International Journal of Mathematical Archive-6(8), 2015, 159-163
Available online through www.ijma.info ISSN 2229 – 5046

ON RANGE QUATERNION HERMITIAN MATRICES

K. GUNASEKARAN
Ramanujan Research centre, PG and Research Department of Mathematics,
Government Arts College (Autonomous), Kumbakonam - 612 002, Tamil Nadu, India.

S. SRIDEVI*
Ramanujan Research centre, PG and Research Department of Mathematics,
Government Arts College (Autonomous), Kumbakonam - 612 002, Tamil Nadu, India.

(Received On: 27-07-15; Revised & Accepted On: 26-08-15)

ABSTRACT

The concept of range quaternion hermitian (q-EP) matrices is introduced as a generalization of quaternion hermitian and EP matrices. Necessary and sufficient conditions are determined for a matrix to be q-EP$_r$ (q - EP and rank r), Equivalent characterization of q - EP matrix are equivalent characterization at q- EP matrixes are discussed. As an application, it is shown that the class of all EP matrices having the same range space form a group under multiplication.

Key words: *Matrix, Quaternion Hermitian, Quaternion matrix.*

1. INTRODUCTION

Let H_{nxn} be the space of nxn quaternion matrices. For $A \in H_{nxn,}$ Let $A^T, A^*, A^\dagger, R(A), N(A)$ and rk(A) denote the transpose, conjugate transpose Moore-Penrose inverse range space, null space and rank of A respectively. We denote the solution of the equation AXA= A by A^- for $A \in H_{nxn,}$ The Moore-Penrose inverse $A^\dagger$ of A is the unique solution of the equations AXA= A, XAX = A, $(AX)^*$ = AX and (XA) = XA [2]. In this paper we introduce the concept of q-EP hermitian and EP matrices and extended many of the basic results on q - hernitian and q- EP matrices [2,4,5], A matrix $A \in C_{nxn}$ is said to be EP or called as range hermitian if N(A)=N(A^*) or equivalently R(A) = R (A^*) [3,P 163] Relation between q- EP and EP matrices are discussed.

2. Q - EP MATRICES

The Concept of range quaternion hemitian (q- EP) matrices introduced as a generalization of q - hermitian and EP matrices. Necessary and sufficient condition are determined for a matrix to be q-EP$_r$ (q - EP and rank r). Equivalently characterizations of a q- EP are discussed. As an application, it is shown that the class of all q - EP matrices having the same range space form a group under multiplication.

Definition: A matrix $A \in \mathbf{H_{nxn}}$ is said to be quaternion EP if R(A) = R (A*) or equivalently N(A) = N (A*).A is said to be quaternion EP$_r$ if A is quaternion EP and of rank r.

Remark 1: If K is any scalar and A is a quaternion matrix then R(KA) = R (KA*).

Remark 2: The concept of q-EP matrix is an analogue of the concept of EP matrix [P. 163, 4].

Remark 3: Further, if A is q -hermitian then A = A* implies that R(A) = R (A*). Automatically holds and therefore A is q -EP. However the converse need not true.

Remark 4: Every quaternion EP matrix is complex matrix if any two axis is zero among i, j and k.

Remarks-5: A is q - EP matrix if only if A is an EP matrix.

*Corresponding Author: S. Sridevi**

Example:

(i) $\begin{bmatrix} 2 & 1+2i+3j+4k & 2+4i+6j+8k \\ 1-2i-3j-4k & 3 & 3+6i+9j+12k \\ 2-4i-6j-8k & 3-6i-9j-12k & 4 \end{bmatrix}$ is a q – hermitian and q-EP.

(ii) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is EP and q - EP not q - Hermitian.

Theorem 1: For A $\in$ H_{nxn} the following are equivalent:

(1) A is q - EP
(2) $A^{\dagger}$ is q - EP
(3) N (A) = N ($A^{\dagger}$)
(4) N (A) = N (A*)
(5) R (A) = R (A^*)
(6) $A^{\dagger}A = AA^{\dagger}$
(7) A = A*H for a non - singular nxn matrix H.
(8) A = HA* for a non - singular nxn matrix H.
(9) A* = HA for a non - singular nxn matrix H.
(10) A* = AH for anon - singular nxn matrix H.
(11) H_n = R(A) $\oplus$ $N(A^*)$
(12) H_n = R (A*) $\oplus$ N (A)

Proof:

(1) $\Leftrightarrow$ (2)

A is q – EP $\Leftrightarrow$ A is EP (by Remark 5)
$\Leftrightarrow$ $A^{\dagger}$ is EP
$\Leftrightarrow$ $A^{\dagger}$ is q- EP

Thus the equivalence of (1) and (2) is proved.

