A Common Fixed Point Theorem in Complex Valued b-Metric Spaces for Four Mappings

TANMOY MITRA*

Asst. Teacher, Akabpur High School, Akabpur, P.O. - Nadanghat, Dist-Burdwan, India.

(Received On: 24-07-15; Revised & Accepted On: 22-08-15)

ABSTRACT

In this paper we prove a common fixed point theorem for four self-mappings in a complete complex valued b-metric space.

Key Words: Complex valued b-metric space, weakly compatible mappings.

AMS Subject Classification (2010): 47H10, 54H25.

1. INTRODUCTION

In 1989, Bakhtin [3] introduced the concept of b-metric space as a generalization of metric spaces. The concept of complex valued b-metric spaces was introduced in 2013 by Rao et al. [10], which was more general than the well-known complex valued metric spaces that were introduced in 2011 by Azam et al. [2]. The main purpose of this paper is to present common fixed point results of four self-mappings satisfying a rational inequality on complex valued b-metric spaces. The results presented in this paper are generalization of work done by Sanjib Kumar Dutta and Sultan Ali in [6].

Definition 1 (see [1]): Let \( X \) be a nonempty set and let \( s \geq 1 \) be a given real number. A function \( d: X \times X \to \mathbb{C} \) is called a b-metric if for all \( x, y, z \in X \), the following conditions are satisfied:

(i) \( d(x, y) = 0 \) if and only if \( x = y \)
(ii) \( d(x, y) = d(y, x) \)
(iii) \( d(x, y) \leq s[d(x, z) + d(z, y)] \).

The pair \( (X, d) \) is called a b-metric space. The number \( s \geq 1 \) is called the coefficient of \( (X, d) \).

Example 2 (see [11]): Let \( (X, d) \) be a metric space and \( \rho(x, y) = (d(x, y))^p \), where \( p > 1 \) is a real number. Then \( (X, \rho) \) is a b-metric space with \( s = 2p^{-1} \).

Let \( \mathbb{C} \) be the set of all complex numbers and \( z_1, z_2 \in \mathbb{C} \). Define a partial order relation \( \preceq \) on \( \mathbb{C} \) as follows:

\( z_1 \preceq z_2 \) if and only if \( Re(z_1) \leq Re(z_2) \) and \( Im(z_1) \leq Im(z_2) \).

Thus \( z_1 \preceq z_2 \) if one of the followings holds:

1. \( Re(z_1) = Re(z_2) \) and \( Im(z_1) = Im(z_2) \),
2. \( Re(z_1) < Re(z_2) \) and \( Im(z_1) = Im(z_2) \),
3. \( Re(z_1) = Re(z_2) \) and \( Im(z_1) < Im(z_2) \) and
4. \( Re(z_1) < Re(z_2) \) and \( Im(z_1) < Im(z_2) \).

We write \( z_1 \preceq z_2 \) if \( z_1 \preceq z_2 \) and \( z_1 \neq z_2 \) i.e., one of (2), (3) and (4) is satisfied and we will write \( z_1 < z_2 \) if only (4) is satisfied.

Remark 1: We can easily check the followings:

1. \( a, b \in \mathbb{R}, a \leq b \Rightarrow az \leq bz \ \forall \ z \in \mathbb{C} \),
2. \( 0 \leq z_1 \leq z_2 \Rightarrow |z_1| < |z_2| \),
3. \( z_1 \preceq z_2 \) and \( z_2 < z_3 \) \( \Rightarrow z_1 < z_3 \).

Corresponding Author: Tanmoy Mitra*

Asst. Teacher, Akabpur High School, Akabpur, P.O. - Nadanghat, Dist-Burdwan, India.
Definition 3 (see [2]): Let \( X \) be a nonempty set. A function \( d: X \times X \rightarrow \mathbb{C} \) is called a complex valued metric on \( X \) if for all \( x, y, z \in X \) the following conditions are satisfied:

(i) \( 0 \leq d(x,y) \) and \( d(x,y) = 0 \) if and only if \( x = y \)
(ii) \( d(x,y) = d(y,x) \)
(iii) \( d(x,y) \leq d(x,z) + d(z,y) \)

The pair \( (X,d) \) is called a complex valued metric space.

Example 4 (see [5]): Let \( X = \mathbb{C} \). Define the mapping \( d: X \times X \rightarrow \mathbb{C} \) by

\[
d(x,y) = i|x - y|, \text{ for all } x,y \in X.
\]

Then \( (X,d) \) is a complex valued metric space.

