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#g-PRECLOSED SETS IN A TOPOLOGICAL SPACES

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ABSTRACT

In this paper, we introduce a new class of #gp- closed sets, and their properties. Applying #g-preclosed sets, we also introduce and study some new classes of spaces, namely, Td# spaces, #Td spaces, Td## spaces, αTd# spaces, αTd## spaces and $\#_s$Td spaces and some interrelationships between these spaces.

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1. INTRODUCTION

N. Levine **[11]** introduced the class of g-closed sets. M. K. R. S. Veerakumar introduced several generalized closed sets namely, g*-closed sets, g#-closed sets, g* p- closed sets and their properties. In this paper we introduce #gp-closed sets and their properties. Applying g#-preclosed sets, we introduce and study some new classes of spaces, namely, Tp# spaces, ${}^{\#}T_d$ *spaces,* $T_d^{\#\#}$ *spaces,* ${}_{\alpha}T_d^{\#}$ *spaces,* ${}_{\alpha}T_d^{\#\#}$ *spaces*, αTp## spaces and #sTp spaces. We obtained some interrelationships between these spaces.

2. PRELIMINARIES

Throughout this paper (X, τ) (or X) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), cl(A), int(A) and Ac denote the closure of A, the interior of A and the complement of A respectively.

We recall the following definitions which are useful in the sequel.

Definition2.1: A subset A of a space (X, τ) is called

1. a preopen set **[16]**if A ⊆ int(cl(A)) and a preclosed set if cl(int(A)) ⊆ A.
2. a semi-open set **[12]** if A ⊆ cl(int(A)) and a semi-closed set if int(cl(A)) ⊆A.
3. an α-open set **[17]** if A ⊆ int(cl(int(A))) and an α-closed set if cl(int(cl(A))) ⊆A.
4. a semi-preopen **[2]** set (=β-open **[1]**) if A⊆cl(int(cl(A))) and a semi-preclosed set **[2]** (=β-closed**[1])** if int(cl(int(A))) ⊆ A.
5. a regular open set **[20]** if A=int(cl(A)) and a regular closed set **[20]** if cl(int(A))=A.

The semi-closure (resp. preclosure, α-closure, semi-preclosure) of a subset A of (X, τ) is the intersection of all semi-closed (resp. preclosed, α-closed, semi-preclosed) sets that contain A and is denoted by scl(A) (resp.pcl(A), αcl(A), splc(A)) .

Definition 2.2: A subset A of a space (X, τ) is called

1. a generalized closed (briefly g-closed) set **[11]** if cl(A)⊆ U whenever A⊆U and U is open in (X, τ).
2. semi-generalized closed **[4]** (briefly sg-closed) set if scl(A)⊆ U whenever A⊆U and U is semi-open in (X, τ). The complement of a sg-closed set is called a sg-open set.
3. a generalized semi-closed (briefly gs-closed) set **[3]** if scl(A)⊆ U whenever A⊆U and U is open in (X, τ).
4. an α-generalized closed (briefly αg-closed) set **[13]** if αcl(A)⊆ U whenever A⊆U and U is open in (X, τ).

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5. a generalized α-closed (briefly gα-closed) set **[14]** if αcl(A)⊆ U whenever A⊆U and U is α-open in (X, τ). The complement of a gα-closed set is called a gα-open **[5]** set.
6. a generalized semi-preclosed (briefly gsp-closed) set **[7]** if spcl(A)⊆ U whenever A⊆U and U is open in (X, τ).
7. a generalized preregular closed (briefly gpr-closed) set **[10]** if pcl(A)⊆ U whenever A⊆U and U is regular open in (X, τ).
8. a g*- pre closed **[18]** (briefly g*p-closed) set if pcl(A)⊆ U whenever A⊆U and U is g-open in (X, τ).
9. g#-closed set **[19]** if cl(A)⊆ U whenever A⊆U and U is αg-open in (X, τ).
10. a generalized pre closed (briefly gp-closed) set **[15]** if pcl(A)⊆ U whenever A⊆U and U is open in (X, τ).

