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# PERFORMANCE ANALYSIS OF STRATEGIC FORM OF COOPERATIVE GAME THEORY

S. M. CHITHRA*, V. VINOBA

*Department of Mathematics, R. M. K College of Engineering and Technology,
Research Scholar in Bharathidasan University, India.

Department of Mathematics, K. N. Govt. Arts College for women, Thanjavur, India.

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***ABSTRACT***

*Game theory provides a mathematical basis for the analysis of interactive decision-making processes. It provides tools for predicting what might happen and possibly what should happen when agents with conflicting interests interact. It is not a single monolithic technique, but a collection of modeling tools that aid in the understanding of interactive decision problems. Generally game theory breaks naturally into two parts: (i) Non-cooperative theory (ii) Cooperative theory. A cooperative game is a game in which the players have complete freedom of preplay communication to make joint binding agreements. These agreements may be of two kinds to coordinate strategies or to share payoffs. In non-cooperative game theory, we focus on the individual player's strategies and their influence on payoffs and try to predict what strategies players will choose (equilibrium concept). In this paper we completely discuss about the performance analysis of strategic form of cooperative game theory.*

***Key Words:*** *Game Theory, Cooperative game theory, Coalition, Bargaining, Core and Shapley.*

## 1. INTRODUCTION

In cooperative game theory, we abstract from individual player's strategies and instead focus on the coalition players may form. We assume each coalition may attain some payoffs, and there we try to predict which coalitions will form. So far we have been concerned with non cooperative models, where the main focus is on the strategic aspects of the interaction among the players. The approach in cooperative game theory is different. Now, it is assumed that players can commit to behave in a way that is socially optimal. The main issue is how to share the benefits arising from cooperation. Important elements in this approach are the different subgroups of players, referred to as coalitions, and the set of outcomes that each coalition can get regardless of what the players outside the coalition do.

When discussing the different equilibrium concepts for non cooperative games, we were concerned about whether a given strategy profile was self-enforcing or not, in the sense that no player had incentives to deviate. We now assume that players can make binding agreements and, hence, instead of being worried about issues like self-enforceability, we care about notions like fairness and equity.

In this chapter, as customary, the set of players is denoted by $I=\{1,2,....n\}$ . As opposed to non cooperative games, where most of the analysis was done at the individual level, coalitions are very important in cooperative model.

For $S\subset I$, we refer to $S$ as a coalition with $|S|$ denoting the number of players in $S$. Coalition $I$ is often referred to as the grand coalition.

We start this chapter by briefly describing the most general class of cooperative games, the so called non-transferable utility games. Then we discuss two important subclasses: Bargaining problems and transferable utility games.

*Corresponding Author: Fuad. S. M. Chithra**
**Department of Mathematics, R. M. K College of Engineering and Technology,
Research Scholar in Bharathidasan University, India.*

## 2. PRELIMINARIES

In this section we formally introduce some basic definitions, concepts and solutions for cooperative game theory.

**Definition 1:** For an n- person game we shall take $I = \{1, 2, ....n\}$ be the set of all players. Any nonempty subset of $I$ is called **coalition**.

For each of these two classes we present the most important solution concepts and some axiomatic characterizations. Finally, we conclude this chapter by presenting some applications of cooperative game theory to the study of operations research problems.

**Non transferable utility games:**

In this section we present a brief introduction to the most general class of cooperative games: non transferable utility games of NTU-games.

The main source of generality comes from the fact that, although binding agreements between the players are implicitly assumed to be possible, utility is not transferable across players. Below, we present the format definition and then we illustrate it with an example.

Given $S \subset I$ and a set $A \subset \mathfrak{R}^S$, we say that $A$ is **comprehensive** if, for each pair $x, y \in \mathfrak{R}^S$ such that $x \in A$ and $y \le x$, we have that $y \in A$. Moreover, the **comprehensive hull** of a set $A$ is the smallest comprehensive set containing $A$.

**Definition 2:** An n-player nontransferable utility game (NTU-game) is a pair $(I, V)$ where $I$ is the set of players and $V$ is a function that assigns, to each coalition $S \subset I$, a set $V(S) \subset \mathfrak{R}^S$. By convention, $V(\varphi) = \{0\}$. Moreover, for each $S \subset I$, $S \ne \varphi$:

(i) $V(S)$ is a nonempty and closed subset of $\mathfrak{R}^S$.