Definition 5 (see[10]): Let \( X \) be a nonempty set and let \( s \geq 1 \) be given real number. A function \( d: X \times X \rightarrow \mathbb{C} \) is called a complex valued b-metric on \( X \) if for all \( x, y, z \in X \) the following conditions are satisfied:

(i) \( 0 \leq d(x,y) \) and \( d(x,y) = 0 \) if and only if \( x = y \)
(ii) \( d(x,y) = d(y,x) \)
(iii) \( d(x,y) \leq s[d(x,z) + d(z,y)] \).

The pair \( (X,d) \) is called a complex valued b-metric space.

Example 6 (see [10]): Let \( X = [0,1] \). Define the mapping \( d: X \times X \rightarrow \mathbb{C} \) by

\[
d(x,y) = |x - y|^2 + i |x - y|^2, \text{ for all } x,y \in X.
\]

Then \( (X,d) \) is a complex valued b-metric space with \( s = 2 \).

Definition 7(see[10]): Let \( (X,d) \) be a complex valued b-metric space. Consider the following .

(i) A point \( x \in X \) is called an interior point of a set \( A \subseteq X \) whenever there exists \( 0 < r \in \mathbb{C} \) such that

\[
B(x,r) = \{ y \in X : d(x,y) < r \} \subseteq A.
\]

(ii) A point \( x \in X \) is called a limit point of a set \( A \) whenever, for every \( 0 < r \in \mathbb{C} \), \( B(x,r) \cap (A - \{x\}) \neq \emptyset \).

(iii) A subset \( A \) of \( X \) is called open whenever each point of \( A \) is an interior point of \( A \).

(iv) A subset \( A \) of \( X \) is called closed whenever each limit point of \( A \) belongs to \( A \).

(v) A subbasis for a Hausdorff topology \( \tau \) on \( X \) is a family

\[
F = \{ B(x,r) : x \in X \text{ and } 0 < r \}.
\]

Definition 8 (see [10]): Let \( (X,d) \) be a complex valued b-metric space and \( \{x_n\} \) a sequence in \( X \) and \( x \in X \). Consider the following.

(i) If for every \( c \in \mathbb{C} \), with \( 0 < c \), there is \( N \in \mathbb{N} \) such that, for all \( n > N \), \( d(x_n,x) < c \), then \( \{x_n\} \) is said to be convergent, if \( \{x_n\} \) converges to \( x \), and \( x \) is the limit point of \( \{x_n\} \). We denote this by \( \lim_{n \to \infty} x_n = x \) or \( x_n \to x \) as \( n \to \infty \).
(ii) If for every \( c \in \mathbb{C} \), with \( 0 < c \), there is \( N \in \mathbb{N} \) such that, for all \( n > N \), \( d(x_m,x_{n+m}) < c \), where \( m \in \mathbb{N} \), then \( \{x_n\} \) is said to be Cauchy sequence.
(iii) If every Cauchy sequence in \( X \) is convergent, then \( (X,d) \) is said to be a complete complex valued b-metric space.

Definition 9 (see [7]): Let \( (X,d) \) be a complex valued metric space. The self-maps \( S \) and \( T \) are said to be commuting if \( STx = TSx \) for all \( x \in X \).

Definition 10 (see [8]): Let \( (X,d) \) be a complex valued metric space. The self-maps \( S \) and \( T \) are said to be compatible if \( \lim_{n \to \infty} d(STx_n,TSx_n) = 0 \) whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} STx_n = \lim_{n \to \infty} TSx_n = t \) for some \( t \in X \).

Definition 11 (see [9]): Let \( (X,d) \) be a complex valued metric space. The self-maps \( S \) and \( T \) are said to be weakly compatible if \( STx = TSx \) whenever \( Sx = Tx \), i.e., they commute at their coincidence points.

Lemma 12 (see [10]): Let \( (X,d) \) be a complex valued b-metric space and let \( \{x_n\} \) be a sequence in \( X \). Then \( \{x_n\} \) converges to \( x \) if and only if \( |d(x_n,x)| \to 0 \text{ as } n \to \infty \).

Lemma 13 (see [10]): Let \( (X,d) \) be a complex valued b-metric space and let \( \{x_n\} \) be a sequence in \( X \). Then \( \{x_n\} \) is a Cauchy sequence if and only if \( |d(x_n,x_{n+m})| \to 0 \text{ as } n \to \infty \), where \( m \in \mathbb{N} \).
Then $S, T, f$ and $g$ have a unique common fixed point.

