International Journal of Mathematical Archive-6(8), 2015, 122-127
Available online through www.ijma.info ISSN 2229 – 5046

A Common Fixed Point Theorem in Complex Valued b-Metric Spaces for Four Mappings

TANMOY MITRA*

Asst. Teacher, Akabpur High School, Akabpur, P.O. - Nadanghat, Dist-Burdwan, India.

(Received On: 24-07-15; Revised & Accepted On: 22-08-15)

ABSTRACT

In this paper we prove a common fixed point theorem for four self-mappings in a complete complex valued b-metric space.

Key Words: *Complex valued b-metric space, weakly compatible mappings.*

AMS Subject Classification (2010): *47H10, 54H25.*

1. INTRODUCTION

In 1989, Bakthtin [3] introduced the concept of b-metric space as a generalization of metric spaces. The concept of complex valued b-metric spaces was introduced in 2013 by Rao *et al.* [10], which was more general than the well-known complex valued metric spaces that were introduced in 2011 by Azam *et al.* [2]. The main purpose of this paper is to present common fixed point results of four self-mappings satisfying a rational inequality on complex valued b-metric spaces. The results presented in this paper are generalization of work done by Sanjib Kumar Dutta and Sultan Ali in [6].

Definition 1 (see [1]): Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d: X \times X \to \mathbb{C}$ is called a b-metric if for all $x, y, z \in X$, the following conditions are satisfied:

(i) $d(x, y) = 0$ if and only if $x = y$
(ii) $d(x, y) = d(y, x)$
(iii) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a b-metric space. The number $s \geq 1$ is called the coefficient of (X, d).

Example 2 (see [11]): Let (X, d) be a metric space and $\rho(x, y) = \left(d(x, y)\right)^p$, where $p > 1$ is a real number. Then (X, ρ) is a b-metric space with $s = 2^{p-1}$.

Let $\mathbb{C}$ be the set of all complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order relation $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

Thus $z_1 \precsim z_2$ if one of the followings holds:

(1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$ and
(4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

We write $z_1 \precnsim z_2$ if $z_1 \precsim z_2$ and $z_1 \neq z_2$ i.e., one of (2), (3) and (4) is satisfied and we will write $z_1 \prec z_2$ if only (4) is satisfied.

Remark 1: We can easily check the followings:

(i) $a, b \in \mathbb{R}, a \leq b \Rightarrow az \precsim bz \ \forall z \in \mathbb{C}$.
(ii) $0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$.
$z_1 \precsim z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

*Corresponding Author: Tanmoy Mitra**
Asst. Teacher, Akabpur High School, Akabpur, P.O. - Nadanghat, Dist-Burdwan, India.

Definition 3 (see [2]): Let X be a nonempty set. A function $d: X \times X \to \mathbb{C}$ is called a complex valued metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $0 \precsim d(x,y)$ and $d(x,y) = 0$ if and only if $x = y$
(ii) $d(x,y) = d(y,x)$
(iii) $d(x,y) \precsim d(x,z) + d(z,y)$

The pair (X,d) is called a complex valued metric space.

Example 4 (see [5]): Let $X = \mathbb{C}$. Define the mapping $d: X \times X \to \mathbb{C}$ by

$$d(x,y) = i|x - y|, \text{ for all } x, y \in X.$$

Then (X,d) is a complex valued metric space.

Definition 5 (see[10]): Let X be a nonempty set and let $s \geq 1$ be given real number. A function $d: X \times X \to \mathbb{C}$ is called a complex valued b-metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $0 \precsim d(x,y)$ *and* $d(x,y) = 0$ if and only if $x = y$
(ii) $d(x,y) = d(y,x)$
(iii) $d(x,y) \precsim s[d(x,z) + d(z,y)]$.

The pair (X,d) is called a complex valued b-metric space.

Example 6 (see [10]): Let $X = [0,1]$. Define the mapping $d: X\, x\, X \to \mathbb{C}$ by

$$d(x,y) = |x-y|^2 + i\,|x-y|^2, \text{ for all } x, y \in X.$$

Then (X,d) is a complex valued b-metric space with $s = 2$.

Definition 7(see[10]): Let (X,d) be a complex valued b-metric space. Consider the following .

