ON THE FUNCTION \( \Delta_r (x, n) \)

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ABSTRACT

Defining the function \( \Delta_r (x, n) \) related to the \( r \)-totatives of \( n \) we study certain properties of it.

Key words: \( r \)-totatives, \( r \)-analogues of Mobius and Euler Functions.

1. INTRODUCTION

Throughout this paper \( r \) denotes a fixed positive integer. For positive integers \( a \) and \( b \), their greatest \( r \)th power common divisor is denoted by \( (a, b)_r \). It is clear that \( (a, b)_1 \) is the greatest common divisor \( (a, b) \) of \( a \) and \( b \); and that \( (a, b)_r = 1 \) if and only if \( (a, b) \) is \( r \)-free (we recall that a positive integer is \( r \)-free if it is not divisible by the \( r \)th power of any prime).

For a positive integer \( n \), a number \( \tau \) with \( (\tau, n)_r = 1 \) will be called a \( r \)-totative of \( n \). Note that 1-totatives of \( n \) are referred as totatives of \( n \) by J.J.Sylvester (see [7], p.124). V.L. Klee [4] has defined the function \( \phi_r (n) \) as the number of integers \( m \) with \( 1 \leq m \leq n \) and \( (m, n)_r = 1 \). Note that \( \phi_r (n) = \phi (n) \), the well-known Euler function; and that \( \phi_r (n) \) is the number of \( r \)-totatives of \( n \) in \( [0, n) \). Denote the number of \( r \)-totatives \( m \) of \( n \) with \( m \leq x \) by \( \phi_r (x, n) \).

Here we define the function

\[
(1.1) \quad \Delta_r (x, n) = \sum_{\substack{m \leq x \nmid n \atop (m, n)_r = 1}} 1 - x \phi_r (n) = \phi_r (xn, n) - x \phi_r (n)
\]

Note that \( \Delta (x, n) := \Delta_1 (x, n) \) was studied by Codeca and Nair [1]. In this paper we present some properties of (1.1) and the results involving this function in section 3.

2. PRELIMINARIES

The \( r \)-analogue of the Mobius function, \( \mu_r (n) \), is defined (see [4]) by

\[
(2.1) \quad \mu_r (n) = \begin{cases} 
1 & \text{if } n = 1 \\
(-1)^k & \text{if } n = p_1^{r_1} p_2^{r_2} \ldots p_t^{r_t} \text{ where } p_i \text{'s are distinct primes} \\
0 & \text{otherwise}
\end{cases}
\]

and showed that it is multiplicative. V.L.Klee [4] has proved that

\[
(2.2) \quad \phi_r (n) = \sum_{d \mid n} \mu_r (d) = \sum_{d \mid n} \mu_r \left( \frac{n}{d} \right) d
\]

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Since \( \phi_r(x, n) \) is the number of \( r \)-totatives \( m \) of \( n \) with \( m \leq x \), it is easy to show that

\[
\phi_r(x, n) = \sum_{d \mid n} \mu_r(d) \left( \frac{x}{d} \right) = \sum_{d \mid n} \mu_r \left( \frac{n}{d} \right) \left( \frac{x}{n} \right)
\]

where \([y]\) is the greatest integer not exceeding \( y \).

Suppose for a given \( n \) let \( N_r = N_r(n) \) is the \( r \)-th power of the maximal square free divisor of \( n \). Then note that \( (a, n) = 1 \Leftrightarrow (N_r, n) = 1 \). Hence we may assume, without loss to generality, that \( n \) itself is an \( r \)-th power of a squarefree number \( m \), say \( n = m^r \). In all that follows \( n \) is always of this form.

Note that

(2.5) \( \Delta_r(x, n) \) is periodic in \( x \) with period 1.

