pg**- CLOSED SETS IN TOPOLOGICAL SPACES

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ABSTRACT

In this paper we introduce a new class of sets called pg**- closed sets in topological spaces which is properly placed in between the class of closed sets and gsp-closed sets. As an application, we introduce new spaces namely, $\rho T_{1/2}^{**}$-space, $ap T_{c}^{**}$-space, $pT_{1/2}^{**}$-space, $\rho T_{c}^{**}$-space and $\rho T_{c}$-space. Further, pg** -continuous, pg**-irresolute mappings are also introduced and investigated.

Key words: pg**-closed set, pg** -continuous map, pg**-irresolute map, $\rho T_{1/2}^{**}$- space, $ap T_{c}^{**}$-space, $pT_{1/2}^{**}$-space, $\rho T_{c}^{**}$-space, $\rho T_{c}$-space and $T_{c}^{**}$-spaces.

1. INTRODUCTION


2. PRILIMINARIES

Throughout this paper $(X,\tau),(Y,\sigma)$ and $(Z,\eta)$ represent non-empty topological spaces of which no separation axioms are assumed unless otherwise stated. For a subset $A$ of a space $(X,\tau)$, cl$(A)$ and int$(A)$ denote the closure and the interior of $A$ respectively. The class of all closed subsets of a space $(X,\tau)$ is denoted by C$(X,\tau)$. The smallest semi-closed (resp. pre-closed and $\alpha$-closed) set containing a subset $A$ of $(X,\tau)$ is called the semi-closure (resp. pre-closure and $\alpha$-closure) of $A$ and is denoted by scl$(A)$ (resp. pcl$(A)$ and $\alpha cl(A)$).

Definition 2.1: A subset $A$ of a topological space $(X,\tau)$ is called

1. a pre-open set [14] if $A \subseteq int(cl(A))$ and a pre-closed set if $cl(int(A)) \subseteq A$.
2. a semi-open set [12] if $A \subseteq cl(int(A))$ and a semi-closed set if $int(cl(A)) \subseteq A$.
3. a semi-preopen set [1] if $A \subseteq cl(int(cl(A)))$ and a semi-preclosed set if $int(int(cl(A))) \subseteq A$.
4. an $\alpha$-open set [16] if $A \subseteq cl(int(A))$ and an $\alpha$-closed set [16] if $cl(int(A)) \subseteq A$.

Definition 2.2: A subset $A$ of a topological space $(X,\tau)$ is called

1. a generalized closed set (briefly g-closed) [1] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X,\tau)$.
2. generalized semi-closed set (briefly gs-closed) [3] if scl$(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X,\tau)$.
3. an $\alpha$-generalized closed set (briefly $\alpha g$-closed) [19] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X,\tau)$.
4. a generalized semi-pre-closed set (briefly gsp-closed) [9] if $sp cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X,\tau)$.
5. a regular generalized closed set (briefly rg-closed) [17] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular open in $(X,\tau)$. 
(6) a generalized pre-closed set (briefly gp-closed) \([13]\) if \(p \text{ cl} (A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \((X, \tau)\).

(7) a generalized pre-regular-closed set (briefly gpr-closed) \([10]\) if \(p \text{ cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular open in \((X, \tau)\).

(8) a generalized \(g^*\)-closed set \([18]\) if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(g^*\)-open in \((X, \tau)\).

(9) a generalized \(g^{**}\)-closed set \([20]\) if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(g^{**}\)-open in \((X, \tau)\).

Definition 2.3: A function \(f: (X, \tau) \rightarrow (Y, \sigma)\) is called

1. \(g\)-continuous \([4]\) if \(f^{-1}(V)\) is a \(g\)-closed set of \((X, \tau)\) for every closed set \(V\) of \((Y, \sigma)\).

2. \(\alpha g\)-continuous \([10]\) if \(f^{-1}(V)\) is an \(\alpha g\)-closed set of \((X, \tau)\) for every closed set \(V\) of \((Y, \sigma)\).

3. \(gs\)-continuous \([7]\) if \(f^{-1}(V)\) is a \(gs\)-closed set of \((X, \tau)\) for every closed set \(V\) of \((Y, \sigma)\).

4. \(gsp\)-continuous \([9]\) if \(f^{-1}(V)\) is a \(gsp\)-closed set of \((X, \tau)\) for every closed set \(V\) of \((Y, \sigma)\).

5. \(rg\)-continuous \([17]\) if \(f^{-1}(V)\) is a \(rg\)-closed set of \((X, \tau)\) for every closed set \(V\) of \((Y, \sigma)\).

6. \(gp\)-continuous \([2]\) if \(f^{-1}(V)\) is a \(gp\)-closed set of \((X, \tau)\) for every closed set \(V\) of \((Y, \sigma)\).

7. \(gpr\)-continuous \([10]\) if \(f^{-1}(V)\) is a \(gpr\)-closed set of \((X, \tau)\) for every closed set \(V\) of \((Y, \sigma)\).

8. \(g^*\)-continuous \([18]\) if \(f^{-1}(V)\) is a \(g^*\)-closed set of \((X, \tau)\) for every closed set \(V\) of \((Y, \sigma)\).