(ii) $V(S)$ is comprehensive. Moreover, for each $i \in I$, $V(\{i\}) \ne \mathfrak{R}$ (i.e) there is $v_i \in \mathfrak{R}$ such that $V(\{i\}) = (-\infty, v_i]$.

(iii) The set $V(S) \cap \{ y \in \mathfrak{R}^S :$ for each $i \in S,\ y_i \ge v_i \}$ is bounded.

**Remark:** Non emptiness and closeness are two technical requirements, which are also fairly natural. Requiring the $V(S)$ sets to be comprehensive is a convenient assumption, whose basic idea is that the players in coalition $S$ can throw away utility if they want to.

Moreover, it is worth mentioning that it is also often assumed that the $V(S)$ sets of convex, which would imply, in particular, that the players inside each coalition $S$ can choose lotteries over the elements of $\mathfrak{R}^S$ and that their utility functions are of the Von Neumann and Morgenstern type.

**Definition 3:** Let $(I, V)$ be an NTU-game. Then, the vector in $\mathfrak{R}^I$ are called allocations. An allocation $x \in \mathfrak{R}^I$ is feasible if there is a partition $\{S_1, S_2, ....S_K\}$ of $I$ satisfying that, for each $l \in \{1, 2.......K\}$ there is $y \in V(S_l)$ such that, for each $i \in S_l$, $y_i = x_i$.

The main objective of the theoretical analysis in this field is to find appropriate rules for choosing feasible allocations for the general class of NTU-games. These rules are referred to as **solution** and aim to select allocations that have desirable properties according to different criteria such as equity, fairness, and stability. If a solution selects a single allocation for each game, then it commonly referred to as an **allocation rule**.

The definition of NTU-game allows us to model a wide variety of situations and yet, at the same time, because of its generality, the study of NTU-games quickly becomes mathematically involved. Thus, because of this, the literature has focused more on studying some special cases than on studying the general frame work.

Now we discuss the two most relevant subclasses of NTU-games: Bargaining games with transferable utility (TU-games).

**Bargaining**
In this section we study a special class of NTU-games, referred to as **bargaining problems**, originally studied in Nash (1950 a). In a bargaining problem, there is a set of possible allocations, the **feasible set** F, and one of them has to be chosen by the players. Importantly, all the players have to agree on the chosen allocation; otherwise, the realized allocation is $\underline{d}$ , the **disagreement point.**

**Definition 4:** An n- player bargaining problem with set of players $I$ is a pair (F, d) whose elements are the following:

Feasible set: F is the comprehensive hull of a compact and convex subset of $\Re^I$ .

Disagreement point: d is an allocation in F. It is assumed that there is $x \in F$ such that $x > d$ .

We denote the set of n-player bargaining problems by $B^I$ . Moreover, given a bargaining problem (F,d)$\in$ $B^I$ , we define the compact set $F_d = \{x \in F : x \geq d\}$ .

**Definition 5:** An allocation rule for n-player bargaining problems is a map $\Phi : B^I \to \Re^I$ such that, for each (F, d)$\in B^I$ , $\varphi(F, d) \in F_d$ .

**Transferable Utility Games:**

We now move to the most widely studied class of cooperative games: those with transferable utility, in short, TU-game. The different coalitions that can be formed among the players in $I$ can enforce certain allocations (possibly through binding agreements); the problem is to decide how the benefits generated by the cooperation of the players(formation of coalitions) have to be shared among them. However, there is one important departure from the general NTU-games framework.

In a TU-game, given a coalition $S$ and an allocation $x \in V(S) \subset \Re^I$ that the players in $S$ can enforce, all the allocations that can be obtained from $x$ by transfers of utility among the players in $S$ also belong to $V(S)$. Hence, $V(S)$ can be characterized by a single number given by $\max\limits_{x \in V(S)} \sum\limits_{i \in S} x_i$ . We denote the last number by $v(S)$, the **worth** of coalition $S$ .

The transferable utility assumption has important implications, both conceptually and mathematically. From the conceptual point of view, it implicitly assumes that there is a enumerative good such that the utilities of all the players are linear with respect to it and that this good can be freely point of view, since the description of a game consists of a number for each coalition of players, TU-games are much more tractable than general NTU-games.