(2.6) Let \( 1 = a_1 < a_2 < \ldots < a_{\phi_r(n)} = n - 1 \) be the \( \phi_r(n) \) \( r \)-totatives of \( n \) in the interval \([1, n]\). We write \( a_o = 0 \) and \( a_{\phi_r(n)+1} = n \). Then \( n - a_i = a_{\phi_r(n)-i+1} \) and \( a_{\phi_r(n)+1} \in [0, 1] \) for \( 0 \leq i \leq \phi_r(n) \). If \( a_i \)'s are defined as in (2.6) we observe that

(2.7) \( \Delta_r \left( \frac{a_i}{n}, n \right) = i - a_i \frac{\phi_r(n)}{n} \) for \( 0 \leq i \leq \phi_r(n) \)

(2.8) \( \Delta_r(x, n) = \Delta_r \left( \frac{a_i}{n}, n \right) - \left( x - \frac{a_i}{n} \right) \phi_r(n) \) which imply that \( \Delta_r(x, n) \) is a piecewise linear function of \( x \) with each line segment in \( \left[ \frac{a_i}{n}, \frac{a_{i+1}}{n} \right] \) having the gradient \( -\phi_r(n) \).

3. MAIN RESULTS

3.1 Lemma: \( \Delta_r(x, n) = -\mu_r(n) \sum_{d \mid n} \mu_r(d) \{ xd \} \)

Proof: By (1.1), (2.3) and (2.2) we get

\[
\Delta_r(x, n) = \sum_{d \mid n} \mu_r(d) \left( \left\{ \frac{xn}{d} \right\} - \left\{ \frac{x}{d} \right\} \right)
= -\sum_{d \mid n} \mu_r(d) \left( \frac{xn}{d} \right) \sum_{d \mid n} \mu_r(d) \{ xd \}
\]

where \( \{y\} \) denotes the fractional part of \( y \). Since the contribution of divisors \( d \) of \( n \) to the sum on the right is non-zero if and only if \( d \) is the \( r \)-th power of square free integer, so that

\[
\Delta_r(x, n) = \sum_{d \mid n} \mu_r(n) \mu_r(d) \{ xd \} = -\mu_r(n) \sum_{d \mid n} \mu_r(d) \{ xd \},
\]

proving the Lemma.

As a consequence of Lemma 3.1, we have the identity:

(3.2) If \( p \nmid n \), \( \Delta_r \left( x, np^r \right) = \Delta_r \left( x, np^r \right) - \Delta_r \left( x, n \right) \)

It is easy to see that

(3.3) \( \Delta_r(x, n) = -\mu_r(n) \sum_{d \mid n} \mu_r(d) \left( \{ xd \} - \frac{1}{2} \right) \)
Theorem A: If \( (\ell, n) = 1 \) then
\[
\sum_{n=0}^{\ell-1} \Delta_r \left( \frac{u+n}{\ell}, n \right) = \Delta_r (u, n)
\]

Proof: By (3.3) we have
\[
\sum_{n=0}^{\ell-1} \left( \frac{u+n}{\ell}, n \right) = -\mu_r (n) \sum_{d|\ell} \mu_r (d) \sum_{n=0}^{\ell-1} \left( \left\{ \frac{ud}{\ell} + \frac{n}{\ell} \right\} - \frac{1}{2} \right)
\]
\[
= -\mu_r (n) \sum_{d|\ell} \mu_r (d) \left( \left\{ ud \right\} - \frac{1}{2} \right).
\]
Since, \( \sum_{n=0}^{\ell-1} \left( \left\{ ud \right\} - \frac{1}{2} \right) = \left\{ ud \right\} - \frac{1}{2} \), by a result of Landau ([5], p.170), we get
\[
\sum_{n=0}^{\ell-1} \Delta_r \left( \frac{u+n}{\ell}, n \right) = \Delta_r (u, n),
\]
proving the theorem.

Theorem B: \( \int_0^{\Delta_r^2 (x, n)} dx = \frac{1}{12} 2^{a(n)} \frac{\phi_r (n)}{n} \)

Proof: By (3.3) we have
\[
\int_0^{\Delta_r^2 (x, n)} dx = \sum_{d_{ij}} \mu_r (d_1) \mu_r (d_2) \int_0^{\left\{ xd_1 \right\}} \left( \left\{ xd_2 \right\} - \frac{1}{2} \right) dx
\]