Definition 2.3: A space (X, τ) is called a,
1. *T1/2 space* [**11]** if every g-closed set is closed,
2. *semi-T1/2 space* **[4] if** every sg-closed set is semi-closed,
3. *semi-pre-T1/2 space* [**7]** if every gsp-closed set is semi-preclosed,
4. *preregular T1/2 space* **[10]** if every gpr-closed set is preclosed.l.

BASIC PROPERTIES OF #g-PRECLOSED SETS

We introduce the following definition.

Definition 3.1: A subset A of a space (X, τ) is called a $^{\#}$*g-preclosed* set(briefly $^{\#}$gp-closed set) if pcl(A) ⊆ U whenever A ⊆ U and U is a αg-open set of (X, τ).

Theorem 3.2: Let (X, τ) be a topological space. Then
1. Every preclosed set is $^{\#}$gp -closed, and thus every gα-closed set, and every α-closed set and every closed set is $^{\#}$gp - closed.
2. Every $^{\#}$gp -closed set is gp-closed, and hence gsp-closed and gpr-closed too.
3. Every $g^{\#}$-closed set is $^{\#}$gp-closed.

Proof:
1. Since pcl(A) = A for any preclosed set A of (X, τ), we have that every preclosed set is $^{\#}$gp -closed. Since every closed (α-closed) set is α-closed (gα-closed) and every gα-closed set is preclosed (Theorem 2.4(ii) of [**11**]), then every gα-closed set, every α-closed set and every closed set is also $^{\#}$gp -closed.
2. Follows from the fact that every open (gp-open) set is g-open (gsp-closed and gpr-closed) (see [**13**]).
3. Let A be a $g^{\#}$-closed set and A⊆ U and U is a αg-open,cl(A) ⊆ U and pcl(A) ⊆ U.Hence A is $^{\#}$gp-closed.

The reverse implications in the above theorem are not true as we see the following examples.

Example 3.3: Let X = {a, b, c} and τ = **{**ϕ, X, {a}, {a, c}**}**. The set A = {a, b} is $^{\#}$gp -closed but not even a preclosed set of (X, τ).

Example 3.4: Let X = {a, b, c} and τ = **{**ϕ, X, {a}**}**. The set B = {a, b} is gp-closed and thus gsp-closed and gpr-closed. But B is not a $^{\#}$gp -closed set of (X, τ).

Example 3.5: Let X = {a, b, c} and τ = **{**ϕ, X, {a, c}**}**. The set C = {c} is $^{\#}$gp -closed but it is not a $g^{\#}$- closed set of (X, τ).

Thus the class of $^{\#}$gp -closed sets properly contains the classes of preclosed sets, gα-closed set, α-closed set,$g^{\#}$-closed set and the class of closed sets. Moreover the class of $^{\#}$gp -closed set is properly contained in the classes of gp-closed sets, gsp-closed sets and in the class of gpr-closed sets.

Theorem 3.6:
1. $^{\#}$gp -closed sets are independent of semi-closed sets and semi-preclosed sets.
2. $^{\#}$gp -closed sets are independent of g-closed sets, αg-closed sets, gs-closed sets and sg-closed sets.

Proof: Follows from the following examples.

Example 3.7: Let X, τ and A be as in the example 3.3. A is a $^{\#}$gp -closed set but not even a semi-preclosed set of (X, τ).

Example 3.8: Let X = {a, b, c} and τ = **{**ϕ, X, {a}, {b}, {a, b}**}**. Let C = {a}. C is semi-closed and hence a semi-preclosed set of (X, τ). But C is not a $^{\#}$gp -closed set of (X, τ). Moreover C is a sg-closed set.

Example 3.9: Let X = {a, b, c} and τ = **{**ϕ, X, {a, b}**}**. Let D = {a}. D is #gp -closed. D is neither sg-closed nor even a gs-closed set of (X, τ).

Example 3.10: Let X, τ and B be as in the example 3.4. B is g-closed and hence αg-closed and gs-closed set in (X, τ). But B is not a #gp -closed set of (X, τ).