2. MAIN RESULT

My theorem is a generalization of Theorem 14 in complex valued b-metric spaces.

**Theorem:** Let $(X, d)$ be a complete complex valued b-metric space with coefficient $s \geq 1$. Let $S, T, f$ and $g$ be self-mappings of $X$ such that

(i) The pairs $\{S, f\}$ and $\{T, g\}$ are weakly compatible,
(ii) $TX \subseteq fX$ and $SX \subseteq gX$,
(iii) $fX$ or $gX$ is a complete subspace of $X$ and
(iv) $d(Sx, Ty) \leq \lambda d(fx, gy) + \frac{\mu d(fx, Sx)d(gy, Ty)}{1+d(fx, gy)}, \forall x, y \in X,$

where $\lambda, \mu$ are non-negative reals with $\lambda + \mu < 1$. Then $S, T, f$ and $g$ have a unique common fixed point.

**Proof:** Let $x_0 \in X$ be arbitrary. Using the condition (ii), we define a sequence $\{y_n\}$ in $X$ as

$y_{2k+1} = gSx_{2k} = Sx_{2k}$,

$y_{2k+2} = fx_{2k+1} = Tx_{2k+1}, \ k = 0, 1, 2, \ldots, \ldots$

Then

$d(y_{2k+1}, y_{2k+2}) = d(Sx_{2k}, Tx_{2k+1})$

$\leq \lambda d(fx_{2k}, gx_{2k+1}) + \frac{\mu d(fx_{2k}, Sx_{2k})d(gx_{2k+1}, Ty_{2k+1})}{1+d(fx_{2k}, gx_{2k+1})}$

$= \lambda d(y_{2k}, y_{2k+1}) + \frac{\mu d(Sx_{2k}, y_{2k+1})d(y_{2k+1}, y_{2k+2})}{1+d(Sx_{2k}, y_{2k+1})}$

$\leq \lambda d(y_{2k}, y_{2k+1}) + \frac{\mu d(Sx_{2k}, y_{2k+1})d(y_{2k+1}, y_{2k+2})}{1+d(Sx_{2k}, y_{2k+1})}$

Thus $d(y_{2k+1}, y_{2k+2}) \leq \frac{\lambda}{1-\mu}d(y_{2k+1}, y_{2k+2}) \tag{1}$

Similarly

$d(y_{2k+2}, y_{2k+3}) = d(Sx_{2k+2}, Tx_{2k+1})$

$\leq \lambda d(fx_{2k+2}, gx_{2k+1}) + \frac{\mu d(fx_{2k+2}, Sx_{2k+2})d(gx_{2k+1}, Ty_{2k+1})}{1+d(fx_{2k+2}, gx_{2k+1})}$

$= \lambda d(y_{2k+2}, y_{2k+1}) + \frac{\mu d(Sx_{2k+2}, y_{2k+1})d(y_{2k+1}, y_{2k+2})}{1+d(Sx_{2k+2}, y_{2k+1})}$

$\leq \lambda d(y_{2k+2}, y_{2k+1}) + \frac{\mu d(Sx_{2k+2}, y_{2k+1})d(y_{2k+1}, y_{2k+2})}{1+d(Sx_{2k+2}, y_{2k+1})}$

Thus $d(y_{2k+2}, y_{2k+3}) \leq \frac{\lambda}{1-\mu}d(y_{2k+2}, y_{2k+3}) \tag{2}$

Now put $\frac{\lambda}{1-\mu}$ since $0 \leq s \lambda + \mu < 1, s \geq 1, \lambda + \mu < 1$ and hence $0 \leq h < 1$.

Thus using (1) and (2) for $n \in \mathbb{N}$, we get that

$d(y_n, y_{n+1}) \leq h^nd(y_{n-1}, y_n) \leq h^2d(y_{n-2}, y_{n-1}) \leq \ldots \ldots \ldots \leq h^{n-1}d(y_1, y_2)$.