9. \(g^*\)-irresolute \([18]\) if \(f^{-1}(V)\) is a \(g^*\)-closed set of \((X, \tau)\) for every \(g^*\)-closed set \(V\) of \((Y, \sigma)\).

10. \(wg\)-continuous \([16]\) if \(f^{-1}(V)\) is a \(wg\)-closed set of \((X, \tau)\) for every closed set \(V\) of \((Y, \sigma)\).

11. \(g^{**}\)-continuous \([20]\) if \(f^{-1}(V)\) is a \(g^{**}\)-closed set of \((X, \tau)\) for every closed set \(V\) of \((Y, \sigma)\).

12. \(g^{**}\)-irresolute \([20]\) if \(f^{-1}(V)\) is a \(g^{**}\)-closed set of \((X, \tau)\) for every \(g^{**}\)-closed set \(V\) of \((Y, \sigma)\).

Definition 2.4: A topological space \((X, \tau)\) is said to be

1. a \(T_{1/2}^\rho\)-space \([11]\) if every \(g\)-closed set in it is closed.

2. a \(T_8\)-space \([6]\) if every \(gs\)-closed set in it is closed.

3. a \(gT_9\)-space \([8]\) if every \(\alpha g\)-closed set in it is closed.

4. a \(T_{1/2}^\gamma\)-space \([18]\) if every \(g^*\)-closed set in it is closed.

5. a \(T_{1/2}^\gamma\)-space \([20]\) if every \(g^{**}\)-closed set is closed.

6. a \(* T_{1/2}\)-space \([20]\) if every \(g^{**}\)-closed set is \(g^*\)-closed.

3. Basic properties of \(pg^{**}\)-closed sets

We introduce the following definition

Definition 3.1: A subset \(A\) of \((X, \tau)\) is said to be a \(pg^{**}\)-closed set if \(\text{pcl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(g^*\)-open in \(X\).

The class of \(pg^{**}\)-closed subset of \((X, \tau)\) is denoted by \(PG^{**}C(X, \tau)\).

Proposition 3.2: Every closed set is \(pg^{**}\)-closed.

Proof follows from the definition.

The following example supports that a \(pg^{**}\)-closed set need not be closed in general.

Proposition 3.3: Every pre closed set is \(pg^{**}\)-closed.

Proof follows from the definition.

Proposition 3.4: Every \(g^{**}\)-closed set is \(pg^{**}\)-closed.

Proof follows from the definition.

Proposition 3.5: Every \(g^*\)-closed set is \(pg^{**}\)-closed.

Proof follows from the definition.

Proposition 3.6: Every \(g\)-closed set is \(pg^{**}\)-closed.

Proof follows from the definition.

The converse of the above propositions need not be true in general.
Example 3.7: Let \( X = \{a, b, c\} \), \( \tau = \{ \emptyset, X, \{a\} \} \). Let \( A = \{a\} \) then \( A \) is a \( pg^{**} \)-closed set but not a closed set and a \( g^{**} \)-closed set of \( (X, \tau) \). So the class of \( pg^{**} \)-closed sets properly contains the class of closed sets and the class of \( g^{**} \)-closed sets. Also \( A = \{a\} \) is not a \( g \)-closed set.

Example 3.8: Let \( X = \{a, b, c\} \), \( \tau = \{ \emptyset, X, \{a\} \} \). Let \( A = \{a, b\} \) then \( A \) is a \( pg^{**} \)-closed set but not a pre closed set and a \( g^{*} \)-closed set of \( (X, \tau) \). So the class of \( pg^{**} \)-closed sets properly contains the class of pre closed sets and the class of \( g^{*} \)-closed sets.

Proposition 3.9: Every \( pg^{**} \)-closed set is (1) \( rg \)-closed (2) \( gpr \)-closed (3) \( gsp \)-closed.

Proof follows from the definition.

The converse of the above propositions need not be true in general as seen in the following examples.

Example 3.10: In example (3.8), let \( A = \{a\} \) is \( gpr \)-closed and \( rg \)-closed but it is not \( pg^{**} \)-closed. Let \( X = \{a, b, c\} \), \( \tau = \{ \emptyset, X, \{a\}, \{b\}, \{a, b\} \} \). Let \( A = \{a\} \) then \( A \) is a \( gsp \)-closed set but not a \( pg^{**} \)-closed set of \( (X, \tau) \). Therefore the class of \( pg^{**} \)-closed sets is properly contained in the class of \( gpr \)-closed, \( rg \)-closed, \( gsp \)-closed sets.

Remark 3.11: \( pg^{**} \)-closedness is independent from \( \alpha \)-closedness, \( semi-closedness \), \( sg \)-closedness, \( ga\alpha \)-closedness, \( ga^{*} \)-closedness and \( semi-preclosedness \).

Let \( X = \{a, b, c\} \), \( \tau = \{ \emptyset, X, \{a\}, \{a, c\} \} \). Let \( A = \{a, b\} \) then \( A \) is a \( pg^{**} \)-closed set. \( A \) is neither \( \alpha \)-closed nor semi-closed, in fact, it is not even a \( semi-preclosed \) set. Also it is not \( sg \)-closed, \( ga \)-closed and \( ga^{*} \)-closed set.

Proposition 3.12: If \( A \) and \( B \) are \( pg^{**} \)-closed sets, then \( A \cup B \) is also a \( pg^{**} \)-closed set.