Remark 3.11: Union of two #gp -closed sets need not be #gp -closed set again.

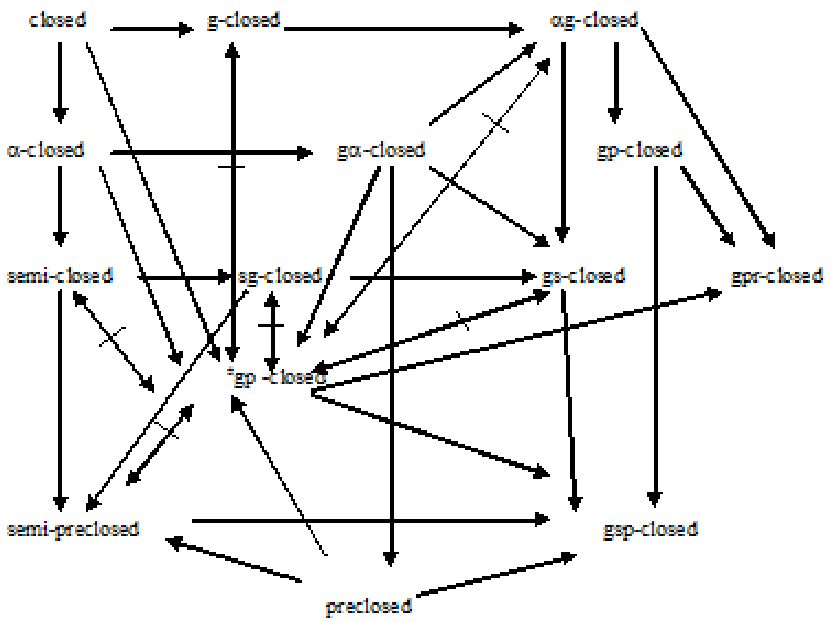
Proof: Let (X, τ) be as in the example 3.9. A = {a} and B = {b} are #gp -closed sets but A $\cup$ B = {a, b}, their union is not a #gp -closed set of (X, τ).

Theorem 3.12: If A is αg-open and #gp -closed set of (X, τ), then A is a preclosed set of (X, τ).

Theorem 3.13: Let A be a #gp -closed set of (X, τ). Then
1. pcl(A)-A does not contain any non-empty αg-closed set.
2. If A $\subseteq$ B $\subseteq$ pcl(A), then B is also a #gp -closed set of (X, τ).

Proof:
1. Let F be a g-closed set contained in pcl(A)-A. pcl(A) $\subseteq$ X-F since X-F is g-open set with A $\subseteq$ X-F and A is a #gp -closed. Then F $\subseteq$ (X-pc(A)) $\cap$ (pc(A)-A) $\subseteq$ (X-pcl(A)) $\cap$ pcl(A) = ϕ. Therefore F = ϕ.
2. Let U be a g-open set of (X, τ) such that B $\subseteq$ U. Then A $\subseteq$ U. Since A $\subseteq$ U and A is #gp -closed, pcl(A) $\subseteq$ U. Then pcl(B) $\subseteq$ pcl(pcl(A)) = pcl(A) since B $\subseteq$ pcl(A). Thus pcl(B) $\subseteq$ pcl(A) $\subseteq$ U. Hence B is also a #gp -closed set of (X, τ).

Theorem 3.14: The following diagram shows the relationships between #gp -closed sets and some other sets.



where A $\longrightarrow$ B (A $\longleftrightarrow$ B) represents A implies B and B need not imply A (A and B are independent of each other).

4. APPLICATIONS OF #g-PRECLOSED SETS

We now introduce and study some new spaces, namely $T_d^{\#}$ spaces, ${}^{\#}T_d$ spaces, $T_d^{\#\#}$ spaces, ${}_{\alpha}T_d^{\#}$ spaces, ${}_{\alpha}T_d^{\#\#}$ spaces and ${}^{\#}{}_{s}T_d$ spaces.

