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ON INTERVAL-VALUED INTUITIONISTIC FUZZY N-FOLD (IMPLICATIVE AND COMMUTATIVE) IDEALS OF BCK-ALGEBRA

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ABSTRACT

In this paper, we apply the concept of interval-valued intuitionistic fuzzy set to n-fold implicative ideal and n-fold (weak) commutative ideal BCK-algebras and introduce the notion of interval-valued intuitionistic fuzzy n-fold implicative ideal, interval-valued intuitionistic fuzzy n-fold (weak) commutative ideal and investigate some of its related properties

Keywords and Phrases: *Interval-valued intuitionistic fuzzy n-fold implicative ideal, Interval-valued intuitionistic fuzzy n-fold (weak) commutative ideal,*

1. INTRODUCTION

Algebraic structure play an important role in mathematics with wide range of applications in many disciplines such as Theoretical Physics, Computer engineering, Control Engineering, Information Sciences, Coding theory etc. In 1999, Hung and Chen [16] introduced the notion of n-fold implicative ideals, n-fold (weak) commutative ideals and investigate some of its properties. Zadeh [23] introduced the notion of Fuzzy sets. At present this concept has been applied to many mathematical branches, such as groups, Near Rings, Semi-rings, Functional Analysis, Probability Theory, Topology and so on. Jun [19], introduced the notions of fuzzyfication of n-fold implicative ideals, n-fold commutative ideals and n-fold weak commutative ideals and investigate some of its properties. Atanassov introduced the ideal of defining a fuzzy set by ascribing a membership degree and non-membership degree separately in such a way that sum of the two degrees must not exceed one, such a pair was given the name of intuitionistic fuzzy set [1]. Satyanarayana and Durga Prasad [22], introduced the notions of intuitionistic fuzzyfication of n-fold implicative ideals, n-fold (weak) commutative ideals of BCK-algebras and some of its related properties are investigated. Atanassov and Gargov [2, 5] introduced the notion of interval valued intuitionistic fuzzy sets which is a generalization of both intuitionistic fuzzy sets and interval valued fuzzy sets. Durga Prasad and Satyanarayana [14] introduced notations of interval-valued intuitionistic fuzzy (implicative and commutative) ideals in BCK-algebra and related properties are investigated

In this paper, we apply the concept of interval-valued intuitionistic fuzzy set to n-fold implicative ideal, n-fold (weak) commutative ideal and introduce the notion of interval-valued intuitionistic fuzzy n-fold implicative ideal, interval-valued intuitionistic fuzzy n-fold (weak) commutative ideals in BCK-algebras. Using level ideals, we give the characterizations of interval-valued intuitionistic fuzzy n-fold implicative ideal and interval-valued intuitionistic fuzzy n-fold (weak) commutative ideals in BCK-algebras. Finally, we establish the extension property for interval-valued intuitionistic fuzzy n-fold commutative ideal in BCK-algebras.

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2. PRELIMINARIES

Definition 2.1: Let X be a set with a binary operation "$*$" and a constant "0". Then $(X, *, 0)$ is called BCK-algebra, if it satisfies the following conditions.
(BCK-1) $((x*y) * (x*z))*(z*y)=0$,
(BCK-2) $(x*(x*y))*y=0$,
(BCK-3) $x*x=0$
(BCK-4) $0*x=0$
(BCK-5) $x*y=0$ and $y*x=0$ imply $x=y$, for all $x, y, z \in X$

We can define a binary relation $\le$ on X by letting $x \le y$ if and only if $x*y=0$. Then $(X,\le)$ is a partially ordered set with least element "0" and $(X, *, 0)$ is a BCK-algebra if and only if, it satisfies the following: For all $x, y, z \in X$
(i) $((x*y)*(x*z)) \le (z*y)$,
(ii) $(x*(x*y)) \le y$,
(iii) $x \le x$,
(iv) $0 \le x$
(v) $x \le y$ and $y \le x$ imply that $x = y$.