(i) A point $x \in X$ is called an interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x,r) = \{y \in X : d(x,y) \prec r\} \subseteq A$.
(ii) A point $x \in X$ is called a limit point of a set A whenever, for every $0 \prec r \in \mathbb{C}$, $B(x,r) \cap (A - \{x\}) \neq \phi$.
(iii) A subset A of X is called open whenever each point of A is an interior point of A.
(iv) A subset A of X is called closed whenever each limit point of A belongs to A.
(v) A subbasis for a Hausdorff topology τ on X is a family

$$F = \{B(x,r) : x \in X \text{ and } 0 \prec r\}.$$

Definition 8 (see [10]): Let (X,d) be a complex valued b-metric space and $\{x_n\}$ a sequence in X and $x \in X$. Consider the following.

(i) If for every c , with $0 \prec c$, there is $N \in \mathbb{N}$ such that, for all $n > N$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, if $\{x_n\}$ converges to x, and x is the limit point of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
(ii) If for every $\in \mathbb{C}$, with $0 \prec c$, there is $N \in \mathbb{N}$ such that, for all $n > N$, $d(x_n, x_{n+m}) \prec c$, where $m \epsilon \mathbb{N}$, then $\{x_n\}$ is said to be Cauchy sequence.
(iii) If every Cauchy sequance in X is convergent, then (X,d) is said to be a complete complex valued b-metric space.

Definition 9 (see [7]): Let (X,d) be a complex valued metric space. The self-maps S and T are said to be commuting if $STx = TSx$ for all $x \in X$.

Definition 10 (see [8]): Let (X,d) be a complex valued metric space. The self-maps S and T are said to be compatible if $\lim\limits_{n\to\infty} d(STx_n , TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} Sx_n = \lim\limits_{n\to\infty} Tx_n = t$ for some $t \in X$.

Definition 11 (see [9]): Let (X,d) be a complex valued metric space. The self-maps S and T are said to be weakly compatible if $STx = TSx$ whenever $Sx = Tx$, i.e., they commute at their coincidence points.

Lemma 12 (see [10]): Let (X,d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.

Lemma 13 (see [10]): Let (X,d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ s a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.

Theorem 14 (see [6]): Let (X, d) be a complete complex valued metric space and let S, T, f and g be self-maps of X such that

(i) The pairs $\{S, f\}$ and $\{T, g\}$ are weakly compatible,
(ii) $TX \subseteq fX$ and $SX \subseteq gX$,
(iii) fX or gX is a complete subspace of X and
(iv) $d(Sx, Ty) \precsim \lambda d(fx, gy) + \frac{\mu d(fx,Sx)d(gy,Ty)}{1+d(fx,gy)}, \forall\, x, y \in X,$
where λ, μ are non-negative reals with $\lambda + \mu < 1$.

Then S, T, f and g have a unique common fixed point.

2. MAIN RESULT

My theorem is a generalization of Theorem 14 in complex valued b-metric spaces.

Theorem: Let (X, d) be a complete complex valued b-metric space with coefficient $s \geq 1$. Let S, T, f and g be self-mappings of X such that

(i) The pairs $\{ S, f \}$ and $\{ T, g \}$ are weakly compatible,
(ii) $TX \subseteq fX$ and $SX \subseteq gX$,
(iii) fX or gX is a complete subspace of X and
(iv) $d(Sx, Ty) \precsim \lambda d(fx, gy) + \frac{\mu d(fx,Sx)d(gy,Ty)}{1+d(fx,gy)}, \forall\, x, y \in X,$ where λ, μ are non-negative reals with $s\lambda + \mu < 1$.

Then S, T, f and g have a unique common fixed point.

Proof: Let $x_0 \in X$ be arbitrary. Using the condition (ii), we defined a sequence $\{y_n\}$ in X as

$$y_{2k+1} = gx_{2k+1} = Sx_{2k}$$
$$y_{2k+2} = fx_{2k+2} = Tx_{2k+1},\ k = 0, 1, 2, \ldots\ldots\ldots.$$

Then

$$\begin{aligned} d(y_{2k+1}, y_{2k+2}) &= d(Sx_{2k}, Tx_{2k+1}) \\ &\precsim \lambda d(fx_{2k}, gx_{2k+1}) + \frac{\mu d(fx_{2k},Sx_{2k})d(gx_{2k+1},Tx_{2k+1})}{1+d(fx_{2k},gx_{2k+1})} \\ &= \lambda d(y_{2k}, y_{2k+1}) + \frac{\mu d(y_{2k},y_{2k+1})d(y_{2k+1},y_{2k+2})}{1+d(y_{2k},y_{2k+1})} \\ &\precsim \lambda\, d(y_{2k}, y_{2k+1}) + \mu d(y_{2k+1}, y_{2k+2}) \end{aligned}$$

Thus $d(y_{2k+1}, y_{2k+2}) \precsim \frac{\lambda}{1-\mu} d(y_{2k+1}, y_{2k+2})$ (1)