Proof follows from the fact that \( pcl(A \cup B) = pcl(A) \cup pcl(B) \).

Proposition 3.13: If \( A \) is both \( g^{*} \)-open and \( pg^{**} \)-closed, then \( A \) is pre closed.

Proof follows from the definition of \( pg^{**} \)-closed sets.

Proposition 3.14: \( A \) is a \( pg^{**} \)-closed of \( (X, \tau) \) if \( pcl(A) \setminus A \) does not contain any non-empty \( g^{*} \)-closed set.

Proof: Let \( F \) be a \( g^{*} \)-closed set of \( (X, \tau) \) such that \( F \subseteq pcl(A) \setminus A \). Then \( A \subseteq X \setminus F \). Since \( A \) is \( pg^{**} \)-closed and \( X \setminus F \) is \( g^{*} \)-open, \( pcl(A) \subseteq X \setminus F \). This implies \( F \subseteq X \setminus pcl(A) \). So, \( F \subseteq (X \setminus pcl(A)) \cap (pcl(A) \setminus A) \subseteq (X \setminus pcl(A)) \cap (pcl(A) = \emptyset \). Therefore \( F = \emptyset \).

Proposition 3.15: If \( A \) is a \( pg^{**} \)-closed set of \( (X, \tau) \) such that \( A \subseteq B \subseteq pcl(A) \), then \( B \) is also a \( pg^{**} \)-closed set of \( (X, \tau) \).

Proof: Let \( U \) be a \( g^{*} \)-open set of \( (X, \tau) \) such that \( B \subseteq U \). Then \( A \subseteq U \), since \( A \) is \( pg^{**} \)-closed, then \( pcl(A) \subseteq U \). Now \( pcl(B) \subseteq pcl(pcl(A)) = pcl(A) \subseteq U \). Therefore \( B \) is also a \( pg^{**} \)-closed set of \( (X, \tau) \).

4. \( pg^{**} \)-continuous and \( pg^{**} \)- irresolute maps.

We introduce the following definitions.

Definition 4.1: A function \( f : (X, \tau) \to (Y, \sigma) \) is called \( pg^{**} \)- continuous if \( f^{-1}(V) \) is a \( pg^{**} \)-closed set of \( (X, \tau) \) for every closed set of \( (Y, \sigma) \).

Theorem 4.2: Every continuous map is \( pg^{**} \)-continuous.

Proof: Let \( f : (X, \tau) \to (Y, \sigma) \) be continuous and let \( F \) be any closed set of \( Y \), then \( f^{-1}(V) \) is closed in \( X \). Since every closed set is \( pg^{**} \)-closed, \( f^{-1}(V) \) is \( pg^{**} \)-closed. Therefore \( f \) is \( pg^{**} \)-continuous.

The following example shows that the converse of the above theorem need not be true in general.

Example 4.3: Let \( X = Y = \{a, b, c\} \), \( \tau = \{ \emptyset, X, \{a\} \} \), \( \sigma = \{ \emptyset, X, \{b\} \} \), \( f : (X, \tau) \to (Y, \sigma) \) is defined as the identity map. The inverse image of all the closed sets of \( (Y, \sigma) \) are \( pg^{**} \)-closed in \( (X, \tau) \). Therefore \( f \) is \( pg^{**} \)-continuous but not continuous.

Thus the class of all \( pg^{**} \)-continuous maps properly contains the class of continuous maps.
Theorem 4.4: Every pg**- continuous map is rg- continuous, gpr- continuous and gsp-continuous maps.

Proof: Let \( f : (X, \tau) \to (Y, \sigma) \) be a pg**- continuous map. Let \( V \) be a closed set of \((Y, \sigma)\). Since \( f \) is pg**- continuous, then \( f^{-1}(V) \) is pg**- closed in \((X, \tau)\). By proposition (3.9), \( f^{-1}(V) \) is rg-closed, gpr-closed and gsp-closed set of \((X, \tau)\).

The converse of the above theorem need not be true as seen in the following example.

Example 4.5: Let \( X = Y = \{a, b, c\} \), \( \tau = \{\phi, X, \{a\}\}, \sigma = \{\phi, X, \{b, c\}\}\). Let \( f : (X, \tau) \to (Y, \sigma) \) be the identity map. Then \( f^{-1}(\{a\}) = \{a\} \) is not pg**- closed in \((X, \tau)\). But \{a\} is rg-closed and gpr-closed. Therefore \( f \) is rg- continuous and gpr-continuous but \( f \) is not pg**- continuous.

Example 4.6: Let \( X = Y = \{a, b, c\}, \tau = \{\phi, X, \{a\}, \{a, b\}\}, \sigma = \{\phi, X, \{b, c\}\}\). Let \( f : (X, \tau) \to (Y, \sigma) \) be the identity map. Then \( f^{-1}(\{a\}) = \{a\} \) is not pg**- closed in \((X, \tau)\). But \{a\} is gsp-closed. Therefore \( f \) is gsp-continuous but \( f \) is not pg**-continuous.

Thus the class of all pg**-continuous maps is properly contained in the classes of rg-continuous, gpr- continuous and gsp-continuous maps.

The following example shows that the compositions of two pg**- continuous maps need not be a pg**- continuous map.