In a BCK-algebra $(X, *, 0)$, we have the following properties:
(P1) $x*0 = x$, (P2) $x*y \le x$, (P3) $(x*y)*z = (x*z)*y$, (P4) $(x*z)*(y*z) \le x*y$,
(P5) $x*(x*(x*y)) = x*y$, (P6) $x \le y \Rightarrow x*z \le y*z$ and $z*y \le z*x$,
(P7) $x*y \le z$ implies $x*z \le y$, for all $x, y, z \in X$.
Throughout this paper X will always mean a BCK-algebra unless otherwise specified.

A non-empty sub-set I of X is said to be sub-algebra of X if for $x,y \in I \Rightarrow x*y \in I$.
A non-empty subset I of X is called an ideal of X if (I_1) $0 \in I$ (I_2) $x*y$ and $y \in I \Rightarrow x \in I$ for every $x, y \in X$, an n-fold implicative ideal if (I_1) and (I_3) there exists a fixed $n \in X$ such that $(x*(y*x^n))*z \in I$ and $z \in I \Rightarrow x \in I$ for every $x, y, z \in X$, an n-fold commutative ideal if (I_1) and (I_4) there exists a fixed $n \in X$ such that $(x*y)*z \in I$ and $z \in I \Rightarrow x*(y*(y*x^n)) \in I$ for every $x, y, z \in X$, an n-fold weak commutative ideal if (I_1) and (I_4) there exists a fixed $n \in X$ such that $(x*(x*y^n))*z \in I$ and $z \in I \Rightarrow y*(y*x) \in I$ for every $x, y, z \in X$.
For any elements x and y of X, $x*y^n$ denotes $(.......((x*y)*y)*.......)*y$ in which y' occurs n-times.

Let X be a given nonempty set. An interval-valued fuzzy set (briefly, i-v fuzzy set) B on X is defined by $B = \{(x, [\mu_B^-(x), \mu_B^+(x)]) : x \in X\}$, Where $\mu_B^-(x)$ and $\mu_B^+(x)$ are fuzzy sets of X such that $\mu_B^-(x) \le \mu_B^+(x)$ for all $x \in x$. Let $\tilde{\mu}_B(x) = [\mu_B^-(x), \mu_B^+(x)]$, then $B = \{(x, \tilde{\mu}_B(x)) : x \in X\}$, Where $\tilde{\mu}_B : X \to D[0,1]$, where $D[0, 1]$ is the set of all closed sub intervals of $[0, 1]$.

Combining the idea of intuitionistic fuzzy set and interval-valued fuzzy set, Atanassov and Gargov [2] introduced the notion of interval-valued intuitionistic fuzzy sets which is a generalization of both intuitionistic fuzzy sets and interval-valued fuzzy sets.
An interval-valued intuitionistic fuzzy set (i-v IFS, shortly) "$\tilde{A}$" over X is an object having the form $\tilde{A} = \{(x, \tilde{\mu}_A, \tilde{\lambda}_A) : x \in X\}$, where $\tilde{\mu}_A(x) : X \to D[0,1]$ and $\tilde{\lambda}_A(x) : X \to D[0,1]$, the intervals $\tilde{\mu}_A(x)$ and $\tilde{\lambda}_A(x)$ denotes the intervals of the degree of membership and the degree of the non-membership of the element x to the set $\tilde{A}$, where $\tilde{\mu}_A(x) = [\mu_A^-(x), \mu_A^+(x)]$ and $\tilde{\lambda}_A(x) = [\lambda_A^-(x), \lambda_A^+(x)]$ for all $x \in X$ with the condition $[0,0] \le \tilde{\mu}_A(x) + \tilde{\lambda}_A(x) \le [1,1]$ for all $x \in X$. For the sake of simplicity, we use the symbol $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$, where $D[0, 1]$ is the set of all closed sub intervals of $[0, 1]$.

Definition 2.2: [15] An i-v IFS $\tilde{A} = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ in X is an interval-valued intuitionistic fuzzy sub-algebra if X if it satisfies

(i-v IF-1) $\tilde{\mu}_A(x) \geq \min\{\tilde{\mu}_A((x*y), \tilde{\mu}_A(y)\}$

(i-v IF-2) $\tilde{\lambda}_A(x) \leq \max\{\tilde{\lambda}_A((x*y), \tilde{\lambda}_A(y)\}$, for all $x, y, z \in X$.