Similarly

$$\begin{aligned} d(y_{2k+2}, y_{2k+3}) &= d(Sx_{2k+2}, Tx_{2k+1}) \\ &\precsim \lambda d(fx_{2k+2}, gx_{2k+1}) + \frac{\mu d(fx_{2k+2},Sx_{2k+2})d(gx_{2k+1},Tx_{2k+1})}{1+d(fx_{2k+2},gx_{2k+1})} \\ &= \lambda d(y_{2k+2}, y_{2k+1}) + \frac{\mu d(y_{2k+2},)y_{2k+3}d(y_{2k+1},y_{2k+2})}{1+d(y_{2k+2},y_{2k+1})} \\ &\precsim \lambda\, d(y_{2k+2}, y_{2k+1}) + \mu d(y_{2k+2}, y_{2k+3}) \end{aligned}$$

Thus $d(y_{2k+2}, y_{2k+3}) \precsim \frac{\lambda}{1-\mu} d(y_{2k+2}, y_{2k+1})$ (2)

Now put $h = \frac{\lambda}{1-\mu}$

Since $0 \leq s\lambda + \mu < 1, s \geq 1,\ \lambda + \mu < 1$ and hence $0 \leq h < 1$.

Thus using (1) and (2) for $n \in \mathbb{N}$, we get that

$$d(y_n, y_{n+1}) \precsim hd(y_{n-1}, y_n) \precsim h^2\, d(y_{n-2}, y_{n-1}) \precsim \ldots\ldots\ldots\ldots\ldots.. \precsim h^{n-1}\, d(y_1, y_2).$$

So for $m, n \in \mathbb{N}$,

$$\begin{aligned} d(y_n, y_{m+n}) &\precsim s[d(y_n, y_{n+1}) + d(y_{n+1}, y_{m+n})] \\ &\precsim sd(y_n, y_{n+1}) + s^2\, [\, d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{m+n})] \\ &\precsim sd(y_n, y_{n+1}) + s^2\, d(y_{n+1}, y_{n+2}) + s^3\, d(y_{n+2}, y_{n+3}) + \cdots \ldots + s^{m-1}\, d(y_{n+m-2}, y_{n+m-1}) \\ &\qquad + s^{m-1}\, d(y_{n+m-1}, y_{n+m}) \\ &\precsim sd(y_n, y_{n+1}) + s^2\, d(y_{n+1}, y_{n+2}) + s^3\, d(y_{n+2}, y_{n+3}) + \ldots\ldots + s^{m-1}\, d(y_{n+m-2}, y_{n+m-1}) \\ &\qquad + s^m\, d(y_{n+m-1}, y_{n+m}) \\ &\precsim sh^{n-1} d(y_1, y_2) + s^2 h^n d(y_1, y_2) + \ldots\ldots\ldots\ldots + s^{m-1} h^{n+m-3} d(y_1, y_2) + s^m\, h^{n+m-2} d(y_1, y_2) \\ &= sh^{n-1}[\, 1 + sh + s^2 h^2 + \ldots\ldots\ldots\ldots\ldots + s^{m-1} h^{m-1}] d(y_1, y_2) \\ &\precsim \frac{sh^{n-1}}{1-sh} d(y_1, y_2) \end{aligned}$$

Thus $|d(y_n, y_{m+n})| \leq \frac{sh^{n-1}}{1-sh}\ |d(y_1, y_2)|\ \rightarrow 0$ as $n\ \rightarrow\ \infty$, where $m \in \mathbb{N}$.

Hence $\{y_n\}$ is a Cauchy sequence in X.

Since X is complete, there exists $z\, \epsilon\, X$ such that $y_n \rightarrow\ z$ as $n\ \rightarrow\ \infty$.

Thus $\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} fx_{2n+2} = z$ (3)

Now if fX is a complete subspace of X, there exists $u \in X$ such that $fu\ =\ z$

From the condition (iv), we have

$$\begin{aligned} d(Su,z) &\precsim sd(Su,Tx_{2n+1}) +\ sd(Tx_{2n+1},z)\\ &\precsim s\left[\ \lambda d(fu,gx_{2n+1}) + \frac{\mu d\,(fu,Su)d(gx_{2n+1},Tx_{2n+1})}{1+d(fu,gx_{2n+1})}\right] + sd(Tx_{2n+1},z)\\ &= s\left[\lambda d(fu,y_{2n+1}) + \frac{\mu d\,(fu,Su)d(y_{2n+1},y_{2n+2})}{1+d(fu,y_{2n+1})}\right] + sd(y_{2n+2},z) \end{aligned}$$

Therefore $|d(Su,z)| \leq s\left[\lambda|d(fu,y_{2n+1})| + \frac{\mu\,|d\,(fu,Su)||d\,(y_{2n+1},y_{2n+2})|}{1+|d(fu,y_{2n+1})|}\right] + s|d(y_{2n+2},z)|$

Letting $n \rightarrow \infty$ and using (3) and Lemma 12, we get that $|d(Su,z)| \leq\ 0$.

