International Journal of Mathematical Archive-6(8), 2015, 111-114
Available online through www.ijma.info ISSN 2229 – 5046

A STUDY OF STRUCTURAL COEFFICIENTS ATTRIBUTED TO STOCHASTIC DIFFERENCE EQUATION MODELS

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(Received On: 21-07-15; Revised & Accepted On: 25-08-15)

ABSTRACT

In this paper, we study of structural coefficients attributed to stochastic difference equation models. We are concerned with the differences in the amplitudes executed by the endogenous variables in a stochastic difference equation model. It is shown that the differences in the amplitudes are partially a function of the structure of the model.

Keywords: *Stochastic difference equation, steady- state of difference equation, structural coefficients, endogenous variable.*

I. INTRODUCATION

Queuing theory plays an important role in modeling real life problems involving congestions in vide areas of applied sciences. Applications of queuing with impatience can be seen in traffic modeling, business and industries, computer-communication, health sectors and medical sciences.

Systems of Linear Simultaneous difference equations have been employed as expressions of economic behavior patterns in important econometric works. Guy Orcutt, and more recently, J. Johnston have discussed the structure, i.e., the network of structural coefficients, of these equation systems and their application to autoregressive analysis [2]. Orcutt emphasizes the remarkable result that if the inhomogeneous parts of the equation system (including the random elements) are constants or zero from their means, then all autoregressive equations will be identical in the sense that they will have identical coefficients in the endogenous variables except for the efeect of the inhomogeneous parts. Johnston has concluded that the model would imply identical and constant periodicity for each of the endogenous variables.

The purpose of this article is twofold. We shall demonstrate that even though each endogenous variable executes identical periods in the homogeneous part of a simultaneous stochastic difference equation model, the amplitudes or proportionate changes (if the roots are real) of the endogenous variables will differ[8]. Differences in the amplitudes or proportionate changes of the endogenous variables can be attributed to two sources: (stochastic) disturbance terms and the structural coefficients. That the disturbance terms model is readily seen. However, it is not immediately obvious just how the structural coefficients enter into the determination of different relative amplitudes of proportionate changes of the variables. Thus, we shall isolate the structural coefficients as a source of differences in amplitudes or proportionate changes of endogenous variables in a linear simultaneous difference equation model.

Rest of the paper is structured as follows: In section 2, queuing model is formulated. The differential-difference equations of the model are derived and solved iteratively in section 3. Measures of effectiveness are derived. The conclusions are presented in section 4.

II. DIFFERENTIAL DIFFERENCE EQUATIONS AND SOLUTION OF THE QUEUING MODEL

Let Pn(t) be the probability that there are n customers in the system at time t [11]. The differential-difference equations are derived by using the general birth death arguments. These equations are solved iteratively in steady-state in order to obtain the steady state solution [5].

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The differential-difference equations of the model are:

$$\frac{d}{dt}P_0(t) = -\lambda P_0(t) + \mu P_1(t) \tag{2.1}$$

$$\frac{d}{dt}P_n(t) = -\left[\left(\frac{\lambda}{n+1}\right) + \mu + (n-1)\xi p\right]P_n(t) + (\mu + n\xi p)P_{n+1}(t) + \left(\frac{\lambda}{n}\right)P_{n-1}(t);\ n = 1,2,3,......,N-1 \tag{2.2}$$

$$\frac{d}{dt}P_N(t) = -\left[\mu\left(N-1\right)\xi P\right]P_N(t) + \left(\frac{\lambda}{N}\right)P_{N-1}(t) \tag{2.3}$$

In steady state, $\lim_{t\to\infty} P_n(t) = P_n$ and therefore $\frac{dP_n(t)}{dt} = 0$ as $t \to \infty$ and hence, The solution of equation (2.1) to (2.3) gives the difference equations

$$0 = -\lambda P_0 + \mu P_1 \tag{2.4}$$

$$0 = -\left[\left(\frac{\lambda}{n+1}\right) + \mu + (n-1)\xi p\right]P_n + (\mu + n\xi p)P_{n+1} + \left(\frac{\lambda}{n}\right)P_{n-1}; n = 1,2,3,...,N-1 \tag{2.5}$$

$$0 = -(\mu + (N-1)\xi p)P_N + \left(\frac{\lambda}{N}\right)P_{N-1} \tag{2.6}$$

Solving iteratively equation (2.4)-(2.6), we get

$$P_n = \left[\frac{1}{n!}\prod_{k-1}^{n}\frac{\lambda}{\mu + (k-1)\xi p}\right]P_0; 1 \le n| \le N \tag{2.7}$$

Using the normalization condition, $\sum_{n-0}^{N} P_n = 1,$ we get

$$P_0 = \frac{1}{1 + \sum_{n-1}^{N}\left(\frac{1}{n!}\prod_{k}^{n}\frac{\lambda}{\mu + (k-1)\xi p}\right)} \tag{2.8}$$

Hence, the steady- state probabilities of the system size are derived explicitly.