Definition 2.3: [15] An i-v IFS $\tilde{A} = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ in X is an interval-valued intuitionistic fuzzy ideal (i-v IFI-ideal) of X if it satisfies

(i-v IFI-1) $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ and $\tilde{\lambda}_A(0) \leq \tilde{\lambda}_A(x)$

(i-v IFI-2) $\tilde{\mu}_A(x) \geq \min\{\tilde{\mu}_A((x*y), \tilde{\mu}_A(y)\}$

(i-v IFI-3) $\tilde{\lambda}_A(x) \leq \max\{\tilde{\lambda}_A((x*y), \tilde{\lambda}_A(y)\}$, for all $x, y, z \in X$.

Theorem 2.4: [15] Let $\tilde{A} = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ be an interval-valued intuitionistic fuzzy ideal of X. If $x \leq y$ in X then $\tilde{\mu}_A(x) \geq \tilde{\mu}_A(y)$ and $\tilde{\lambda}_A(x) \leq \tilde{\lambda}_A(y)$

Theorem 2.5: [15] Let $\tilde{A} = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ be interval-valued intuitionistic sub-algebra of X.
Then $\tilde{A}$ is an interval-valued intuitionistic fuzzy ideal of X if and only if for $x,y,z \in X$, $x*y \leq z$ then $\tilde{\mu}_A(x) \geq \min\{\tilde{\mu}_A(y), \tilde{\mu}_A(z)\}$ and $\tilde{\lambda}_A(x) \leq \max\{\tilde{\lambda}_A(y), \tilde{\lambda}_A(z)\}$

Definition 2.6: [15] An i-v IFS $\tilde{A} = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ in X is an interval-valued intuitionistic fuzzy implicative ideal (i-v IFI-ideal) of X if it satisfies

(i-v IFI-1) $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ and $\tilde{\lambda}_A(0) \leq \tilde{\lambda}_A(x)$

(i-v IFI-2) $\tilde{\mu}_A(x) \geq \min\{\tilde{\mu}_A((x*(y*x))*z), \tilde{\mu}_A(z)\}$

(i-v IFI-3) $\tilde{\lambda}_A(x) \leq \max\{\tilde{\lambda}_A((x*(y*x))*z), \tilde{\lambda}_A(z)\}$, for all $x, y, z \in X$.

Definition 2.7: [15] An i-v IFS $\tilde{A} = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ in X is an interval-valued intuitionistic fuzzy commutative ideal (i-v IFC-ideal) of X if it satisfies

(i-v IFCI 1) $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ and $\tilde{\lambda}_A(0) \leq \tilde{\lambda}_A(x)$

(i-v IFCI 2) $\tilde{\mu}_A(x*(y*(y*x)) \geq \min\{\tilde{\mu}_A((x*y)*z), \tilde{\mu}_A(z)\}$

(i-v IFCI 3) $\tilde{\lambda}_A(x*(y*(y*x)) \leq \max\{\tilde{\lambda}_A((x*y)*z), \tilde{\lambda}_A(z)\}$ for all $x, y, z \in X$.

3. INTERVAL-VALUED INTUITIONISTIC FUZZY N-FOLD IMPLICATIVE IDEALS OF BCK-ALGEBRAS

In this section, we apply the concept of interval-valued intuitionistic fuzzy set to n-fold implicative ideals of BCK-algebras and introduce the notions of interval-valued intuitionistic fuzzy n-fold implicative ideals of BCK-algebras and investigate the some of its related properties.

Definition 3.1: An i-v IFS $\tilde{A} = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ in X is an interval-valued intuitionistic fuzzy n-fold implicative $(\text{i-vIFI}^n-\text{ideal})$ ideal of X if it satisfies

(i-v IFI^{n}1) $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$, $\tilde{\lambda}_A(0) \leq \tilde{\lambda}_A(x)$ and there exists a fixed $n \in N$ such that

(i-v IFI^{n}2) $\tilde{\mu}_A(x) \geq \min\{\tilde{\mu}_A((x*(y*x^n))*z), \tilde{\mu}_A(z)\}$

(i-v IFI^{n}3) $\tilde{\lambda}_A(x) \leq \max\{\tilde{\lambda}_A((x*(y*x^n))*z), \tilde{\lambda}_A(z)\}$ for every $x, y, z \in X$.