Definition 4.1: A topological space (X, τ) is said to be
1. a $T_d^{\#}$ space if every #gp -closed set in it is closed.
2. a ${}^{\#}T_d$ space if every gp-closed set in it is #gp -closed.
3. a $T_d^{\#\#}$ space if every #gp -closed set in it is gα-closed.
4. a ${}_{\alpha}T_d^{\#}$ space if every #gp -closed set in it is preclosed.
5. a ${}_{\alpha}T_d^{\#\#}$ space if every #gp -closed set in it is α-closed.
6. a ${}^{\#}{}_{s}T_d$ space if every gsp-closed set in it is #gp -closed.

Theorem 4.2:

1. Every ${}_{\alpha}T_d^{\#\#}$ space is ${}_{\alpha}T_d^{\#}$ space and $T_d^{\#\#}$ space.
2. Every $T_d^{\#}$ space is ${}_{\alpha}T_d^{\#\#}$ space and hence $T_d^{\#\#}$ space ${}_{\alpha}T_d^{\#}$ space.
3. Every $T_d^{\#\#}$ space is ${}_{\alpha}T_d^{\#}$ space.

Proof:

1. Follows from the fact that every α-closed set is gα-closed and hence a preclosed set.
2. Follows from (1) and the fact that every closed set is an α-closed set.
3. Follows from the fact that every gα-closed set is preclosed.

Following examples show that the reverse implications in the above theorem need not be true in general.

Example 4.3: Let X = {a, b, c} and τ = **{**ϕ, X, {a}, {b, c}**}**. {b} is a #gp-closed set but not even an α-closed set. So (X, τ) is neither $T_d^{\#}$ nor ${}_{\alpha}T_d^{\#\#}$. However (X, τ) is a $T_d^{\#\#}$ space.

Example 4.4: Let X = {a, b, c} and τ = **{**ϕ, X, {a}**}**. (X, τ) is not a $T_d^{\#}$ space since {b} is a #gp -closed set but not a closed set. However (X, τ) is an ${}_{\alpha}T_d^{\#}$ space.

Example 4.5: Let X = {a, b, c} and τ = **{**ϕ, X, {a, b}**}**. (X, τ) is not even a $T_d^{\#\#}$ space since {a} is a #gp -closed set but not a gα-closed set. However (X, τ) is an ${}_{\alpha}T_d^{\#}$ space.

Thus $T_d^{\#}$ spaces is properly contained in the classes of ${}_{\alpha}T_d^{\#\#}$ spaces, $T_d^{\#\#}$ spaces and in the class of ${}_{\alpha}T_d^{\#}$ spaces. The class of ${}_{\alpha}T_d^{\#\#}$ spaces is properly contained in the classes of ${}_{\alpha}T_d^{\#}$ spaces and $T_d^{\#\#}$ spaces. Moreover the class of ${}_{\alpha}T_d^{\#}$ spaces properly contains the class of $T_d^{\#\#}$ spaces.

Theorem 4.6:

1. Every ${}^{\#}{}_{s}T_d$ space is a ${}^{\#}T_d$ space.
2. Every semi-pre-$T_{1/2}$ space is an ${}_{\alpha}T_d^{\#}$ space and a ${}^{\#}T_d$ space.

Proof:

1. Follows from the fact that every gp-closed set is gsp-closed.
2. Follows from the fact that every #gp -closed (resp. preclosed) set is gp-closed (resp. #gp -closed) set and the Lemma 2.4 of [**3**].

Following examples show that the reverse implications in the above theorem need not be true in general.

Example 4.7: Let X = {a, b, c} and τ = **{**ϕ, X, {a}, {b}, {a, b}**}**. {a, b} is a gsp-closed set but not a #gp -closed set of (X, τ). Thus (X, τ) is not a ${}^{\#}{}_{s}T_d$ space. However (X, τ) is a ${}^{\#}T_d$ space.

Example 4.8: Let (X, τ) be as in the example 4.4. (X, τ) is not a semi-pre-$T_{1/2}$ space since {a, b} is a gsp-closed set but not a semi-preclosed set. However (X, τ) is an ${}_{\alpha}T_d^{\#}$ space.