Example 4.7: Let \( X = Y = Z = \{a, b, c\} \) and let \( f : (X, \tau) \to (Y, \sigma) \), \( g : (Y, \sigma) \to (Z, \eta) \), be the identity maps. \( \tau = \{\phi, X, \{a\}, \{a, c\}\}, \sigma = \{\phi, X, \{a\}\}, \eta = \{\phi, X, \{b\}\}\). \((f \circ g)^{-1}(\{a, c\}) = f^{-1}(g^{-1}(\{a, c\})) = f^{-1}(\{a, c\}) = \{a, c\}\) is not pg**- closed in \((X, \tau)\). But \( f \) and\( g \) are pg**-continuous maps.

Theorem 4.8: Every g*- continuous map is pg**- continuous map.

Proof: Let \( f : (X, \tau) \to (Y, \sigma) \) be g*-continuous and let \( V \) be a closed set of \( Y \). Then \( f^{-1}(V) \) is g*-closed and hence by proposition (3.5), it is pg**- closed. Hence \( f \) is pg**-continuous map.

The following example shows that the converse of the above theorem is not true in general.

Example 4.9: Let \( X = Y = \{a, b, c\} \), \( \tau = \{\phi, X, \{a\}\}, \sigma = \{\phi, X, \{b\}\}\). Let \( f : (X, \tau) \to (Y, \sigma) \) be the identity map. Then \( A = \{a, c\} \) is closed in \((Y, \sigma)\) and is pg**- closed in \((X, \tau)\) but not g*-closed in \((X, \tau)\). Therefore isf \( f \) is pg**-continuous but not g*-continuous.

Theorem 4.10: Every g- continuous map is pg**- continuous map.

Proof: Let \( f : (X, \tau) \to (Y, \sigma) \) be g-continuous and let \( V \) be a closed set of \( Y \). Then \( f^{-1}(V) \) is g-closed and hence by proposition (3.6), it is pg**-closed. Hence \( f \) is pg**-continuous map.

The following example shows that the converse of the above theorem is not true in general.

Example 4.11: Let \( X = Y = \{a, b, c\}, \tau = \{\phi, X, \{a\}, \{a, c\}\}, \sigma = \{\phi, X, \{a, b\}\}\). Let \( f : (X, \tau) \to (Y, \sigma) \) be the identity map. Then \( A = \{c\} \) is closed in \((Y, \sigma)\) and is pg**-closed in \((X, \tau)\) but not g- closed in \((X, \tau)\). Therefore isf \( f \) is pg**-continuous but not g-continuous.

Theorem 4.12: Every g**- continuous map is pg**- continuous map.

Proof: Let \( f : (X, \tau) \to (Y, \sigma) \) be g**-continuous and let \( V \) be a closed set of \( Y \). Then \( f^{-1}(V) \) is g**-closed and hence by proposition (3.4), it is pg**-closed. Hence \( f \) is pg**-continuous map.

The following example shows that the converse of the above theorem is not true in general.

Example 4.13: Let \( X = Y = \{a, b, c\}, \tau = \{\phi, X, \{a\}, \{a, c\}\}, \sigma = \{\phi, X, \{b, c\}\}\). Let \( f : (X, \tau) \to (Y, \sigma) \) be the identity map. Then \( A = \{a\} \) is closed in \((Y, \sigma)\) and is pg**-closed in \((X, \tau)\) but not g**-closed in \((X, \tau)\). Therefore isf \( f \) is pg**-continuous but not g**-continuous.

Definition 4.14: A function \( f : (X, \tau) \to (Y, \sigma) \) is called pg**- irresolute if \( f^{-1}(V) \) is a pg**-closed set of \((X, \tau)\) for every pg**-closed set \( V \) of \((Y, \sigma)\).
Definition 4.15: Let \((X, \tau)\) and \((Y, \sigma)\) be two topological spaces and \(f : (X, \tau) \to (Y, \sigma)\) is said to be pg**- resolute if 
\(f(U)\) is pg**- open in \(Y\) whenever \(U\) is pg**- open in \(X\).

Definition 4.16: A function \(f : (X, \tau) \to (Y, \sigma)\) is called pg**-homeomorphism if
(i) \(f\) is one – one and onto.
(ii) \(f\) is pg**- irresolute and pg**- resolute.

Theorem 4.17: Every pg**- irresolute function is pg**- continuous.
Proof follows from the definition.

Theorem 4.18: Every g - irresolute function is pg**- continuous.
Proof follows from the definition.

Theorem 4.19: Every g* - irresolute function is pg**- continuous.
Proof follows from the definition.

Theorem 4.20: Every g** - irresolute function is pg**- continuous.
Proof follows from the definition.

Converse of the above theorems need not be true in general as seen in the following example.

Example 4.21: Let \(X = \{a, b, c\}, \tau = \{\phi, X, \{a\}\}, \sigma = \{\phi, X, \{a\}\}\). Let \(f : (X, \tau) \to (Y, \sigma)\) by \(f(a) = b, f(b) = a, f(c) = c\), \(\{b, c\}\) is the only closed set of \(Y\). \(f^{-1}(\{b, c\}) = \{a, c\}\) is pg**- closed in \((X, \tau)\). Therefore \(f\) is pg**- continuous. \(\{b, c\}\) is pg**- closed in \((X, \tau)\). Therefore \(f\) is not g – closed, g* - closed and g**- closed set in \(X\). Therefore \(f\) is not a pg**- irresolute. Hence \(f\) is pg**- continuous but not pg**- irresolute.