Note that, an interval-valued intuitionistic fuzzy 1-fold implicative ideal is an interval-valued intuitionistic fuzzy implicative ideal.

Theorem 3.2: Every interval-valued intuitionistic fuzzy n-fold implicative ideal of X is an interval-valued intuitionistic fuzzy ideal.

Proof: Put $y = 0$ in $(\text{i-v IFI}^{n}2)$ and $(\text{i-v IFI}^{n}3)$.

We get, $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy ideal of X.

The following example shows that the converse of Theorem 3.2 may not be true.

Example 3.3: Let $X = N \cup \{0\}$, where N is the set of natural numbers , in which the operation $*$ is defined by $x * y = \max\{0,\ x\text{-}y\}$ for all $x, y \in X$. Then X is a BCK-algebra.

Let $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ be an i-v IFS in X given by $\tilde{\mu}_A(0) = [0.7, 0.8] > [0.3, 0.4] = \tilde{\mu}_A(x)$ and $\tilde{\lambda}_A(0) = [0.1, 0.2] < [0.4, 0.45] = \tilde{\lambda}_A(x)$, for all $x (\neq 0) \in X$. Then $\tilde{A}$ is an interval-valued intuitionistic fuzzy ideal of X but $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is not an interval-valued intuitionistic fuzzy 2-fold implicative ideal of X because

$$\tilde{\mu}_A(3) = [0.3, 0.4] < [0.7, 0.8] = \tilde{\mu}_A(0) = \min\{\tilde{\mu}_A((3*(14*3^2))*0), \tilde{\mu}_A(0)\} \text{ and}$$

$$\tilde{\lambda}_A(3) = [0.4, 0.45] > [0.1, 0.2] = \tilde{\lambda}_A(0) = \max\{\tilde{\lambda}_A((3*(14*3^2))*0), \tilde{\lambda}_A(0)\}$$

We give a condition for an interval-valued intuitionistic fuzzy ideal to be an interval-valued intuitionistic fuzzy n-fold implicative ideal.

Theorem 3.4: Let $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ be an `interval-valued intuitionistic fuzzy ideal of X. Then $\tilde{A}$ is an interval-valued intuitionistic fuzzy n-fold implicative ideal of X if and only if it satisfies the inequalities $\tilde{\mu}_A(x) \geq \tilde{\mu}_A(x*(y*x^n))$ and $\tilde{\lambda}_A(x) \leq \tilde{\lambda}_A(x*(y*x^n))$ for all $x, y \in X$.

Theorem 3.5: Let $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ be an interval-valued intuitionistic fuzzy ideal of X.Then the following conditions are equivalent:

- **(i)** $\tilde{A}$ is an interval-valued intuitionistic fuzzy n-fold implicative ideal of X.
- **(ii)** $\tilde{\mu}_A(x) \geq \tilde{\mu}_A(x*(y*x^n))$ and $\tilde{\lambda}_A(x) \leq \tilde{\lambda}_A(x*(y*x^n))$ for all $x, y \in X$.
- **(iii)** $\tilde{\mu}_A(x) = \tilde{\mu}_A(x*(y*x^n))$ and $\tilde{\lambda}_A(x) = \tilde{\lambda}_A(x*(y*x^n))$ for all $x, y \in X$.

Proof: $(i) \Rightarrow (ii)$ Assume (i), that is**.** Let $\tilde{A}$ be an interval-valued intuitionistic fuzzy n-fold implicative ideal of X. By Theorem 3.4, the condition (ii) holds.

$(ii) \Rightarrow (iii)$ Observe that in X , $x*(y*x^n) \leq x$ by $(P3)$ and Theorem 2.4.

We have $\tilde{\mu}_A(x*(y*x^n)) \geq \tilde{\mu}_A(x)$ and $\tilde{\lambda}_A(x*(y*x^n)) \leq \tilde{\lambda}_A(x)$. It follows from (ii) that $\tilde{\mu}_A(x*(y*x^n)) = \tilde{\mu}_A(x)$ and $\tilde{\lambda}_A(x*(y*y^n)) = \tilde{\lambda}_A(x)$, for all $x, y \in X$.