So for $m, n \in \mathbb{N}$,

$d(y_n, y_m) \leq s[d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots + d(y_{n+m-1}, y_m)]$

$\leq sd(y_n, y_{n+1}) + s^2[d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots + d(y_{n+m-1}, y_m)]$

$\leq sd(y_n, y_{n+1}) + s^2d(y_{n+1}, y_{n+2}) + \ldots + s^{m-1}d(y_{n+m-2}, y_m)$

$\leq sd(y_n, y_{n+1}) + s^2d(y_{n+1}, y_{n+2}) + \ldots + s^{m-1}d(y_{n+m-2}, y_m)$

$\leq s[d^{h^{n-1}}(y_1, y_2)] + s^2h^nd(y_1, y_2) + \ldots + s^{m-1}h^{n-m-3}d(y_1, y_2) + \ldots + s^{m-1}h^{n-m-2}d(y_1, y_2)$

$\leq s^{h^{n-1}}[1 + sh + s^2h^2 + \ldots + s^{m-1}h^{n-1}]d(y_1, y_2)$

$\leq \frac{sh}{1-sh}d(y_1, y_2)$

© 2015, IJMA. All Rights Reserved
Thus $|d(y_n, y_{m+n})| \leq \frac{s^h - 1}{1 - s^h} |d(y_1, y_2)|$ as $n \to \infty$, where $m \in \mathbb{N}$.

Hence $\{y_n\}$ is a Cauchy sequence in $X$.

Since $X$ is complete, there exists $z \in X$ such that $y_n \to z$ as $n \to \infty$.

Thus $\lim_{n \to \infty} Sx_{2n} = \lim_{n \to \infty} gx_{2n+1} = \lim_{n \to \infty} Tx_{2n+1} = \lim_{n \to \infty} f x_{2n+2} = z$ (3)

Now if $fX$ is a complete subspace of $X$, there exists $u \in X$ such that $fu = z$.

From the condition (iv), we have

$$d(Su, z) \leq sd(Su, Tx_{2n+1}) + sd(Tx_{2n+1}, z)$$

$$\leq s \left[ \lambda d(fu, gx_{2n+1}) + \frac{\mu d(fu, Su)d(gx_{2n+1}, Tx_{2n+1})}{1+d(fu, gx_{2n+1})} \right] + sd(Tx_{2n+1}, z)$$

$$= s \left[ \lambda d(fu, y_{2n+1}) + \frac{\mu d(fu, Su)d(y_{2n+1}, y_{2n+2})}{1+d(fu, y_{2n+1})} \right] + sd(y_{2n+2}, z)$$

Therefore $|d(Su, z)| \leq s \left[ \lambda |d(fu, y_{2n+1})| + \frac{\mu |d(fu, Su)||d(y_{2n+1}, y_{2n+2})|}{1+|d(fu, y_{2n+1})|} \right] + s|d(y_{2n+2}, z)|$

Letting $n \to \infty$ and using (3) and Lemma 12, we get that $|d(Su, z)| \leq 0$.

Thus $|d(Su, z)| = 0$. i.e. $d(Su, z) = 0$ and hence $Su = z$.

Since $SX \subseteq gX$, there exists $v \in X$ such that $gv = z$.

Again from condition (iv), we have

$$d(z, Tv) = d(Su, Tv)$$

$$\leq \lambda d(fu, gv) + \frac{\mu d(fu, Su)d(gv, Tv)}{1+d(fu, gv)}$$

$$= 0$$

Thus $d(z, Tv) = 0$ and hence $Tv = z$.

Thus $fu = Su = z = gv = Tv$.

Since $f$ and $S$ are weakly compatible,

$fz = fSu = Sz$.

Now we will show that $Sz = z$.

From condition (iv),

$$d(Sz, z) = d(Sz, Tv)$$

$$\leq \lambda d(fz, gv) + \frac{\mu d(fz, Sz)d(gv, Tv)}{1+d(fz, gv)}$$

$$= \lambda d(Sz, z)$$

Thus $(1 - \lambda)|d(Sz, z)| \leq 0$.

Thus $d(Sz, z) = 0$ and hence $Sz = z$.

Similarly since $g$ and $T$ are weakly compatible,

$gz = gTv = Tgv = Tz$.

Also $d(z, Tz) = d(Sz, Tz)$

$$\leq \lambda d(fz, gz) + \frac{\mu d(fz, Sz)d(gz, Tz)}{1+d(fz, gz)}$$

$$= \lambda d(z, Tz)$$

Thus $d(z, Tz) = 0$ and hence $Tz = z$.

Thus $Sz = fz = gz = Tz = z$. 
i.e. $z$ is a common fixed point of four mappings $S, T, f$ and $g$.

Now we show that $z$ is the unique common fixed point.

Let $z^* \in X$ such that $fz^* = Sz^* = Tz^* = gz^* = z^*$.