(2) $\Leftrightarrow$ (3)

$A^{\dagger}$ is q – EP $\Leftrightarrow$ A is q - EP
$\Leftrightarrow$ N(A) = N(A*)
$\Leftrightarrow$ N (A) = N($A^{\dagger}$)

(3) $\Leftrightarrow$ (4)

N (A) = N($A^{\dagger}$) $\Leftrightarrow$ $A^{\dagger}$ is q - EP
$\Leftrightarrow$ A is q - EP (by definition q - EP)
$\Leftrightarrow$ N(A) = N(A*)

Similarly by the definition (4) $\Leftrightarrow$ (5). Thus equivalence of (3), (4) and (5).

(5) $\Leftrightarrow$ (6)

R (A) = R (A*) $\Leftrightarrow$ A is q - EP
$\Leftrightarrow$ A is EP
$\Leftrightarrow$ A $A^{\dagger}$ = $A^{\dagger}$ A

(6) $\Leftrightarrow$ (7)

$AA^{\dagger} = A^{\dagger}A$ $\Leftrightarrow$ R (A) = R (A*)
$\Leftrightarrow$ A is q - EP
$\Leftrightarrow$ A* =AH_1 for a non singular nxn matrix H_1
$\Leftrightarrow$ A = A* $(H_1)^{-1}$
$\Leftrightarrow$ A = A* H, where H = $(H_1)^{-1}$
$\Leftrightarrow$ A = A* H, where H = $(H_1)^{-1}$ is a non - singular nxn matrix.

(6) $\Leftrightarrow$ (8):

$AA^{\dagger} = A^{\dagger}A$ $\Leftrightarrow$ A is q- EP

$\Leftrightarrow$ A* = H_1A for a non-singular nxn matrix H.,

$\Leftrightarrow$ A = $H_1^{-1}A*$

$\Leftrightarrow$ A = HA*, where H = $(H_1)^{-1}$ is a non – singular matrix.

Thus equivalence of (7) $\Leftrightarrow$ (9) and (8) $\Leftrightarrow$ (10) follows immediately by taking conjugate transpose.

(9) $\Leftrightarrow$ (11): $A*$ = HA for a non - singular nxn matrix H.

$\Leftrightarrow$ A*A = HAA

$\Leftrightarrow$ A*A = H A^2

$\Leftrightarrow$ rk (A*A) = rk (HA^2)

$\Leftrightarrow$ rk (A*A) = rk (A^2)

Over the complex field, A*A and A have the same rank. Therefore,

rk $((A)^2) = rk(A*A) = rk(A) = rk(A*)$

$\Leftrightarrow$ R (A*) $\bigcap$ N (A*) = $\{0\}$

$\Leftrightarrow$ R (A*) $\bigcap$ N (A) = $\{0\}$

$\Leftrightarrow$ H_n = R(A*) $\oplus$ N (A)

This can be proved along the same line and using rk (A*) = rk (A). Thus (11) $\Leftrightarrow$ (12)

(11) $\Leftrightarrow$ (1): If H_n = R (A*) $\oplus$ N(A) then R (A*) $\bigcap$ N (A) = $\{0\}$. For x $\in$ N(A), x $\notin$ R(A)* $\Leftrightarrow$ x $\in$ N(A)* = N (A*)

Hence N(A) $\subseteq$ N (A*) and rk (A) = rk (A*)

$\Leftrightarrow$ N (A) = N (A*)

$\Leftrightarrow$ A is q- EP

Thus (11) $\Leftrightarrow$ (1) holds. Similarly, we can prove (12) $\Leftrightarrow$ (1). Hence the theorem.

Theorem 2: If A $\in$ H_{nxn} is normal and AA* is q - EP then A is q- EP.

Proof: Since A is normal, A is EP moreover AA* is q-EP.

$\Rightarrow$ R(AA*) = R ((AA*)*)

$\Rightarrow$ R (A) = R ((A)*)

$\Rightarrow$ R (A) = R(A*)

$\Rightarrow$ A is q- EP.

Hence the theorem.

Theorem 3: Let 'E' be quaternion hermitian idempotent. Then Hq(E) = {A:A is q-EP and R(A) = R(E)}forms a maximal subgroup at H_{nxn} containing E as identity.

Proof: Since E as identity is quaternion hermitian, it is automatically q - Ep. Thus E $\in$ H_q (E).

Next we shall prove that for any A $\in$ Hq(E) then $A^{\dagger}$ $\in$ H_q(E). Now for any A $\in$ H_q (E) $\Leftrightarrow$ A is q - EP and R(A) = R(E).

$$\begin{aligned} R(A^{\dagger}) &= R(A)^{\dagger} = R(A)* \\ &= R(A*) \\ &= R(A) \\ &= R(E) \end{aligned}$$

Thus $A^{\dagger}$ $\in$ H_q (E). Since E = E* = E^2.