Hence the condition (iii) holds.

$(iii) \Rightarrow (i)$ Suppose the condition (iii) holds.

Using (i-v IF2) and (i-v IF3) we get

We have $\tilde{\mu}_A(x*(y*x^n))) \geq \min\{\tilde{\mu}_A((x*(y*x^n))*z), \tilde{\mu}_A(z)\}$ and

$$\tilde{\lambda}_A(x*(y*x^n))) \leq \max\{\tilde{\lambda}_A((x*(y*x^n))*z), \tilde{\lambda}_A(z)\} \text{ for all } x, y, z \in X.$$

Combining (iii) we obtain $\tilde{\mu}_A(x) \geq \min\{\tilde{\mu}_A((x*(y*x^n))*z), \tilde{\mu}_A(z)\}$ and

$$\tilde{\lambda}_A(x) \leq \max\{\tilde{\lambda}_A((x*(y*x^n))*z), \tilde{\lambda}_A(z)\} \text{ for all } x, y, z \in X.$$

Obviously $\tilde{A}$ satisfies $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ and $\tilde{\lambda}_A(0) \leq \tilde{\lambda}_A(x)$ for all $x \in X$.

Therefore $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy n-fold implicative ideal of X. Hence the condition (i) holds. This proof is complete.

Theorem 3.7: An i-v IFS $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ of X is an interval-valued intuitionistic fuzzy n-fold implicative ideal, if for $x,y,z,u \in X$, $(x*(y*x^n))*z \leq u$ implies that $\tilde{\mu}_A(x) \geq \min\{\tilde{\mu}_A(z), \tilde{\mu}_A(u)\}$ and $\tilde{\lambda}_A(x) \leq \max\{\tilde{\lambda}_A(z), \tilde{\lambda}_A(u)\}$.

Theorem 3.8: Let $\tilde{A} = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ be an i-v IF set of X, then the following conditions are equivalent.

- **(i)** $\tilde{A}$ is an i-v intuitionistic fuzzy n-fold implicative ideal of X.
- **(ii)** The non-empty sets $U(\tilde{\mu}_A; [s_1, s_2])$ and $L(\tilde{\lambda}_A; [t_1, t_2])$ are interval valued n-fold implicative ideals of X, for all $[s_1, s_2], [t_1, t_2] \in D[0,1]$

Proof: The proof is straight forward**.**

4. INTERVAL-VALUED INTUITIONISTIC FUZZY N-FOLD COMMUTATIVE IDEALS OF BCK-ALGEBRAS

In this section, we apply the concept of interval-valued intuitionistic fuzzy sets to n-fold commutative ideals of BCK-algebras and introduced the notions of interval-valued intuitionistic fuzzy n-fold (weak) commutative ideals of BCK-algebras and investigate the some of its related properties.

Definition 4.1: An i-v IFS $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ in X is an interval-valued intuitionistic fuzzy n-fold commutative ideal (i-v IFCI^n − ideal) of X if it satisfies

(i-v IFCI^{n}1) $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$, $\tilde{\lambda}_A(0) \leq \tilde{\lambda}_A(x)$ and there exists a fixed $n \in N$ such that

(i-v IFCI^{n}2) $\tilde{\mu}_A(x*(y*(y*x^n))) \geq \min\{\tilde{\mu}_A((x*y)*z), \tilde{\mu}_A(z)\}$

(i-v IFCI^{n}3) $\tilde{\lambda}_A(x*(y*(y*x^n))) \leq \max\{\tilde{\lambda}_A((x*y)*z), \tilde{\lambda}_A(z)\}$ for all $x, y, z \in X$.

Definition 4.2: An i-v IFS $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ in X is an interval-valued intuitionistic fuzzy n-fold weak commutative ideal (i-v IFWC^n − ideal) of X if it satisfies

(i-v IFWCI^n-1) $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$, $\tilde{\lambda}_A(0) \leq \tilde{\lambda}_A(x)$ and there exist a fixed $n \in N$ such that

(i-v IFWCI^n-2) $\tilde{\mu}_A(y*(y*x)) \geq \min\{\tilde{\mu}_A((x*(x*y^n))*z), \tilde{\mu}_A(z)\}$

(i-v IFWCI^n-3) $\tilde{\lambda}_A(y*(y*x)) \leq \max\{\tilde{\lambda}_A((x*(x*y^n))*z), \tilde{\lambda}_A(z)\}$ for all $x, y, z \in X$.