Thus $|d(Su,z)|\ =\ 0$. i.e. $d(Su,z)\ =\ 0$ and hence $Su\ =\ z$.

Since $SX\ \subseteq gX$, there exists $v \in X$ such that $gv = z$.

Again from condition (iv), we have

$$\begin{aligned} d(z,Tv)\ &=\ d(Su,Tv)\\ &\precsim \lambda d(fu,gv) + \frac{\mu d\,(fu,Su)d(gv,Tv)}{1+d(fu,gv)}\\ &= 0 \end{aligned}$$

Thus $d(z,Tv)\ =\ 0$ and hence $Tv\ =\ z$.

Thus $fu\ =\ Su\ =\ z\ =\ gv\ =\ Tv$.

Since f and S are weakly compatible,

$$fz\ =\ fSu\ =\ Sfu\ = Sz.$$

Now we will show that $Sz\ =\ z$.

From condition (iv),

$$\begin{aligned} d(Sz,z)\ &=\ d(Sz,Tv)\\ &\precsim \lambda d(fz,gv) + \frac{\mu d\,(fz,Sz)d(gv,Tv)}{1+d(fz,gv)}\\ &= \lambda d(Sz,z) \end{aligned}$$

Thus $(1-\lambda)|d(Sz,z)|\ \leq 0$

Thus $d(Sz,z) = 0$ and hence $Sz\ =\ z$.

Similarly since g and T are weakly compatible,

$$gz\ =\ gTv\ =\ Tgv\ =\ Tz.$$

Also $d(z,Tz) =\ d(Sz,Tz)$

$$\begin{aligned} &\precsim \lambda d(fz,gz) + \frac{\mu d\,(fz,Sz)d(gz,Tz)}{1+d(fz,gz)}\\ &= \lambda d(z,Tz) \end{aligned}$$

Thus $d(z,Tz)\ =\ 0$ and hence $Tz\ =\ z$.

Thus $Sz\ =\ fz\ =\ gz\ =\ Tz\ =\ z$.

© 2015, IJMA. All Rights Reserved

i.e. z is a common fixed point of four mappings S, T, f and g.

Now we show that z is the unique common fixed point.

Let $z^* \in X$ such that $fz^* = Sz^* = gz^* = Tz^* = z^*$.

Then we have,

$$\begin{aligned} d(z, z^*) &= d(Sz, Tz^*) \\ &\precsim \lambda d(fz, gz^*) + \frac{\mu d(fz, Sz) d(gz^*, Tz^*)}{1 + d(fz, gz^*)} \\ &= \lambda d(z, z^*) \end{aligned}$$

Thus $d(z, z^*) = 0$ and so $z = z^*$**.** Thus z is the unique common fixed point of S, T, f and g.

If gX is complete, we can similarly prove the theorem.

Corollary 1: Let (X, d) be a complete complex valued b-metric space with coefficient $s \geq 1$. Let S, T be self-mappings of X such that

$$d(Sx, Ty) \precsim \lambda d(x, y) + \frac{\mu d(x, Sx) d(y, Ty)}{1 + d(x, y)}, \forall\, x, y \in X,$$

where λ, μ are non-negative reals with $s\lambda + \mu < 1$.

Then S, T have a unique common fixed point.

Proof: Taking $f(x) = x$ and $g(x) = x$, $\forall\, x \in X$ in the above theorem we get the result.

Corollary 2: Let (X, d) be a complete complex valued b-metric space with coefficient $s \geq 1$. Let T, f and g be self-mappings of X such that

(i) The pairs $\{T, f\}$ and $\{T, g\}$ are weakly compatible
(ii) $TX \subseteq fX$ and $TX \subseteq gX$
(iii) fX or gX is a complete subspace of X and
(iv) $d(Tx, Ty) \precsim \lambda d(fx, gy) + \frac{\mu d(fx, Tx) d(gy, Ty)}{1 + d(fx, gy)}, \forall\, x, y \in X$, where λ, μ are non-negative reals with $s\lambda + \mu < 1$.

Then T, f and g have a unique common fixed point.