**Definition 6:** A TU-game is a pair $(I, v)$, where $I$ is the set of players of $\upsilon : 2^I \to \Re$ is the **characteristic function** of the game. By convention, $\upsilon(\phi) = 0$.

In general, we interpret $\upsilon(S)$, the worth of coalition $S$ ,as the benefit that $S$ can generate when no confusion arises, we denote the game $(I, v)$ by $v$ . Also, we denote $\upsilon(\{i\})$ and $\upsilon(\{i, j\})$ by $\upsilon(i)$ and $\upsilon(i, j)$ respectively. Let $G^I$ be the class of TU-games with n-players.

**Remark:** A TU-game $(I, v)$ can be seen as an NTU-game $(I, V)$ by defining, for each nonempty coalition $S \subset I$ , $V(S) = \{y \in \Re^S : \sum\limits_{i \in S} y_i \leq \upsilon(S)\}$.

**Definition 7:** Let $(I, v) \in G^I$ and let $S \subset I$ . The restriction of $(I, v)$ to the coalition $S$ is the TU-game $(S, \upsilon_S)$, where, for each $T \subset S$ , $\upsilon_S(T) = \upsilon(T)$.

**The Core and Shapley**:

In this section we study the most important concept dealing with stability: **the core.** First, we introduce some properties of the allocations associated with a TU-game.

Let $\upsilon \in G^I$ and let $x \in \Re^I$ be an allocation. Then $x$ is efficient if $\sum_{i \in N} x_i = \upsilon(I)$. The allocation $x$ is individually rational if, for each $i \in N, x_i \geq \upsilon(i)$, that is, no player gets less than what he can get by himself. The set of imputations of a TU-game, $I(\upsilon)$, consists of all the efficient and individually rational allocations.

**Definition 8:** Let $\upsilon \in G^I$. The set of **imputations** of $\upsilon$, $I(\upsilon)$ is defined by
$I(\upsilon) = \left\{ x \in \Re^N : \sum_{i \in N} x_i = \upsilon(N), i \in N, x_i \geq \upsilon(i) \right\}$.

**Definition 9:** Let $\upsilon \in G^I$. The **core** of $\upsilon$, $C(\upsilon)$ is defined by $C(\upsilon) = \left\{ x \in I(\upsilon) : S \subset I, \sum_{i \in S} x_i \geq \upsilon(S) \right\}$.

The elements of $C(\upsilon)$ are usually called **Core allocation.** The core is always a subset of the set of imputations.

**Definition 10:** Let $\upsilon \in G^I$. Let $S \subset I$, $S \neq \phi$, and let $x, y \in I(\upsilon)$. We say that y dominates x through S if (i) for each $i \in S, y_i > x_i$ and (ii) $\sum_{i \in S} y_i \leq \upsilon(S)$.We say that y dominates x if there is a nonempty coalition $S \subset I$ such that y dominates x through S. Finally x is an un dominated imputation of $\upsilon$ if there is no $y \in I(\upsilon)$ such that y dominates x.

Proposition: Let $\upsilon \in G^I$. Then (i) if $x \in C(\upsilon), x$ is un dominated. (ii) if $\upsilon \in SG^N$, $C(\upsilon) = \{x \in D(\upsilon)\}$ : x is dominated.

**The Shapley Value**:

In the previous section we studied the core of a TU-game, which is the most important set valued solution concept for TU-games. Now, we present the most important allocation rule: The Shapley value (Shapley 1953). Formally, an allocation rule is defined as follows.

**Definition 11:** An allocation rule for n- player TU-game is just a map $\phi : G^I \rightarrow \Re^I$ .

Shapley (1953), following an approach similar to the one taken by Nash when studying the Nash solution for bargaining problems, gave some appealing properties that an allocation rule should satisfy and proved that they characteristic a unique allocation rule. First, we need to introduce two other concepts.

**Definition 12:** The Shapley value $\Phi$ is defined, for each $\upsilon \in G^I$ and each $i \in I$, by
$\Phi_i(\upsilon) = \sum_{S \subset I/\{i\}} \frac{|S|!(n-|S|-1)!}{n!} \left( \upsilon(S \cup \{i\}) - \upsilon(S) \right)$. Therefore, in the Shapley value, each player gets a weighted average of the contributions he makes to the different coalitions.