Note that, an interval-valued intuitionistic fuzzy 1-fold commutative ideal is an interval-valued intuitionistic fuzzy commutative ideal.

Theorem 4.3: Every interval-valued intuitionistic fuzzy n-fold commutative ideal is an interval-valued intuitionistic fuzzy deal of X.

Proof: Put $y = 0$ in $(\text{i-vIFCI}^n\text{-2})$ and $(\text{IFCI}^n\text{-3})$.

We get, $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy ideal of X.

Theorem 4.4: Every interval-valued intuitionistic fuzzy n-fold weak commutative ideal is an interval-valued intuitionistic fuzzy ideal of X.

Proof: Put $y = x$ in $(\text{i-vIFWCI}^n\text{-2})$ and $(\text{i-vIFWCI}^n\text{-3})$.

We get, $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy ideal of X.

Theorem 4.5: Let $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ be an interval-valued intuitionistic fuzzy ideal of X. Then

- **(i)** $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy n-fold commutative ideal of X If and only if $\tilde{\mu}_A(x*(y*(y*x^n))) \geq \tilde{\mu}_A(x*y)$ and $\tilde{\lambda}_A(x*(y*(y*x^n))) \leq \tilde{\lambda}_A(x*y)$ for all $x, y \in X$.
- **(ii)** $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy n-fold weak commutative ideal of X if and only if $\tilde{\mu}_A(y*(y*x)) \geq \tilde{\mu}_A(x*(x*y^n))$ and $\tilde{\lambda}_A(y*(y*x)) \leq \tilde{\lambda}_A(x*(x*y^n))$, for all $x, y \in X$.

Proof: Let $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ be an interval-valued intuitionist fuzzy ideal of X.

(i) Assume that $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy n-fold commutative ideal of X.

Put $z = 0$ in $(\text{i-vIFCI}^n 2)$ & $(\text{i-vIFCI}^n 3)$, we get

$$\tilde{\mu}_A(x*(y*(y*x^n))) \geq \min\{\tilde{\mu}_A((x*y)*0), \tilde{\mu}_A(0)\} = \tilde{\mu}_A(x*y) \text{ and}$$

$$\tilde{\lambda}_A(x*(y*(y*x^n))) \leq \max\{\tilde{\lambda}_A((x*y)*0), \tilde{\lambda}_A(0)\} = \tilde{\lambda}_A(x*y)$$

Therefore, $\tilde{\mu}_A(x*(y*(y*x^n))) \geq \tilde{\mu}_A(x*y)$ and $\tilde{\lambda}_A(x*(y*(y*x^n))) \leq \tilde{\lambda}_A(x*y)$ for all $x, y \in X$.

Conversely assume that $\tilde{\mu}_A(x*(y*(y*x^n))) \geq \tilde{\mu}_A(x*y)$ and $\tilde{\lambda}_A(x*(y*(y*x^n))) \leq \tilde{\lambda}_A(x*y)$ for all $x, y \in X$.

Using (i-vIF2) and (i-vIF3) we get

$$\tilde{\mu}_A(x*(y*(y*x^n)) \geq \tilde{\mu}_A(x*y) \geq \min\{\tilde{\mu}_A((x*y)*z), \tilde{\mu}_A(z)\} \text{ and}$$

$$\tilde{\lambda}_A(x*(y*(y*x^n)) \leq \tilde{\lambda}_A(x*y) \leq \max\{\tilde{\lambda}_A((x*y)*z), \tilde{\lambda}_A(z)\}, \text{ for all } x, y, z \in X.$$

Thus $\tilde{A}$ is an interval-valued intuitionistic fuzzy n-fold commutative ideal of X

(ii) Assume that $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy n-fold weak commutative ideal of X. Put $z = 0$ in $(\text{i-vIFWCI}^n 2)$ & $(\text{i-vIFWCI}^n 3)$,

We get $\tilde{\mu}_A(y*(y*x)) \geq \min\{\tilde{\mu}_A((x*(x*y^n)*0),\tilde{\mu}_A(0)\} = \tilde{\mu}_A((x*(x*y^n)$ and
$\tilde{\lambda}_A(y*(y*x)) \leq \max\{\tilde{\lambda}_A((x*(x*y^n)*0),\tilde{\lambda}_A(0)\} = \tilde{\lambda}_A((x*(x*y^n)$.

