ and I is defined as follows: for $A \subseteq X$, $A^*(I, \tau) = \{x \in X | U \cap A \notin I \text{ for every } U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau | x \in U\}$.

Definition 2.2 [5]: A subset A of an ideal space $(X, \tau, I)$ is said to be $^*\cdot$ - closed if $A^* \subseteq A$.

Definition 2.3 [1]: A subset A of a space $(X, \tau)$ is said to be $g$-closed if cl(A) $\subseteq U$ whenever A $\subseteq U$ and U is semi-open.

Definition 2.4: A subset A of a space $(X, \tau)$ is said to be $g^*$-open if its complement is $g^*$-closed.
Definition 2.5 [1]: A subset $A$ of an ideal space $(X, \tau, I)$ is said to be $I_g$-closed if $A^* \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open.

Definition 2.6 [10]: A subset $A$ of a space $(X, \tau)$ is said to be $^g$-closed if $A^* \subseteq U$ whenever $A \subseteq U$ and $U$ is $^g$-open.

3. Weakly $I_g$ closed sets and weakly $I_{g*}$-closed sets with respect to an Ideal

Definition 3.1: A subset $A$ of an ideal space $(X, \tau, I)$ is said to be $wI_g$-closed if $\text{int}(A^*) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open.

Definition 3.2: A subset $A$ of an ideal space $(X, \tau, I)$ is said to be $wI_{g*}$-closed if $X - A$ is $wI_g$-closed.

Theorem 3.3: If $(X, \tau, I)$ is an ideal space, then every $I_g$-closed set is $wI_g$-closed but not conversely.

Proof: Let $A$ be an $I_g$-closed set. Let $A \subseteq U$ and $U$ is semi-open. Since $A$ is $I_g$-closed, $A^* \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open. We have $\text{int}(A^*) \subseteq A^* \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open. Therefore, $A$ is $wI_g$-closed.

Example 3.4: Let $X = \{a, b, c\}$, $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$ and $I = \{\varnothing, \{c\}\}$. It is clear that $\{a\}$ is $wI_g$-closed but it is not $I_g$-closed.

Theorem 3.5: Every $^*$-closed set is $wI_g$-closed but not conversely.

Proof: Let $A$ be a $^*$-closed, then $A^* \subseteq A$. Let $A \subseteq U$ where $U$ is semi-open. Hence $\text{int}(A^*) \subseteq A^* \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open. Therefore, $A$ is $wI_g$-closed.

Example 3.6: Let $X = \{a, b, c\}$, $\tau = \{\varnothing, X, \{b\}, \{c\}, \{b, c\}\}$ and $I = \{\varnothing, \{a\}\}$. It is clear that $\{c\}$ is $wI_g$-closed but not $^*$-closed.

Theorem 3.7: Let $(X, \tau, I)$ be an ideal space. For every $A \in I$, $A$ is $wI_g$-closed.

Proof: Let $A \subseteq U$ where $U$ is semi-open. Since $A^* = \varnothing$ for every $A \in I$, then $\text{cl}^*(A) = A \cup A^* = A$, since $A^* = \varnothing$. $\text{int}(A^*) \subseteq \text{cl}^*(A) = A \subseteq U$. Therefore, $A$ is $wI_g$-closed.

Theorem 3.8: If $(X, \tau, I)$ is an ideal space, then $A^*$ is always $wI_g$-closed for every subset $A$ of $X$.

Proof: Let $A^* \subseteq U$ where $U$ is semi-open. Since $\text{int}(A^*) \subseteq (A^*)^* \subseteq A^*$, we have $(A^*)^* \subseteq U$ whenever $A^* \subseteq U$ and $U$ is semi-open. Hence $A^*$ is $^*$-closed.

Theorem 3.9: Let $(X, \tau, I)$ be an ideal space. Then every $^g$-closed set is $wI_g$-closed set but not conversely.

Proof: Let $A$ be a $^g$-closed set. Then $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open. We have $\text{int}(A^*) \subseteq \text{cl}^*(A) \subseteq \text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open. Hence $A$ is $wI_g$-closed.

Example 3.10: Let $X = \{a, b, c\}$, $\tau = \{\varnothing, X, \{b\}, \{c\}, \{b, c\}\}$ and $I = \{\varnothing, \{b\}\}$. It is clear that $\{c\}$ is $wI_g$-closed set but not $^g$-closed set.

Theorem 3.11: Let $(X, \tau, I)$ be an ideal space and $A \subseteq X$. If $A \subseteq B \subseteq \text{int}(A^*)$, then $A^* = B^*$ and $B$ is $^*$-dense in itself.

Proof: Since $A \subseteq B$, then $A^* \subseteq B^*$ and since $B \subseteq \text{int}(A^*)$, then $B^* \supseteq \text{int}(A^*) \supseteq (A^*)^* \subseteq A^*$. Since $A^* \subseteq B^*$ and $B^* \subseteq A^*$, $A^* = B^*$. Since $B \subseteq \text{int}(A^*) \supseteq A^* = B^*$. Therefore, $B \subseteq B^*$. Hence $B$ is $^*$-dense in itself. Hence proved.