III. MEASURES EFFECTIVENESS OF STOCHASTIC DIFFERENCE EQUATIONS

The structure of a stochastic equation system may be identified with the network of coefficients; these coefficients indicate the nature in which, and the extent to which, each endogenous variable is "related" to each other endogenous variable. Since we are dealing with a linear variables are fixed and invariant, e.g., with respect to time and space. Furthermore, the extent to which each variable is related to each other variable, i.e., the value of the structural coefficients, is determined "outside" the system, e.g., by technological and econometric considerations. Now, the structure partially determines the differences in the amplitudes or proportionate changes of the endogenous variables [10]. Thus, our second purpose is to demonstrate that certain structures may contribute more or less to these differences; specifically, we shall show that a particular structure in which each endogenous variable is related to each other endogenous variable to a small extent generates large differences among the variances of the endogenous variables; the obverse holds.

In a numerical example, we shall spell out some of the implications of this for a "sectoral" interpretation of the accelerator theory, with special reference to a "Hicksian" business cycle model.
Consider two simultaneous difference equations of the first order:

$$R_{11}x_1 + R_{12}x_2 = Y_1$$

$$R_{21}x_1 + R_{22}x_2 = Y_2 \tag{3.1}$$

Where the R_{ij} are polynomials in "E" and the Yi are exogenous terms for our purposes, the Yi may equal zero, since we are primarily interested in the structure of the homogeneous system. The xi are stated in terms of deviations; i.e., x_i=X_i,t$\sqrt{X_i}$, where $\bar{X}$ is the mean of X_i. We shall call X_{12} and X_{21} "coupling polynomials," since they link the endogenous variables together, whereas $X_{11}$and X_{22} are the "self polynomials," since, e.g., X_{11} represents the nature in which, and the extent to which, lagged and current values of x_1 determine the magnitude of x_1, t.

The general solution of (3.1) is:

$$x_1.t = \delta_1^{'}\lambda_1^t + \delta_2^{'}\lambda_2^t + Y_{1,t}$$

$$x_2.t = \delta_1^{'}\lambda_1^t + \delta_2^{'}\lambda_2^t + Y_{2,t} \tag{3.2}$$

Where the $Y_{i,t}$ are solutions to the inhomogeneous parts of (1). The equation in $x_{2,t}$ contains two constants, $\delta_1 and\ \delta_2$.

Similarly, the equation $x_{1,t}$ contains the constants, $\delta_1^{'}\ and\ \delta_2^{'}$. If the $\delta_i^{'} = \delta_i$ for each I, then clearly the time paths of all endogenous variables are identical. But if they are unequal, then $\delta_i^{'}\lambda_i^t = \delta_i\lambda_i^t$, and the difference between the variables would be reflected in the amplitudes or proportionate changes of each endogenous variable.

Now, the two constants in $x_{2,t}$ are not independent of the constants in $x_{1,t}$. After the λ_i are determined and the δ_i are arbitrarily chosen-they represent the initial conditions-the $\delta_i^{'}$ are determined by the structural coefficients [11]. An initial condition is that $x_2(0) = \delta_1 + \delta_2 + Y_1(0)$, and the $\delta_i^{'}$ are determined by

$$\delta_i^{'} = -\delta_i \frac{R_{22}(\lambda_i)}{R_{21}(\lambda_i)} \tag{3.3}$$

If the $\delta_i^{'} = \delta_i$, then the function $x_{1,t}$ and $x_{2,t}$ will not differ with respect to periods or amplitudes or proportionate changes in the endogenous variables, if the roots are real. But in order for this to hold, we must have $R_{22}(\lambda_i) = R_{21}(\lambda_i)$. To recall, the R_{ij}, which include the structural coefficients, are polynomials in E which operate on the endogenous variables, and the λ_i are the roots of the equation system which are determined by the structural coefficients. But the $\delta_i^{'}$ are determined by the initial conditions δ_i , the roots λ_i the coupling polynomial R_{21}, and the self polynomial R_{22}. Rewriting (3.3)

$$\frac{\delta_i^{'}}{\delta_i} = \frac{R_{22}(\lambda_i)}{R_{21}(\lambda_i)} \tag{3.3.1}$$

It can be seen that the larger R_{22} is to R_{21}, the greater is the ratio of $\delta_i^{'}$ to δ_i, and, in turn, the greater is the difference between the variances of the endogenous variables. In other words, the less "related" or "coupled" the variables are to each other, the greater will be the differences in variances between the endogenous variables. Intuitively, this is the case, since the less related the variables are to each other, the less impact will variations in one variable have on the other, and therefore the greater will be the in-dependence in the variances of each.