**The Nucleolus**

In this section we present the nucleolus (Schmeidler 1969), perhaps the second most important allocation rule for TU-games, just behind the Shapley value.

Let $\upsilon \in G^I$ and let $x \in \Re^N$ be an allocation. Given a coalition $S \subset I$, the excess of coalition S with respect to $x$ is defined by $e(S, x) : \upsilon(S) - \sum_{i \in S} x_i$.

This is a measure of the degree of dissatisfaction of coalition S when the allocation $x$ is realized. Note that, for each $S \subset I$, $e(S,x) \le 0$.

In n-person non-cooperative game theory, we can consider a cooperative game as presented in extensive or normal forms. The practical problem of either of these is obvious; to present a finite n-person game in normal form requires an n-dimensional matrix. Moreover, in a cooperative game new consideration arise. We are especially interested in which coalitions are likely to form. Since payoffs are assumed to be in Monetary form we can take it that coalitions will by and large act, by coordinating strategies, to maximize their joint payoff. Because agreements are binding we assume that coalitions once formed, by whatever bargaining process, remain stable for the duration of the game.

Although the information concerning which coalitions are likely to form can be recovered from the extensive or normal form, it is obviously more desirable to have it available explicitly. The stage is now set for us to move to the next level of abstraction, in game theory: the characteristic function form of a game.

Let $\Gamma$ be an n-person game with a set of players $I = \{1,2,....n\}$. Any subset $S \subseteq I$ will be called a coalition. The characteristic function $\upsilon$ of the game $\Gamma$ is a function $\upsilon : \phi(I) \to \Re$ such that $\upsilon(S)$ represents the largest joint payoff which the coalition $S$ is guaranteed to obtain if the members coordinate their strategies by preplay agreement. We define $\upsilon(\varphi) = 0$.

**Theorem 1:** For any finite cooperative game $\Gamma$, $\upsilon(S \cup T) \ge \upsilon(S) + \upsilon(T)$ for $S,T \subseteq I$ and $S \cap T = \Phi$ (1).

**Proof:** We know that $\upsilon(S) = \max\limits_{x \in X_S} \min\limits_{y \in X_{I\setminus S}} P_i(x,y)$ where $X_S, X_{I\setminus S}$ respectively, and $P_i(x,y)$ denotes the expected payoff to player $i$ when the mixed strategies $x \in X_S$, $y \in X_{I\setminus S}$ are employed.

$$\therefore \upsilon(S \cup T) = \max_{x \in X_{S\cup T}} \min_{y \in X_{I\setminus(S\cup T)}} \sum_{i \in S\cup T} P_i(x,y),$$

Where $X_{S\cup T}$ denotes the set of coordinated mixed strategies for the coalition $S \cup T$ etc. If we restrict our attention to independent mixed strategies $\alpha \in X_S$, $\beta \in X_T$ the range of maximization will decrease and so the value of the maximum above can only decrease. Hence

$$\therefore \upsilon(S \cup T) \ge \max_{\alpha \in X_S} \min_{\beta \in X_T} \min_{y \in X_{I\setminus(S\cup T)}} \sum_{i \in S\cup T} P_i(\alpha,\beta,y),.$$

Hence for each $\alpha \in X_S$, $\beta \in X_T$

$$\upsilon(S \cup T) \ge \min_{y \in X_{I\setminus(S\cup T)}} \sum_{i \in S\cup T} P_i(\alpha,\beta,y)$$

$$\ge \min_{y \in X_{I\setminus(S\cup T)}} \left( \sum_{i \in S} P_i(\alpha,\beta,y) + \sum_{i \in T} P_j(\alpha,\beta,y) \right)$$

Since $S \cap T = \Phi$. Hence

$$\upsilon(S \cup T) \ge \min_{\beta \in X_T} \min_{y \in X_{I\setminus(S\cup T)}} \sum_{i \in S} P_i(\alpha,\beta,y) + \min_{\alpha \in X_S} \min_{y \in X_{I\setminus(S\cup T)}} \sum_{i \in T} P_i(\alpha,\beta,y)$$

In the first term on the right the minimum is taken with respect to mixed strategies $\beta \in X_T$, $y \in X_{I\setminus(S\cup T)}$.