Theorem 3.12: Let $(X, \tau, I)$ be an ideal space. If every semi-open set is $^*$-closed, then every subset of $X$ is $wI_g$-closed.

Proof: Let $U$ be a semi-open set. Let $A \subseteq X$ be a subset of $X$. Let $A \subseteq U$ where $U$ is semi-open. If $U$ is semi-open set such that $A \subseteq U \subseteq X$, then $A^* \subseteq U^* \subseteq U$. Therefore, $\text{int}(A^*) \subseteq A^* \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open. Hence $A$ is $wI_g$-closed.

Theorem 3.13: Let $(X, \tau, I)$ be an ideal space. Then every $I_g$-closed, open set is $wI_g$-closed.
Proof: Let A be an \( I_g \)-closed, open set. Let \( A \subseteq A \) where A is open. Since A is open and \( I_g \)-closed, \( A^*_g \subseteq A \). Let \( A \subseteq U \) where U is semi-open. Since \( A^*_g \subseteq A \), \( \text{int}(A^*_g) \subseteq A^*_g \subseteq A \subseteq U \) which implies \( \text{int}(A^*_g) \subseteq U \) whenever U is semi-open. Hence A is \( w_l g \)-closed.

**Theorem 3.14:** Let \( (X, \tau, I) \) be an ideal space. Then every \( g \)-closed set is \( w_l g \)-closed but not conversely.

**Proof:** Let \( A \subseteq A \) where A is open. Since A is \( g \)-closed, \( \text{cl}(A) \subseteq A \). We have \( A \subseteq \text{cl}(A) \) which implies \( \text{cl}(A) = A \). Let \( A \subseteq U \) and U is semi-open. Therefore, \( \text{int}(A^*_g) \subseteq \text{cl}(A) \subseteq A \subseteq U \). Hence A is \( w_l g \)-closed.

**Example 3.15:** Let \( X = \{a, b, c\} \), \( \tau = \{\varphi, X, \{a\}, \{b\}, \{c\} \} \) and \( I = \{\varphi, \{b\} \} \). It is clear that \( \{c\} \) is \( w_l g \)-closed set but not \( g \)-closed set.

**Definition 3.16:** A subset A of an ideal space \( (X, \tau, I) \) is said to be \( w_l g \)-closed if \( \text{int}(A^*_g) \subseteq U \) whenever \( A \subseteq U \) and U is \( g \)-open.

**Definition 3.17:** A subset A of an ideal space \( (X, \tau, I) \) is said to be \( w_l g \)-open if \( X - A \) is \( w_l g \)-closed.

**Theorem 3.18:** If \( (X, \tau, I) \) is an ideal space, then every \( I_{g} \)-closed set is \( w_l g \)-closed but not conversely.

**Proof:** Let \( A \subseteq U \) and U is \( g \)-open. Since A is \( I_{g} \)-closed, \( A^*_g \subseteq U \) whenever \( A \subseteq U \) and U is \( g \)-open. We have \( \text{int}(A^*_g) \subseteq A^*_g \subseteq U \) whenever \( A \subseteq U \) and U is \( g \)-open. Therefore, A is \( w_l g \)-closed.

**Example 3.19:** Let \( X = \{a, b, c\} \), \( \tau = \{\varphi, X, \{a\}, \{b\}, \{a, b\} \} \) and \( I = \{\varphi, \{c\} \} \). It is clear that \( \{a\} \) is \( w_l g \)-closed but it is not \( I_{g} \)-closed.

**Theorem 3.20:** Every \( * \)-closed set is \( w_l g \)-closed but not conversely.

**Proof:** Let A be a \( * \)-closed, then \( A^*_g \subseteq A \). Let \( A \subseteq U \) where U is \( g \)-open. Therefore, \( \text{int}(A^*_g) \subseteq A^*_g \subseteq U \) whenever \( A \subseteq U \) and U is \( g \)-open. Therefore, A is \( w_l g \)-closed.

**Example 3.21:** Let \( X = \{a, b, c\} \), \( \tau = \{\varphi, X, \{a\}, \{b\}, \{a, b\} \} \) and \( I = \{\varphi, \{c\} \} \). It is clear that \( \{b\} \) is \( w_l g \)-closed but not \( * \)-closed.

**Theorem 3.22:** Let \( (X, \tau, I) \) be an ideal space. For every \( A \subseteq I \), A is \( I_{g} \)-closed.

**Proof:** Let \( A \subseteq U \) where U is \( g \)-open. Since \( A^*_g = \varphi \) for every \( A \subseteq I \), then \( \text{cl}(A) = A \cup A^*_g = A \), since \( A^*_g = \varphi \). Therefore, \( \text{int}(A^*_g) \subseteq \text{cl}(A) \subseteq A \subseteq U \). Hence \( \text{int}(A^*_g) \subseteq U \) whenever \( A \subseteq U \) and U is \( g \)-open. Therefore, A is \( w_l g \)-closed.