Theorem 4.22: Let \(f : (X, \tau) \to (Y, \sigma)\) and \(g : (Y, \sigma) \to (Z, \eta)\), be any two functions then,
(i) \(g \circ f\) is pg**- continuous if \(g\) is continuous and \(f\) is pg**- continuous.
(ii) \(g \circ f\) is pg**- irresolute if both \(f\) and \(g\) are pg**- irresolute.
(iii) \(g \circ f\) is pg**- continuous if \(g\) is pg**- continuous and \(f\) is pg**- irresolute.

5. Applications of pg**- closed sets

As applications of pg**- closed sets, new spaces, namely, \(p_{T^{\text{1/2}}_{*}}\) space, \(p_{T^{*}_{*}}\) space, \(p_{T^{1/2}_{*}}\) space, \(p_{T^{*}_{*}}\) space and \(p_{T_{c}}\) space are introduced.

We introduce the following definition.

Definition 5.1: A space \((X, \tau)\) is called a \(p_{T^{*}_{*}}\) space if every pg**- closed set is closed.

Theorem 5.2: Every \(p_{T^{*}_{*}}\) space is \(T_{1/2}\) space.
Proof follows from the definition.

Theorem 5.3: Every \(p_{T^{1/2}_{*}}\) space is \(T_{1/2}\) space.
Proof follows from the definition.

The converse need not be true in general as seen in the following example.

Example 5.4: Let \(X = \{a, b, c\}, \tau = \{\phi, X, \{a\}\}, G^*(C(X,\tau)) = \{\phi, X, \{b, c\}\} = C(X,\tau)\). Therefore \((X, \tau)\) is a \(T_{1/2}\) space but not \(p_{T^{1/2}_{*}}\) space since \(\{a, b\}\) is a pg**- closed set but not a closed set of \((X, \tau)\).
**Theorem 5.5:** Every $T_2$-space is $\mathcal{P}T^{**}_{1/2}$-space.

Proof follows from the definition.

The converse need not be true in general as seen in the following example.

**Example 5.6:** Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. $(X,\tau)$ is a $\mathcal{P}T^{**}_{1/2}$-space but not a $T_2$-space since $\{a\}$ is gs-closed but not closed.

**Remark 5.7:** $T_d$-ness is independent of $\mathcal{P}T^{**}_{1/2}$-ness as it can be seen from the following example.

**Example 5.8:** Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. $(X,\tau)$ is a $\mathcal{P}T^{**}_{1/2}$-space but not a $T_d$-space since $\{a\}$ is gs-closed but not g-closed.

**Example 5.9:** Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$. $(X,\tau)$ is a $T_d$-space but not a $\mathcal{P}T^{**}_{1/2}$-space since $\{c\}$ is pg**-closed but not closed.

**Theorem 5.10:** The following conditions are equivalent in topological space $(X,\tau)$.

(i) $(X,\tau)$ is a $\mathcal{P}T^{**}_{1/2}$-space.

(ii) Every singleton of $X$ is either g*-closed or open.

Proof:

(i) $\Rightarrow$ (ii): Let $(X,\tau)$ be a $\mathcal{P}T^{**}_{1/2}$-space. Let $x \in X$ and suppose $\{x\}$ is not g*-closed. Then $X \setminus \{x\}$ is not g*-open. This implies that $X$ is the only g*-open set containing $X \setminus \{x\}$. Therefore $X \setminus \{x\}$ is closed since $(X,\tau)$ is a $\mathcal{P}T^{**}_{1/2}$-space. Therefore $\{x\}$ is open in $(X,\tau)$.

(ii) $\Rightarrow$ (i): Let $A$ be a pg**-closed set of $(X,\tau)A \subseteq pcl(A) \subseteq cl(A)$ and let $x \in pcl(A)$ this implies $x \in cl(A)$.

We introduce the following definition.

**Definition 5.11:** A space $(X,\tau)$ is called an $apT^*_c$-space if every $\alpha g$-closed set of $(X,\tau)$ is pg**-closed.

**Theorem 5.12:** Every $\alpha g$-closed set is an $apT^*_c$-space but not conversely.

**Example 5.13:** Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}\}$. $(X,\tau)$ is an $apT^*_c$-space but not $\alpha g$-closed but not closed.

**Definition 5.14:** A subset A of a space $(X,\tau)$ is called a pg**-open set if its complement is a pg**-closed set of $(X,\tau)$.

**Theorem 5.15:** If $(X,\tau)$ is an $apT^*_c$-space for each $x \in X$, $\{x\}$ is either $\alpha g$-closed or pg**-open.

Proof: Let $x \in X$ suppose that $\{x\}$ is not an $\alpha g$-closed set of $(X,\tau)$. Then $\{x\}$ is not a closed set since every closed set is an $\alpha g$-closed set. Therefore $X \setminus \{x\}$ is not open. Therefore $X \setminus \{x\}$ is an $\alpha g$-closed set since $X$ is the only open set which contains $X \setminus \{x\}$. Since $(X,\tau)$ is an $apT^*_c$-space, $X \setminus \{x\}$ is a pg**-closed set or $\{x\}$ is pg**-open.