Proof: Taking $S = T$ in the above theorem, we get the result.

Corollary 3: Let (X, d) be a complete complex valued b-metric space with coefficient $s \geq 1$. Let T, f and g be self-mappings of X and n is a possitive integers, satisfying the following conditions

(i) The pairs $\{T^n, f\}, \{T^n, g\}$, $\{T, f\}$ and $\{T, g\}$ are weakly compatible
(ii) $T^n X \subseteq fX$ and $T^n X \subseteq gX$
(iii) fX or gX is a complete subspace of X and
(iv) $d(T^n x, T^n y) \precsim \lambda d(fx, gy) + \frac{\mu d(fx, T^n x) d(gy, T^n y)}{1 + d(fx, gy)}, \forall\, x, y \in X$, where λ, μ are non-negative reals wit $s\lambda + \mu < 1$.

Then T, f and g have a unique common fixed point.

Proof: Applying corollary 2, we get a unique common fixed point z of T^n, f and g.

Therefore $T^n z = fz = gz = z$.

Now we note that $T^n Tz = TT^n z = Tz$.

Also since the pairs $\{T, f\}$ and $\{T, g\}$ are weakly compatible,

$$fTz = Tfz = Tz \text{ and } gTz = Tgz = Tz.$$

Thus we see that Tz is also a common fixed point of T^n, f and g.

Thus by uniqueness of z, we have $Tz = z$.

Hence z is a common fixed point of T, f and g.

Since any common fixed point of T, f and g is also a common fixed point of T^n, f and g, the common fixed point z of T, f and g is unique.

This complete the proof.

Corollary 4: Let (X, d) be a complete complex valued b-metric space with coefficient $s \geq 1$. Let T be a self-mappings of X and n is a possitive integers , such that

$$d(T^n x, T^n y) \precsim \lambda d(x, y) + \frac{\mu d(x, T^n x) d(y, T^n y)}{1 + d(x, y)}, \forall\, x, y \in X,$$

where λ, μ are non-negative reals with $s\lambda + \mu < 1$.

Then T has a unique common fixed point.

Proof: In corollary 3, if we take $f(x) = x$ and $g(x) = x$, for all $x \in X$, then the required result follows.

REFERENCES

1. H. Aydi, M. Bota, E. Karapinar and S. Mitrovic, "A fixed point theorem for set valued quasi-contractions in b-metric spaces", Fixed point theory and its applications, vol. 2012, article 88, 2012.
2. A. Azam, B. Fisher and M. Khan, "Common fixed point theorems in complex valued metric spaces", Numerical Functional Analysis and Optimization, vol 32, no. 3, pp. 243 – 253, 2011.
3. I. A. Bakhtin, "The contraction principle in quasimetric spaces", Functional Analysis, vol 30., pp. 26 – 37, 1989.
4. S. Banach, "Sur les operations dans less ensembles abstraits et leur applications aux equations integrals", Fundamenta Mathematicae, vol 3, pp 133 – 181, 1922.
5. S. Datta and S. Ali, "A common fixed point theorem under contractive condition in complex valued metric spaces", Internationa Journal of Advanced Scintific and Technical Research, vol. 6, no. 2, pp. 467 – 475, 2012.
6. S. K. Datta and S. Ali, "Common fixed point theorems for four mappings in complex valued metric spaces", Internat. J. Functional Analysis, Operator Theory and Applications, vol. 5, no. 2, pp. 91 – 102, 2013
7.] G. Jungck: Commuting maps and fixed points, Amer. Math. Monthly, Vol. 83(1976), pp. 261-263.
8. G. Jungck: Compatible mappings and common fixed points, Int. J. Math. & Math. Sci., Vol. 9, No. 4(1986), pp. 771-779.
9. G. Jungck: Common fixed points for non-continuous non-self mappings on non-metric spaces, Far East J. Math. Sc., Vol. 4, No. 2(1996), pp. 199-212.
10. K. Rao, P. Swamy and J. Prasad, "A common fixed point theorem complex valued b-metric spaces", Bulletin of Mathematics and Statistics Research, vol. 1, no. 1, 2013.
11. J. Roshan, V. Parvaneh, S. Sedghi, N. Shobkolaei and W. Shatanawi, "Common fixed point of almost generalized $(\psi, \phi)_s$ – contractive mappings in ordered b-metric spaces", Fixed point theory and applications, vol. 2013, article 159, 2013.

Source of support: Nil, Conflict of interest: None Declared

[Copy right © 2015. This is an Open Access article distributed under the terms of the International Journal of Mathematical Archive (IJMA), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.]