**Theorem 3.23:** If \( (X, \tau, I) \) is an ideal space, then \( A^*_g \) is always \( w_l g \)-closed for every subset A of X.

**Proof:** Let \( A^*_g \subseteq U \) where U is \( g \)-open. Since \( (A^*_g)^g \subseteq (A^*_g)^g \subseteq A^*_g \), we have \( (A^*_g)^g \subseteq U \) whenever \( A^*_g \subseteq U \) and U is \( g \)-open. Hence \( A^*_g \) is \( w_l g \)-closed.

**Remark 3.24:** Every \( w_l g \)-closed set need not be a \( g \)-closed set. This can be shown from the following example.

**Example 3.25:** Let \( X = \{a, b, c\} \), \( \tau = \{\varphi, X, \{a\}, \{b\}, \{a, b\} \} \) and \( I = \{\varphi, \{c\} \} \). It is clear that \( \{b\} \) is \( w_l g \)-closed set but not \( g \)-closed set.

**Theorem 3.26:** Let \( (X, \tau, I) \) be an ideal space and \( A \subseteq X \). If \( A \subseteq B \subseteq \text{int}(A^*_g) \), then \( A^*_g = B^*_g \) and B is \( * \)-dense in itself.

**Proof:** Since \( A \subseteq B \), then \( A^*_g \subseteq B^*_g \) and since \( B \subseteq \text{int}(A^*_g) \), then \( B^*_g \subseteq \text{int}(A^*_g) \subseteq (A^*_g)^g \subseteq A^*_g \). Since \( A^*_g \subseteq B^*_g \) and \( B^*_g \subseteq A^*_g \), \( A^*_g = B^*_g \). Since \( B \subseteq \text{int}(A^*_g) \subseteq A^*_g = B^*_g \). Therefore, B \( \subseteq B^*_g \). Hence B is \( * \)-dense in itself. Hence proved.

**Theorem 3.27:** Let \( (X, \tau, I) \) be an ideal space. If every \( g \)-closed set is \( * \)-closed, then every subset of X is \( w_l g \)-closed.

**Proof:** Let U be a \( g \)-open set. Let \( A \subseteq X \) be a subset of X. Let \( A \subseteq U \) where U is \( g \)-open. If U is \( g \)-open set such that \( A \subseteq U \subseteq X \), then \( A^*_g \subseteq U^*_g \subseteq U \). Therefore, \( \text{int}(A^*_g) \subseteq A^*_g \subseteq U \) whenever \( A \subseteq U \) and U is \( g \)-open. Hence A is \( w_l g \)-closed.
Theorem 3.28: Let \((X, \tau, I)\) be an ideal space. Then every \(I_g\)-closed, \(g\)-open set is \(wI_g\)-closed.

**Proof:** Let \(A\) be an \(I_g\)-closed, open set. Let \(A \subseteq A\) where \(A\) is open. Since \(A\) is open and \(I_g\)-closed, \(A^* \subseteq A\). Let \(A \subseteq U\) where \(U\) is \(g\)-open. Since \(A^* \subseteq A\), \(\text{int}(A^*) \subseteq A^* \subseteq A \subseteq U\) which implies \(\text{int}(A^*) \subseteq U\) whenever \(U\) is \(g\)-open. Hence \(A\) is \(wI_g\)-closed.

Theorem 3.29: Let \((X, \tau, I)\) be an ideal space. Then every \(g\)-closed set is \(wI_g\)-closed but not conversely.

**Proof:** Let \(A \subseteq A\) where \(A\) is open. Since \(A\) is \(g\)-closed, \(\text{cl}(A) \subseteq A\). We have \(A \subseteq \text{cl}(A)\) which implies \(\text{cl}(A) = A\). Let \(A \subseteq U\) and \(U\) is \(g\)-open. Therefore, \(\text{int}(A^*) \subseteq \text{cl}^*(A) \subseteq \text{cl}(A) \subseteq A \subseteq U\) whenever \(A\) is \(g\)-closed.

Example 3.30: Let \(X = \{a, b, c\}\), \(\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}\) and \(I = \{\varnothing, \{c\}\}\). It is clear that \(\{a\}\) is \(wI_g\)-closed set but not \(g\)-closed set.

Theorem 3.31: Let \((X, \tau, I)\) be an ideal space. Then every \(*g\)-closed set is \(wI_g\)-closed set but not conversely.

**Proof:** Let \(A\) be a \(*g\)-closed set. Then \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(g\)-open. We have \(\text{int}(A^*) \subseteq \text{cl}^*(A) \subseteq \text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(g\)-open. Hence \(A\) is \(wI_g\)-closed.

Example 3.32: Let \(X = \{a, b, c\}\), \(\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}\) and \(I = \{\varnothing, \{c\}\}\). It is clear that \(\{b\}\) is \(wI_g\)-closed set but not \(*g\)-closed set.

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