**Remark 5.16:** The converse of the above theorem is not true as it can be seen from the following example.

**Example 5.17:** Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}\}$. $(X,\tau)$ is not a $apT^*_c$-space but $\{b\}$ is $\alpha g$-closed and $\{a\}$ and $\{c\}$ are pg**-open.

We introduce the following definition.

**Definition 5.18:** A space $(X,\tau)$ is called a $\mathcal{P}T^{**}_{1/2}$-space if every pg**-closed set of $(X,\tau)$ is a g*-closed set.
Theorem 5.19: Every $p_{1/2}^T$ space is $\gamma p_{1/2}$-space.

Proof: Let $(X,\tau)$ be a $p_{1/2}^T$ space. Let $A$ be a $pg^{**}$- closed set of $(X,\tau)$. Since $(X,\tau)$ is a $p_{1/2}^T$-space, $A$ is closed. But since every closed set is $g^*$- closed, $A$ is $g^*$- closed. Therefore $(X,\tau)$ is a $p_{1/2}^T$-space.

Theorem 5.20: Every $T_b$-space is a $\gamma p_{1/2}$-space.

Proof: Let $(X,\tau)$ be a $T_b$-space. Then by theorem (4.5), it is a $p_{1/2}^T$ space. Therefore by theorem (4.19), it is $\gamma p_{1/2}$-space.

The converse of the above theorem need not be true as seen in the following example.

Example 5.21: Let $X = \{a, b, c\}$, $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$. $(X,\tau)$ is a $\gamma T_{1/2}$-space but not a $T_b$-space since $A = \{a\}$ is $gs$-closed but not closed.

Theorem 5.22: Every $\gamma T_{1/2}$-space is a $T_{1/2}$-space.

Proof: Let $(X,\tau)$ be a $\gamma T_{1/2}$- space. Let $A$ be a $g^*$- closed set of $(X,\tau)$. Then by proposition (3.6), $A$ is $pg^{**}$- closed. Since $(X,\tau)$ is an $\gamma T_{1/2}$- space, $A$ is $g^*$- closed. Therefore it is a $T_{1/2}$- space.

The converse of the above theorem need not be true as seen in the following example.

Example 5.23: Let $X = \{a, b, c\}$, $\tau = \{\varnothing, X, \{a, c\}\}$. $(X,\tau)$ is a $T_{1/2}$-space but not a $\gamma T_{1/2}$- space since $A = \{c\}$ is $g^*$- closed but not $g^{**}$- closed.

Theorem 5.24: Every $\gamma T_{1/2}$-space is a $\gamma^* T_{1/2}$-space.

Proof: Let $(X,\tau)$ be a $\gamma T_{1/2}$- space. Let $A$ be a $g^{**}$- closed set of $(X,\tau)$. Then by proposition (3.4), $A$ is $pg^{**}$- closed. Since $(X,\tau)$ is a $\gamma T_{1/2}$- space, $A$ is $g^*$- closed. Therefore it is a $\gamma^* T_{1/2}$- space.

The converse of the above theorem need not be true as seen in the following example.

Example 5.25: Let $X = \{a, b, c\}$, $\tau = \{\varnothing, X, \{a\}\}$. $(X,\tau)$ is a $\gamma^* T_{1/2}$-space but not a $\gamma T_{1/2}$- space since $A = \{c\}$ is $g^*$- closed but not $g^{**}$- closed.

Theorem 5.26: If $(X,\tau)$ is a $\gamma^* T_{1/2}$- space, then for each $x \in X$, $\{x\}$ is either closed or $g^*$-open.

Proof: Suppose $(X,\tau)$ is a $\gamma^* T_{1/2}$- space. Let $x \in X$ and let $\{x\}$ not be closed. Then $X \setminus \{x\}$ is not open set. Therefore $X \setminus \{x\}$ is a $g$-closed set since $X$ is the only open set which contains $X \setminus \{x\}$. By theorem (3.6) $X \setminus \{x\}$ is a $pg^{**}$- closed set. Since $(X,\tau)$ is a $\gamma^* T_{1/2}$- space, $X \setminus \{x\}$ is $g^*$- closed set. Therefore $\{x\}$ is $g^*$-open.

Definition 5.27: A space $(X,\tau)$ is called an $p_{1/2}$-space if every $pg^{**}$-closed set of $(X,\tau)$ is $g$-closed.

Theorem 5.28: Every $p_{1/2}^T$-space is a $p_{1/2}^T$-space.

Proof: Let $(X,\tau)$ be a $p_{1/2}^T$-space. Let $A$ be a $pg^{**}$- closed set of $(X,\tau)$. Then $A$ is closed since $(X,\tau)$ is a $p_{1/2}^T$-space. But every closed set is $g$-closed set. Therefore $(X,\tau)$ is a $p_{1/2}^T$-space.

The converse of the above theorem need not be true as seen in the following example.

Example 5.29: Let $X = \{a, b, c\}$, $\tau = \{\varnothing, X, \{a\}\}$. $(X,\tau)$ is a $p_{1/2}^T$-space but not a $p_{1/2}^T$-space since $A = \{c\}$ is $pg^{**}$- closed but not $g^{**}$- closed.