Then we have,

$$d(z, z^*) = d(Sz, Tz^*)$$

$$\leq \lambda d(fz, gz^*) + \frac{\mu d(Sx, Ty)}{1+d(Sx, Ty)}$$

Thus $d(z, z^*) = 0$ and so $z = z^*$. Thus $z$ is the unique common fixed point of $S, T, f$ and $g$.

If $gX$ is complete, we can similarly prove the theorem.

**Corollary 1:** Let $(X, d)$ be a complete complex valued $b$-metric space with coefficient $s \geq 1$. Let $S, T$ be self-mappings of $X$ such that

$$d(Sx, Ty) \leq \lambda d(x, y) + \frac{\mu d(Sx, Ty)}{1+d(Sx, Ty)}, \forall \ x, y \in X,$$

where $\lambda, \mu$ are non-negative reals with $s\lambda + \mu < 1$.

Then $S, T$ have a unique common fixed point.

**Proof:** Taking $f(x) = x$ and $g(x) = x, \forall \ x \in X$ in the above theorem we get the result.

**Corollary 2:** Let $(X, d)$ be a complete complex valued $b$-metric space with coefficient $s \geq 1$. Let $T, f$ and $g$ be self-mappings of $X$ such that

(i) The pairs $\{T, f\}$ and $\{T, g\}$ are weakly compatible

(ii) $TX \subseteq fX$ and $TX \subseteq gX$

(iii) $fX$ or $gX$ is a complete subspace of $X$ and

(iv) $d(Tx, Ty) \leq \lambda d(fx, gy) + \frac{\mu d(fx, gy)}{1+d(fx, gy)}, \forall \ x, y \in X$, where $\lambda, \mu$ are non-negative reals with $s\lambda + \mu < 1$.

Then $T, f$ and $g$ have a unique common fixed point.

**Proof:** Taking $S = T$ in the above theorem, we get the result.

**Corollary 3:** Let $(X, d)$ be a complete complex valued $b$-metric space with coefficient $s \geq 1$. Let $T, f$ and $g$ be self-mappings of $X$ and $n$ is a positive integer, satisfying the following conditions

(i) The pairs $\{T^n, f\}, \{T^n, g\}, \{T, f\}$ and $\{T, g\}$ are weakly compatible

(ii) $T^n X \subseteq fX$ and $T^n X \subseteq gX$

(iii) $fX$ or $gX$ is a complete subspace of $X$ and

(iv) $d(T^n x, T^n y) \leq \lambda d(fx, gy) + \frac{\mu d(fx, gy)}{1+d(fx, gy)}, \forall \ x, y \in X$, where $\lambda, \mu$ are non-negative reals with $s\lambda + \mu < 1$.

Then $T, f$ and $g$ have a unique common fixed point.

**Proof:** Applying corollary 2, we get a unique common fixed point $z$ of $T^n, f$ and $g$.

Therefore $T^n z = fz = gz = z$.

Now we note that $T^n Tz = TT^n z = Tz$.

Also since the pairs $\{T, f\}$ and $\{T, g\}$ are weakly compatible,

$fTz = Tfz = Tz$ and $gTz = Tgz = Tz$.

Thus we see that $Tz$ is also a common fixed point of $T^n, f$ and $g$.

Thus by uniqueness of $z$, we have $Tz = z$.

Hence $z$ is a common fixed point of $T, f$ and $g$. 
Since any common fixed point of $T, f$ and $g$ is also a common fixed point of $T^n, f$ and $g$, the common fixed point $z$ of $T, f$ and $g$ is unique.

This complete the proof.

**Corollary 4:** Let $(X, d)$ be a complete complex valued b-metric space with coefficient $s \geq 1$. Let $T$ be a self-mappings of $X$ and $n$ is a positive integers , such that

$$d(T^n x, T^n y) \leq \lambda d(x,y) + \mu d(x,T^n x)d(y,T^n y), \forall x, y \in X,$$

where $\lambda, \mu$ are non-negative reals with $s\lambda + \mu < 1$.

Then $T$ has a unique common fixed point.

**Proof:** In corollary 3, if we take $f(x) = x$ and $g(x) = x$, for all $x \in X$, then the required result follows.

**REFERENCES**


**Source of support:** Nil, **Conflict of interest:** None Declared

*[Copy right © 2015. This is an Open Access article distributed under the terms of the International Journal of Mathematical Archive (IJMA), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.]*