Therefore $\tilde{\mu}_A(y*(y*x)) \geq \tilde{\mu}_A(x*(x*y^n))$ and $\tilde{\lambda}_A(y*(y*x)) \leq \tilde{\lambda}_A(x*(x*y^n))$, for all $x, y \in X$.

Conversely assume that $\tilde{\mu}_A(y*(y*x)) \geq \tilde{\mu}_A(x*(x*y^n))$ and $\tilde{\lambda}_A(y*(y*x)) \leq \tilde{\lambda}_A(x*(x*y^n))$, for all $x, y \in X$. Since (i-vIF2) and (i-vIF3), we obtain

$$\tilde{\mu}_A(y*(y*x)) \geq \tilde{\mu}_A(x*(x*y^n)) \geq \min\{\tilde{\mu}_A((x*(x*y^n))*z),\tilde{\mu}_A(z)\} \text{ and}$$
$$\tilde{\lambda}_A(y*(y*x)) \leq \tilde{\lambda}_A(x*(x*y^n) \leq \max\{\tilde{\lambda}_A((x*(x*y^n))*z),\tilde{\lambda}_A(z)\}, \text{ for all } x,y,z \in X.$$

Thus $\tilde{A}$ is an interval-valued intuitionistic fuzzy n-fold weak commutative ideal of X.

- Observe that $x*y \leq x*(y*(y*x^n))$ and using Theorem 2.4. we have
$$\tilde{\mu}_A(x*y) \geq \tilde{\mu}_A(x*(y*(y*x^n))) \text{ and } \tilde{\lambda}_A(x*y) \leq \tilde{\lambda}_A(x*(y*(y*x^n))).$$

Hence proposition 4.5(i) can be improved as follows.

Theorem 4.6: An interval-valued intuitionistic fuzzy ideal $\tilde{A} = (\tilde{\mu}_A,\tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy n-fold commutative ideal if and only if

$$\tilde{\mu}_A(x*(y*(y*x^n))) = \tilde{\mu}_A(x*y) \text{ and } \tilde{\lambda}_A(x*(y*(y*x^n))) = \tilde{\lambda}_A(x*y) \text{ for all } x, y \in X.$$

Theorem 4.7: Let $\tilde{A} = (X,\tilde{\mu}_A,\tilde{\lambda}_A)$ be an i-v IF set of X, Then the following conditions are equivalent.

(i) $\tilde{A}$ is an i-v intuitionistic fuzzy n-fold commutative ideal of X.

(ii) The non-empty sets $U(\tilde{\mu}_A; [s_1, s_2])$ and $L(\tilde{\lambda}_A;[t_1, t_2])$ are interval valued n-fold commutative ideal of X, for all $[s_1, s_2],[t_1, t_2] \in D[0,1]$.

Proof: The proof is straight forward.

Theorem 4.8: Let $\tilde{A} = (X,\tilde{\mu}_A,\tilde{\lambda}_A)$ be an i-v IF set of X, Then the following conditions are equivalent.

(i) $\tilde{A}$ is an i-v intuitionistic fuzzy n-fold weak commutative ideal of X.

(ii) The non-empty sets $U(\tilde{\mu}_A; [s_1, s_2])$ and $L(\tilde{\lambda}_A;[t_1, t_2])$ are interval valued n-fold weak commutative ideal of X, for all $[s_1, s_2],[t_1, t_2] \in D[0,1]$

Proof: The proof is straight forward.