Theorem 5.30: The space $(X,\tau)$ is a $p_{1/2}^T$-space if and only if it is a $p_{1/2}^T$-space and a $T_{1/2}$- space.

Proof: Necessity: Let $(X,\tau)$ be a $p_{1/2}^T$-space. Let $A$ be a $g$-closed set of $(X,\tau)$. Then by theorem (3.6) $A$ is $pg^{**}$- closed. Also since $(X,\tau)$ is a $p_{1/2}^T$-space, $A$ is a closed set. Therefore $(X,\tau)$ is a $T_{1/2}$- space. By theorem (4.24) $(X,\tau)$ is a $p_{1/2}^T$-space.

Sufficiency: Let $(X,\tau)$ be a $T_{1/2}$- space and a $p_{1/2}^T$-space. Let $A$ be a $pg^{**}$- closed set. Then $A$ is $g$-closed since $(X,\tau)$ is a $p_{1/2}^T$-space. Also since $(X,\tau)$ is a $T_{1/2}$-space, $A$ is a closed set. Therefore $(X,\tau)$ is a $p_{1/2}^T$-space.
Theorem 5.31: Every \( \rho T_{1/2} \)-space is a \( \rho T_{1/2}^* \)-space.

Let \((X, \tau)\) be a \( \rho T_{1/2} \)-space. Let \( A \) be a pg**-closed set. Then \( A \) is g*-closed since \((X, \tau)\) is a \( \rho T_{1/2} \)-space. But every g*-closed set is g-closed, and hence \( A \) is a g-closed set. Therefore \((X, \tau)\) is a \( \rho T_{1/2}^* \)-space.

The converse of the above theorem need not be true as seen in the following example.

Example 5.32: Let \( X = \{a, b, c, d\}, \tau = \{\emptyset, X, \{a\}\} \), \((X, \tau)\) is a \( \rho T_{1/2} \)-space but not a \( \rho T_{1/2}^* \)-space since \( A = \{c\} \) is pg**-closed but not a g*-closed set.

We introduce the following definition

Definition 5.33: A space \((X, \tau)\) is called a \( \rho T_c \)-space if every gs-closed set of \((X, \tau)\) is a pg**-closed set.

Theorem 5.34: Every \( T_c \)-space is a \( \rho T_c \)-space.

Proof: Let \((X, \tau)\) be a \( T_c \)-space. Let \( A \) be a gs-closed set of \((X, \tau)\). Then \( A \) is g*-closed since \((X, \tau)\) is a \( T_c \)-space. But by proposition (3.5) \( A \) is pg**-closed set. Therefore \((X, \tau)\) is a \( \rho T_c \)-space.

The converse of the above theorem need not be true as seen in the following example.

Example 5.35: Let \( X = \{a, b, c\}, \tau = \{\emptyset, X, \{a, c\}\} \), \((X, \tau)\) is a \( \rho T_c \)-space but not a \( T_c \)-space since \( A = \{c\} \) is gs-closed but not g*-closed set.

Theorem 5.36: Every \( T_h \)-space is a \( \rho T_c \)-space.

Proof: Let \((X, \tau)\) be a \( T_h \)-space. Let \( A \) be a gs-closed set of \((X, \tau)\). Then \( A \) is g*-closed since \((X, \tau)\) is a \( T_h \)-space. But by proposition (3.2) \( A \) is pg**-closed set. Therefore \((X, \tau)\) is a \( \rho T_c \)-space.

The converse of the above theorem need not be true as seen in the following example.

Example 5.37: Let \( X = \{a, b, c\}, \tau = \{\emptyset, X, \{a, c\}\} \), \((X, \tau)\) is a \( \rho T_c \)-space but not a \( T_h \)-space since \( A = \{c\} \) is gs-closed but not aclosed set.

Theorem 5.38: If \((X, \tau)\) is a \( \rho T_c \)-space and a \( \rho T_{1/2}^* \)-space, then it is a \( aT_d \)-space.

Proof: Let \((X, \tau)\) be a \( \rho T_c \)-space and a \( \rho T_{1/2}^* \)-space. Let \( A \) be a pg**-closed set of \((X, \tau)\). Then \( A \) is also g*-closed. Since \((X, \tau)\) is a \( \rho T_c \)-space, \( A \) is pg**-closed. Also since \((X, \tau)\) is a \( \rho T_{1/2}^* \)-space, \( A \) is a g-closed set. Therefore \((X, \tau)\) is a \( aT_d \)-space.

The converse of the above theorem need not be true as seen in the following example.

Example 5.39: Let \( X = \{a, b, c\}, \tau = \{\emptyset, X, \{a, c\}\} \), \((X, \tau)\) is a \( aT_d \)-space but not a \( \rho T_{1/2}^* \)-space since \( A = \{c\} \) is pg**-closed but not ag-closed set.

Theorem 5.40: If \((X, \tau)\) is a \( \rho T_c \)-space and a \( \rho T_{1/2}^* \)-space, then it is a \( aT_b \)-space.