Now using the result of Franel [3], namely
\[
\int_0^{\left\{ xd_1 \right\}} \left( \left\{ xd_2 \right\} - \frac{1}{2} \right) dx = \frac{1}{12} (d_1, d_2)^2
\]

it follows that
\[
(3.4) \quad \int_0^{\Delta_r^2 (x, n)} dx = \frac{1}{12} \sum_{d_{ij}} \mu_r (d_1) \mu_r (d_2) \frac{(d_1, d_2)^2}{d_1 d_2}
\]

Let \( D = (d_1, d_2) \) so that \( d_1 = D\delta_1, d_2 = D\delta_2 \) and \( (\delta_1, \delta_2) = 1 \). Then (3.4) gives
\[
\int_0^{\Delta_r^2 (x, n)} dx = \frac{1}{12} \sum_{D|n} \sum_{\delta_1, \delta_2 | D} \mu_r (\delta_1) \mu_r (\delta_2) \frac{(\delta_1, \delta_2)}{\delta}
\]

(3.5) \( \int_0^{\Delta_r^2 (x, n)} dx = \frac{1}{12} g(n) \),

where \( g(n) = \sum_{D|n} f \left( \frac{n}{D} \right) \) in which \( f(m) = \sum_{d|m} \mu_r (d) \tau_r (d) \frac{1}{d} \),

clearly \( f(m) \) is a multiplicative arithmetic function and \( f(p^r) = 1 - \frac{1}{p^r} \).

Therefore \( g(p^r) = f(p^r) + f(1) = 2 \left( 1 - \frac{1}{p^r} \right) = \frac{1}{p^r} \).
Again since \( g(n) \) is multiplicative, it gives that \( g(n) = 2^{\omega(n)} \phi_r(n) \).

Hence \( \int_0^1 \Delta_r^2(x, n) \, dx = \frac{1}{12} 2^{\omega(n)} \phi_r(n) \), proving the theorem.

We need the following Lemma proved in [1] (Corollary, p.347) for our next result:

**3.6 Lemma:** Let \( \alpha_1 < \alpha_2 < \alpha_3 < \ldots < \alpha_r \) be the points in \((0,1)\) such that they are symmetric about \( \frac{1}{2} \) and if

\[
s(x) = \sum_{i=1}^{r} 1 - x \ell \quad \text{then} \quad \frac{1}{r} \sum_{i=1}^{r} s^2(\alpha_i) = \int_0^1 s^2(x) \, dx + \frac{1}{6}.
\]

**Theorem C:** For \( n > 1 \) and if \( a_1 < a_2 < \ldots < a_{\phi_r(n)} \) are the \( r \)-totatives of \( n \) then

\[
\frac{1}{\phi_r(n)} \sum_{i=1}^{\phi_r(n)} \Delta_r^2 \left( \frac{a_i}{n}, n \right) = \frac{1}{12} 2^{\omega(n)} \phi_r(n) + \frac{1}{6}
\]

**Proof:** Since \( (\tau, n) = 1 \Longleftrightarrow (n - \tau, n) = 1 \), the intervals \( \left[ 0, \frac{n}{2} \right] \) and \( \left[ \frac{n}{2}, n \right] \) have the same number of \( r \)-totatives, it follows that the numbers \( \frac{a_i}{n} \) are symmetrically distributed about \( \frac{1}{2} \) in \((0,1)\). Taking \( \alpha_i = \frac{a_i}{n} \) for \( 1 \leq i \leq \phi_r(n) \) in Lemma 3.6 and noting \( s \left( \frac{a_i}{n} \right) = \Delta_r \left( \frac{a_i}{n}, n \right) \), we get

\[
\frac{1}{\phi_r(n)} \sum_{i=1}^{\phi_r(n)} \Delta_r^2 \left( \frac{a_i}{n}, n \right) = \int_0^1 \Delta_r^2(x, n) \, dx + \frac{1}{6}.
\]

Using Thereom B, we have

\[
\frac{1}{\phi_r(n)} \sum_{i=1}^{\phi_r(n)} \Delta_r^2 \left( \frac{a_i}{n}, n \right) = \frac{1}{12} 2^{\omega(n)} \phi_r(n) + \frac{1}{6}
\]

proving the theorem.

**REFERENCES**

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