Example 4.9: Let (X, τ) be as in the example 3.3. (X, τ) is not a semi-pre-$T_{1/2}$ space since {a, b} is a gsp-closed set but not a semi-preclosed set. However (X, τ) is a ${}^{\#}T_d$ space.

Thus the class of ${}^{\#}T_d$ spaces properly contains the class of ${}^{\#}{}_{s}T_d$ spaces. Moreover the class of semi-pre-$T_{1/2}$ spaces is properly contained in the class of ${}_{\alpha}T_d^{\#}$ spaces and in the class of ${}^{\#}T_d$ spaces.

Dontchev [**10**] proved that the dual of the class of semi-$T_{1/2}$ spaces to the class of $T_{1/2}$ spaces is the class of semi-pre-$T_{1/2}$ spaces.

Theorem 4.10: A space (X, τ) is semi-pre-$T_{1/2}$ if and only if it is ${}_{\alpha}T_d^{\#}$ and ${}^{\#}T_d$.

Proof: Follows from the theorem 2.4 of [**4**].

Thus the class of ${}^{\#}T_d$ spaces is the dual of the class of semi-pre-$T_{1/2}$ spaces to the class of ${}_{\alpha}T_d^{\#}$ spaces.

The following theorem gives a tridecomposition for $T_{1/2}$ spaces.

Theorem 4.11: A space (X, τ) is $T_{1/2}$ if and only if it is semi-$T_{1/2}$, ${}_{\alpha}T_d^{\#}$ and ${}^{\#}T_d$.

Proof: Follows from the above theorem 4.10 and the theorem 4.5 of Dontchev [**10**].

Theorem 4.12:
1. ${}_{\alpha}T_d^{\#}$ spaces are independent of ${}^{\#}T_d$ spaces and ${}^{\#}{}_{s}T_d$ spaces.
2. ${}^{\#}T_d$ spaces are independent of $T_d^{\#\#}$ spaces and ${}_{\alpha}T_d^{\#\#}$ spaces.
3. ${}^{\#}{}_{s}T_d$ spaces are independent of $T_d^{\#}$ spaces, $T_d^{\#\#}$ spaces, ${}_{\alpha}T_d^{\#}$ spaces, ${}_{\alpha}T_d^{\#\#}$ spaces and semi-pre-$T_{1/2}$ spaces.
4. ${}_{\alpha}T_d^{\#\#}$ spaces are independent of semi-pre-$T_{1/2}$ spaces and preregular $T_{1/2}$ spaces.
5. $T_d^{\#}$ spaces are independent of preregular $T_{1/2}$ spaces.

Proof: The space (X, τ) in the example 3.8 is a $T_d^{\#}$ space but it is not ${}^{\#}{}_{s}T_d$. Moreover (X, τ) is not preregular $T_{1/2}$. The space (X, τ) in the example 4.3 is ${}^{\#}{}_{s}T_d$, preregular $T_{1/2}$, semi-pre-$T_{1/2}$, ${}^{\#}T_d$ but not even an ${}_{\alpha}T_d^{\#\#}$ space. The space (X, τ) in the example 4.4 is an ${}_{\alpha}T_d^{\#\#}$ space and thus ${}_{\alpha}T_d^{\#}$ and also $T_d^{\#\#}$ but it is not even a ${}^{\#}T_d$ space . The space (X, τ) in the example 4.5 is a ${}^{\#}{}_{s}T_d$ space and thus a ${}^{\#}T_d$ space but it is not even a $T_d^{\#\#}$ space. The space in the example 4.9 is ${}^{\#}T_d$ but not ${}_{\alpha}T_d^{\#}$.

Theorem 4.13:
1. Every preregular $T_{1/2}$ space is a ${}^{\#}T_d$ space but not conversely.
2. Every preregular $T_{1/2}$ space is an ${}_{\alpha}T_d^{\#}$ space but not conversely.