Proof: Let \((X, \tau)\) be a \( \rho T_c \)-space and a \( \rho T_{1/2}^* \)-space. Let \( A \) be a pg**-closed set of \((X, \tau)\). Then \( A \) is also g*-closed. Since \((X, \tau)\) is a \( \rho T_c \)-space, \( A \) is pg**-closed. But every pg**-closed set is closed since \((X, \tau)\) is a \( \rho T_{1/2}^* \)-space, \( A \) is a closed set. Therefore \((X, \tau)\) is a \( aT_b \)-space.

Theorem 5.41: If \((X, \tau)\) is a \( \rho T_c \)-space and a \( \rho T_{1/2}^* \)-space, then it is a \( T_d \)-space.

Proof: Let \((X, \tau)\) be a \( \rho T_c \)-space and a \( \rho T_{1/2}^* \)-space. Let \( A \) be a gs-closed set of \((X, \tau)\). Since \((X, \tau)\) is a \( \rho T_c \)-space, \( A \) is pg**-closed. Also since \((X, \tau)\) is a \( \rho T_{1/2}^* \)-space, \( A \) is g-closed set. Therefore \((X, \tau)\) is a \( T_d \)-space.

The converse of the above theorem need not be true as seen in the following example.

Example 5.42: Let \( X = \{a, b, c\}, \tau = \{\emptyset, X, \{a, c\}\} \), \((X, \tau)\) is a \( T_d \)-space but not a \( \rho T_{1/2}^* \)-space since \( A = \{c\} \) is pg**-closed but not ag-closed set.
Theorem 5.43: If \( (X, \tau) \) is a \( T^*_c \)-space, then for each \( x \in X \), \( \{x\} \) is either semi-closed or pg**-open in \( (X, \tau) \).

Proof: Suppose \( (X, \tau) \) is a \( T^*_c \)-space. Let \( x \in X \) and let \( \{x\} \) not be semi-closed. Then \( X \setminus \{x\} \) is gs-closed. Also \( X \setminus \{x\} \) is pg**-closed set. Therefore \( \{x\} \) is pg**-open.

Theorem 5.44: Let \( f : (X, \tau) \to (Y, \sigma) \) be a pg**-continuous map. If \( (X, \tau) \) is \( p^*_T/1/2 \)-space then \( f \) is continuous.

Theorem 5.45: Let \( f : (X, \tau) \to (Y, \sigma) \) be a pg**-continuous map. If \( (X, \tau) \) is \( p^*_T/1/2 \)-space then \( f \) is \( g^* \)-continuous.

Theorem 5.46: Let \( f : (X, \tau) \to (Y, \sigma) \) be a pg**-continuous map. If \( (X, \tau) \) is \( p^*_T/1/2 \)-space then \( f \) is \( g \)-continuous.

Theorem 5.47: Let \( f : (X, \tau) \to (Y, \sigma) \) be a gs-continuous map. If \( (X, \tau) \) is \( p^*_T/1/2 \)-space then \( f \) is pg**-continuous.

Theorem 5.48: Let \( f : (X, \tau) \to (Y, \sigma) \) be ag**- irresolute map and a pre-closed map. Then \( f(A) \) is a pg**-closed set of \( (Y, \sigma) \) for every pg**-closed set \( A \) of \( (X, \tau) \).

Proof: Let \( A \) be a pg**-closed set of \( (X, \tau) \). Let \( U \) be a \( g^* \)-open set of \( (Y, \sigma) \) such that \( f(A) \subseteq U \). Since \( f \) is \( g^* \)-irresolute, \( f^{-1}(U) \) is \( g^* \)-open in \( (X, \tau) \). Now \( f^{-1}(U) \) is \( g^* \)-open and \( A \) is pg**-closed set of \( (X, \tau) \), then \( pcl(A) \subseteq f^{-1}(U) \). Then \( f(pcl(A)) = pcl(f(pcl(A))) \). Therefore \( pcl(f(A)) \subseteq pcl(f(pcl(A))) = f(pcl(A)) \subseteq U \). Therefore \( f(A) \) is a pg**-closed set of \( (Y, \sigma) \).

Theorem 5.49: Let \( f : (X, \tau) \to (Y, \sigma) \) be onto, pg**- irresolute and closed. If \( (X, \tau) \) is \( p^*_T/1/2 \) then \((Y, \sigma)\) is also a \( p^*_T/1/2 \)-space.

Definition 5.50: A function \( f : (X, \tau) \to (Y, \sigma) \) is called a pg**-closed map if \( f(A) \) is a pg**-closed set of \( (Y, \sigma) \) for every pg**-closed set \( A \) of \( (X, \tau) \).

Theorem 5.51: Let \( f : (X, \tau) \to (Y, \sigma) \) be onto, pg**- irresolute and pre - \( g^* \)- closed. If \( (X, \tau) \) is \( p^*_T/1/2 \), then \((Y, \sigma)\) is also a \( p^*_T/1/2 \)-space.

Theorem 5.52: Let \( f : (X, \tau) \to (Y, \sigma) \) be onto, gs - irresolute and pg**-closed map. If \( (X, \tau) \) is \( p^*_T/1/2 \), then \((Y, \sigma)\) is also a \( p^*_T/1/2 \)-space.

Theorem 5.53: Let \( f : (X, \tau) \to (Y, \sigma) \) be onto, pg** - irresolute and g-closed map. If \( (X, \tau) \) is \( p^*_T/1/2 \), then \((Y, \sigma)\) is also a \( p^*_T/1/2 \)-space.

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