The pair $(\beta, y) \in X_T \times X_{I\setminus(S\cup T)}$ defines a mixed strategy in $X_{I\setminus S}$. Since $X_T \times X_{I\setminus(S\cup T)} = \{S_T\} \times \lfloor S_{I\setminus(S\cup T)} \rfloor \subseteq [S_{I\setminus S}] = X_{I\setminus S}$. If instead we minimize with respect to an arbitrary mixed strategy in $X_{I\setminus S}$, the range of minimization will increase and so the value of this minimum can only decrease. A similar remark applies to the second term on the right if the minimum is taken with respect to an arbitrary mixed strategy in $X_{I\setminus T}$. Thus

$$\upsilon(S \cup T) \ge \min_{\gamma \in X_{I\setminus S}} \sum_{i \in S} P_i(\alpha,\gamma) + \min_{\delta \in x_{i\setminus T}} \sum_{i \in T} P_i(\beta,S)$$

 

For all $\alpha \in X_S$, $\beta \in X_T$. Hence

$$\upsilon(S \cup T) \geq \max_{\alpha \in X_S} \min_{\gamma \in X_{I \setminus S}} \sum_{i \in S} P_i(\alpha, \gamma) + \max_{\beta \in X_T} \min_{\delta \in x_{i \setminus T}} \sum_{i \in T} P_i(\beta, S)$$

Whence from (1), as required.

**Definition 13:** A cooperative game with an additive characteristic function is called **inessential**. Other cooperative games are called **essential**.

**Theorem 2**: A finite n-person cooperative game $\Gamma$ is inessential if and only if

$$\sum_{i \in I} \upsilon(\{i\}) = \upsilon(I) \tag{2}$$

**Proof:** From Theorem 1 $\upsilon(S \cup T) \geq \upsilon(S) + \upsilon(T)$ which implies Theorem (2). It remains to show that equation (2) is sufficient to prove equation 1. From Theorem 1 we have

$$\upsilon(S \cup T) + \upsilon(I \setminus (S \cup T)) \leq \upsilon(I) \tag{3}$$

From (2) $\upsilon(I) = \sum_{i \in I} \upsilon(\{i\}) = \sum_{i \in S} \upsilon(\{i\}) + \sum_{i \in T} \upsilon(\{i\}) + \sum_{i \in I \setminus (S \cup T)} \upsilon(\{i\})$

$$\upsilon(I) \leq \upsilon(S) + \upsilon(T) + \upsilon(I \setminus (S \cup T)) \tag{4}$$

again by theorem, (1). If we now combine the inequalities (3) and (4) and again use (1) we obtain
$\upsilon(S \cup T) \geq \upsilon(S) + \upsilon(T)$ for $S, T \subseteq I$ and $S \cap T = \Phi$

**Theorem 3**: For any finite constant sum cooperative game $\Gamma$, $\upsilon(S) + \upsilon(I \setminus S) = \upsilon(I)$ for each $S \subseteq I$ (5)

The converse is false; that is, there are non-constant sum game which also satisfy (5). The theorem is false if the game $\Gamma$ is not required to be finite.

**Proof:** For any constant sum cooperative game $\upsilon(I) = \sum P_i(x_1, x_2, .........x_n) = C$ for every mixed strategy n-tuple $(x_1, x_2 .....x_n)$.

Hence from (1)

$$\begin{aligned}
\upsilon(S) &= \max_{x \in X_S} \min_{y \in X_{I \setminus S}} \sum_{i \in S} P_i(x, y) \\
&= \max_{x \in X_S} \min_{y \in X_{I \setminus S}} \left( C - \sum_{i \in I \setminus S} P_i(x, y) \right) \\
&= C - \min_{x \in X_S} \max_{y \in X_{I \setminus S}} \sum_{i \in I \setminus S} P_i(x, y) \\
&= C - \max_{y \in X_{I \setminus S}} \min_{x \in X_S} \sum_{i \in I \setminus S} P_i(x, y) \\
&= C - \upsilon(I \setminus S)
\end{aligned}$$

as required.

## 3. CONCLUSION

The basic idea in cooperative game theory is that the net gains that a coalition can generate are divided equally among its members. Since its inception, there have been developed numerous axiomatic characterizations of the Shapley value and it has emerged as an exceptionally important concept that balances coalition's power and fairness in a very intricate fashion. Thus, the main objective of cooperative game theory is to determine a "just" or "well-supported" contract between all players to divided the total wealth generated collectively.

 

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