For a numerical example, consider a two sector theoretical model such as
$x_{2\ t} = v_{21}(x_{1.t} - x_{1.t-1}) + v_{22}(x_{2.t} - x_{2.t-1}) + y_{2.t,}$

Where $x_{1,t}$ is the output of consumer goods in period t, $x_{2,t}$ is the output of investment goods in period t, a_{ij} is j's marginal propensity to consume i's product, V_{ij} is the accelerator coefficient representing the amount of j's output required per unit of i's output, $y_{1.t,}$ is an autonomous consumption term, and $y_{2.t,}$ is an autonomous investment term. Assume that we have the following values for the coefficients: $a_{11} = .8,\ a_{12} = .6, v_{21} = 1.5, v_{22} = 1.8$. Again, the solution has the form:

$$x_1.t = \delta_1^{'}\lambda_1^t + \delta_2^{'}\lambda_2^t + Y_{1,t}$$

$$x_2.t = \delta_1\lambda_1^t + \delta_2\lambda_2^t + Y_{2,t}$$

If we suppose that the autonomous terms in are constants or rising at a steady rate, their role in the solution is simply to determine the level about which fluctuations take place. Thus, for present purposes, we can ignore the solution to the non-homogeneous part. Focusing on the homogeneous part of we find the values of the roots and the $\delta_i^{'}$ to be: $\lambda_{1,2} = -.3 \pm 3.6i$, and $\delta_i^{'} = -.3\delta_1, \delta_i^{'} = -.8\delta_2$. We may infer that fluctuations in the investment goods sector are more violent than fluctuations in the investment goods sector are more violent than fluctuations in the consumer goods sector. This result conforms to both theoretical and empirical expectations. Therefore, in a model such as, the difference in the proportionate changes between the consumer and investment goods sectors is reflected in the solution by the $\delta_1$and $\delta_i^{'}$, which are determined, inter alia, by the structural coefficients.

Suppose that the simultaneous equation system where to be examined by a "Hicksian" business cycle analyst; he would expect the model to contain explosive roots, which would be generated by the large values of the accelerator coefficients. However, he would not expect both sectors to explode toward the ceiling of production at the same rate; the investment goods sector x_2 would have to advance more relentlessly than the consumer goods sector x_1.

IV. CONCLUSION

In this paper we discussed A Study of Structural Coefficients Attributed to Stochastic Difference Equation Models it would be unlikely that the time paths of endogenous variables in a simultaneous difference equation model would be identical with respect to amplitudes or proportional changes, even though the time paths would be identical with respect to periods. Furthermore, we can state that the less related are the variables to each other, the greater will be the differences in the amplitudes or proportionate changes, i.e., in general, different structural relationships between the endogenous variables generate certain differences in the amplitudes or proportionate changes of the endogenous variables. Given the structural coefficients, we are able to determine these differences. These results assume that the (stochastic) disturbance terms are zero or that they do not affect each endogenous variable in a different way.

REFERENCES

1. Allen, E.J., 1999. Stochastic differential equations and persistence time for two interacting populations. Dynam. Contin., Discrete and Impulsive Systems 5, 271–281.
2. Allen, L.J.S., 2003. An Introduction to Stochastic Processes with Applications to Biology. Prentice Hall, Upper Saddle River, NJ. Allen, E.J., Victory, H.D., 2003. Modelling and simulation of a schistosomiasis infection with biological control. Acta Trop. 87, 251–267.
3. Busenberg, S., Cooke, K., 1993. Vertically Transmitted Diseases: Models and Dynamics. Springer, Berlin, Heidelberg, NewYork.
4. Caswell, H., 2001. Matrix Population Models, 2nd Edition. Sinauer Assoc. Inc. Pub., Sunderland, Mass.
5. Dennis, B., Costantino, R.F., 1988. Analysis of steady-state populations with the gamma abundance model: application to Tribolium. Ecology 69, 1200–1213.
6. Dushoff, J., 2000. Carrying capacity and demographic stochasticity: scaling behavior of the stochastic logistic model. Theor. Popul. Biol. 57, 59–65.
7. Guy Orcutt, “A Study of the Autoregressive Nature of the Time Series Used for Tinbergen’s Model of the Economic System of the United States, 1919-1932,” Journal of the Royal Statistical Society, Series B, Vol. X, 1948, pp.1-45.
8. Gard, T.C., 1988. Introduction to Stochastic Differential Equations. Marcel Dekker, Inc., New York and Basel.
9. Karlin, S., Taylor, H.M., 1981. A Second Course in Stochastic Processes. Academic Press, San Diego, NewYork, Boston, London.
10. Kloeden, P.E., Platen, E., 1992. Numerical Solution of Stochastic Differential Equations. Springer, NewYork.
11. Linda J.S. Allen and Edward J. Allen A comparison of three different stochastic population models with regard to persistence time Department of Mathematics and Statistics, Texas Tech University, MS 1042, 117I, Lubbock, TX 79409-1042, USA Received 21 September 2002
12. Roberts, J.B., 1986. First passage probabilities for randomly excited systems: diffusion methods. Prob. Eng. Mech. 1, 66–81.
13. Schurz, H., 2002. Numerical analysis of stochastic differential equations without tears. In: Kannan, D., Lakshmikantham, V. (Eds.), Handbook of Stochastic Analysis and Applications. Marcel-Dekker, NewYork, pp. 237–359.
14. Ter, C., Hanson, F.B., 1981.Persistence in density dependent stochastic populations. Math. Biosci.53, 89– 117.

Source of support: Nil, Conflict of interest: None Declared