Proof:
1. The first assertion follows from the fact that every preclosed set is a $^{\#}$gp -closed set. The space (X, τ) in the example 3.3 is a ${}^{\#}T_d$ space. But (X, τ) is not a preregular $T_{1/2}$ space.
2. The first assertion follows from the theorem 4.06(2) and the corollary 5.8 of [**13**].The space (X, τ) in the example 3.8 supports the second assertion.

Thus the class of preregular $T_{1/2}$ spaces is properly contained in the class of ${}_{\alpha}T_d^{\#}$ spaces and in the class of ${}^{\#}T_d$ spaces.

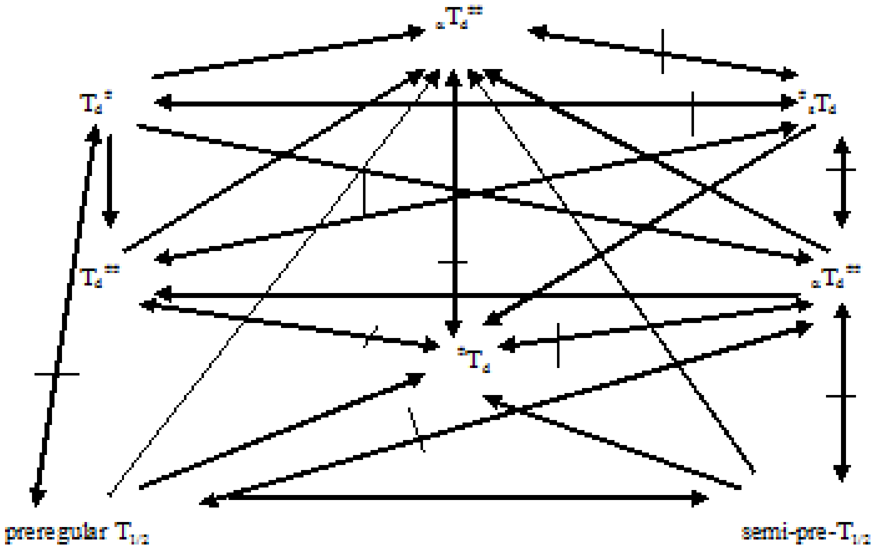
Definition 4.14: A subset A of a space (X, τ) is called a $^{\#}$gp *-open* if C(A) is $^{\#}$gp -closed.

Theorem 4.15: The following statements are true but the respective converses are not true in general.
1. If (X, τ) is a $T_d^{\#}$ space, then every singleton of X is either αg-closed or open.
2. If (X, τ) is a ${}^{\#}T_d$ space, then every singleton of X is either closed or $^{\#}$gp -open.
3. If (X, τ) is a $T_d^{\#\#}$ space, then every singleton of X is either αg -closed or gα-open.
4. If (X, τ) is an ${}_{\alpha}T_d^{\#}$ space, then every singleton of X is either αg -closed or pre-open.
5. If (X, τ) is an ${}_{\alpha}T_d^{\#\#}$ space, then every singleton of X is either αg -closed or α-open.
6. If (X, τ) is a ${}^{\#}{}_{s}T_d$ space, then every singleton of X is either closed or $^{\#}$gp -open.

Proof: (1)**:** Let x ∈ X and suppose that {x} is not a g-closed set of (X, τ). This implies X-{x} is not a g-open set. So X is the only g-open set such that X-{x} ⊆ X. Then X-{x} is a $^{\#}$gp -closed set of (X, τ). Since (X, τ) is a $T_d^{\#}$ space, then X-{x} is closed or equivalently {x} is open. The space in the example 4.03 shows that the converse need not be true.

The proofs for the first assertions of (2) to (6) are similar to as that of the first assertion of (1). The space (X, τ) in the example 4.4 shows that converse of (2) need not be true. The space (X, τ) in the example 4.3 shows that the converse of (4) need not be true. The space (X, τ) in the example 4.0 shows that the converses of (3), (5) and (6) need not be true.

Theorem 4.16: The following diagram shows some relationships among the spaces considered in this paper.



where A $\longrightarrow$ B (resp. A $\longleftrightarrow$ B) represents A implies B but B need not imply A (resp. A and B are independent of each other).

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