Theorem 4.9: (Extension property for interval-valued intuitionistic fuzzy n-fold commutative ideals). Let $\tilde{A} = (\tilde{\mu}_A,\tilde{\lambda}_A)$ and $\tilde{B} = (\tilde{\mu}_B,\tilde{\lambda}_B)$ are interval-valued intuitionistic fuzzy ideals of X such that $\tilde{A}(0) = \tilde{B}(0)$ and $\tilde{A} \subseteq \tilde{B}$, that is, $\tilde{\mu}_A(0) = \tilde{\mu}_B(0), \tilde{\lambda}_A(0) = \tilde{\lambda}_B(0)$ and $\tilde{\mu}_A(x) \leq \tilde{\mu}_B(x)$ $\tilde{\lambda}_A(x) \geq \tilde{\lambda}_B(x)$ for all $x \in X$. If $\tilde{A}$ is an interval-valued intuitionistic fuzzy n-fold commutative ideal of X, then so is $\tilde{B}$.

Proof: Let $x, y \in X$. Taking $a = x*(x*y)$, We have $\tilde{\mu}_B(0) = \tilde{\mu}_A(0) = \tilde{\mu}_A(a*y) \le \tilde{\mu}_A(a*(y*(y*a^n)))$ [By Theorem 4.5(i)]

$$\le \tilde{\mu}_B(a*(y*(y*a^n)))$$
$$\le \tilde{\mu}_B((x*(x*y))*(y*(y*a^n))) \quad \text{[By P3]}$$
$$\le \tilde{\mu}_B((x*(y*(y*a^n)))*(x*y)).$$

Therefore, $\tilde{\mu}_B(0) \le \tilde{\mu}_B((x*(y*(y*a^n)))*(x*y))$, for all $x, y \in X$

Since $x*(y*(y*x^n)) \le x*(y*(y*a^n))$ and since $\tilde{\mu}_B$ is an order reversing, it follows that

$$\tilde{\mu}_B(x*(y*(y*x^n))) \ge \tilde{\mu}_B(x*(y*(y*a^n)))$$
$$\ge \min\{\tilde{\mu}_B((x*(y*(y*a^n)))*(x*y)), \tilde{\mu}_B(x*y)\}$$
$$\ge \min\{\tilde{\mu}_B(0), \tilde{\mu}_B(x*y)\} = \tilde{\mu}_B(x*y).$$

Let $x, y \in X$. Taking $b = x*(x*y)$,

we have $\tilde{\lambda}_B(0) = \tilde{\lambda}_A(0) = \tilde{\lambda}_A(b*y) \ge \tilde{\lambda}_A(b*(y*(y*b^n)))$ [By Theorem 4.5 (i)]

$$\ge \tilde{\lambda}_B(b*(y*(y*b^n)))$$
$$\ge \tilde{\lambda}_B((x*(x*y))*(y*(y*b^n))) \quad \text{[By P3]}$$
$$\ge \tilde{\lambda}_B((x*(y*(y*b^n)))*(x*y))$$

Therefore, $\tilde{\lambda}_B(0) \ge \tilde{\lambda}_B((x*(y*(y*b^n)))*(x*y))$, for all $x, y \in X$

Since $x*(y*(y*x^n)) \le x*(y*(y*b^n))$ and since $\tilde{\lambda}_B$ is an order preserving,
it follows that $\tilde{\lambda}_B(x*(y*(y*x^n))) \le \tilde{\lambda}_B(x*(y*(y*b^n)))$

$$\le \max\{\tilde{\lambda}_B((x*(y*(y*b^n)))*(x*y)), \tilde{\lambda}_B(x*y)\}$$
$$\le \max\{\tilde{\lambda}_B(0), \tilde{\lambda}_B(x*y)\} = \tilde{\lambda}_B(x*y) \text{ and}$$

Therefore, $\tilde{\mu}_B(x*(y*(y*x^n))) \ge \tilde{\mu}_B(x*y)$ and $\tilde{\lambda}_B(x*(y*(y*x^n))) \le \tilde{\lambda}_B(x*y)$ for all $x, y \in X$.

Hence by Theorem 4.5(i), $\tilde{B} = (\tilde{\mu}_B, \tilde{\lambda}_B)$ is an interval-valued intuitionistic fuzzy n-fold commutative ideal of X.

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