E being hermitian idempotent with R (A) = R (E).E is Projection on R(A).

Therefore
E = $AA^{\dagger}$ = $A^{\dagger}A$ that is E = for any $A \in H_q$ (E).

Now EA = A =AE $\Rightarrow$ for every $A \in H_q$ (E) which shows that 'E' is identity, for Hq (E). Now for any $A \in H_q$ (E) we have $AA^{\dagger} = E \Rightarrow A^{\dagger}$

That is $AA^{\dagger} = E \Rightarrow A^{\dagger}$ is the inverse of A.

Suppose A, B $\in H_q(E)$ $\Rightarrow$ A and B are q - EP with R(A) = R(E) = R(B).

Also rk (A) = rk (A^2). AB is q - EP_r. $\Rightarrow$ Moreover,

Thus E= $AA^{\dagger} = A^{\dagger}A = BB^{\dagger} = B^{\dagger}B$

Now
R(AB) $\subseteq$ R(A) = R(E)
R(AB) $\subseteq$ R(E)

Therefore, AB H_q (E) is closed under multiplication Thus we have shows that H_q(E) is a subgroup of H_{nxn} with identity E. Maximality of H_q(E) follows from the theorem "H(E)=$\{A; A\, is\, EP\, and\, R(A)=R(E)\}$ forms a maximal subgroup containing E as identity" Hence the theorem.

Remark 6: Let $F = F^2 = F^*$ be symmetric idempotent in H_{nxn} Then
H(F) = $\{B \in H_{nxn} : B\, is\, q - EP\, and\, R(B) = R(F)\}$ is maximal Subgroup of H_{nxn} Containing F as identity theorem 2.1, (4).

Theorem 4: H_q (E) and H (F) are isomorphic Subgroups of $H_{nxn.}$

Proof: By defining the mapping ϕ: H_q (E) $\rightarrow$ H(F) Such that ϕ (A) = A. One can Prove that ϕ is well defined, 1-1, onto homomorphism. That is, ϕ is an isomorphism. Thus H_q(E) and H(F) are isomorphic subgroups of H_{nxn}. Hence the theorem.

Remark 7: For $A \in H_{nxn}$ there exists q - hermitian matrices P and Q such that A=P+Q where $Q = xi + yj + zk$,Q is a matrix then $P = \frac{1}{2}(A + A^*)$ and $Q = \frac{1}{2}(A - A^*)$. In the following theorem equivalent condition for matrix A to be q - EP.

Theorem 5: For $A \in H_{nxn}$, A is q - EP $\Leftrightarrow$ N(A) $\subseteq$ N(P) where P is the q - hermitian part of A.

Proof: If A is q-EP, then by the definition $N(A) = N(A^*) \Rightarrow N(A^{\dagger}) = N(A^*)$ Then for $x \in$ N(A), both Ax = 0 and A*x=0 which implied that px = $\left[\frac{1}{2}(A + A^*)\right]x = 0$

Thus N(A) $\subseteq$ N(P). Conversely, let N(A) $\subseteq$ N(P); then Ax = 0 $\Rightarrow$ Px = 0 and hence Qx= 0. Therefore, N(A) $\subseteq$ N (Q). Thus N(A) $\subseteq$ N(P) $\bigcap$ N(Q).Since both P and Q are q- hermition P = P*, Q = Q*

Hence N(P) = N(P*) and N(Q)=N(Q*)
Now,
N(A) $\subseteq$ N(P) $\cap$ N(Q)
= N(P*) $\cap$ N(Q*)
$\subseteq$ N(P*-Q*)
Therefore, $N(A) \subseteq N(A^*)$ and rk (A) = rk (A*) .Hence N (A) = N (A*). Thus A is q - EP.
Hence the the theorem.

REFERENCES

1. Baokett. TS and katz. IJ: Theorems on product EPr matrices; lin. Alg. Appl., 2, 87 - 103 (1969).
2. Ben Isreal.A and Greville.TNE: Generalized Inverses, Theory and Applications: wiley and sons, NewYork (1974).
3. Meenakshi.AR: On Ep_r matrices with entries from an arbitrary field; lin. and multi. Alg., 9, 159-164 (1980).
4. Schwedtfeger. H: Introduction to lin. Alg. and Theory of matrices; P. Noordhogt, Groningen (1962).
5. Zhang.F, Quaternions and matrices of quaternions, linear Algebra and its Application, 251 (1997), 21 - 57.

Source of support: Nil, Conflict of interest: None Declared