CR- SUBMANIFOLD OF NEARLY HYPERBOLIC COSYMPLECTIC MANIFOLD WITH A QUARTER SYMMETRIC NON METRIC CONNECTION

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ABSTRACT

In present paper, we study some properties of CR-submanifold of a nearly hyperbolic cosymplectic manifold with a quarter symmetric non metric connection, obtain some result on ξ-horizontal and ξ-vertical CR-submanifold of a nearly hyperbolic cosymplectic manifold with a quarter symmetric non metric connection. We also find the integrability conditions of some distributions and study parallel distributions (horizontal & vertical distributions) on CR-submanifold of a nearly hyperbolic cosymplectic manifold with a quarter symmetric non metric connection.

Keywords and Phrases: CR-submanifold, nearly hyperbolic cosymplectic manifold with a quarter symmetric non metric connection, parallel distribution, and integrability condition.

INTRODUCTION

The notion of CR-submanifolds of a Kaehler manifold was introduced and studied by A. Bejancu in ([1], [2]). Since then, several paper on Kaehler manifold were published. CR-submanifolds of Sasakian manifold was studied by C.J.Hsu in [3] and M.Kobayashi in [4]. Later, several geometers (see, [5], [6], [7], [8], [9], [10]) enrich the study of CR-submanifolds of almost contact manifolds. On the other hand, almost hyperbolic (f, g, η, ξ)-structure was defined and studied by M.D.Upadhyay and K.K.Dube in [11]. L.Bhatt and K.K.Dube studied CR-submanifolds of a trans-hyperbolic Sasakian manifold in [12]. Ahmad M. and Ali K. study CR-submanifold of a nearly hyperbolic cosymplectic manifold [13]. In this paper, we study some properties of CR-Submanifold of nearly hyperbolic cosymplectic manifold with a quarter symmetric non metric connection.

The paper is organized as follows. In section 2, we give a brief description of nearly hyperbolic cosymplectic manifold with a quarter symmetric non metric connection. In section 3, some properties of CR-Submanifold of nearly hyperbolic cosymplectic manifold with a quarter symmetric non metric connection are investigated. In section 4, some result on parallel distribution on ξ-horizontal and ξ-vertical CR-Submanifold of nearly hyperbolic cosymplectic manifold with a quarter symmetric non metric connections are obtained.

2. PRELIMINARIES

Let M be an n-dimensional almost hyperbolic contact metric manifold with the almost hyperbolic contact metric structure (θ, ξ, η, g), where a tensor θ of type (1,1) a vector field ξ, called structure vector field and η, the dual 1-form of ξ satisfying the following

\[ \theta^2 X = X + \eta(X) \xi, \quad g(X, \xi) = \eta(X), \]
\[ \eta(\xi) = -1, \quad \theta(\xi) = 0, \quad \eta \theta = 0, \]
\[ g(\theta X, \theta Y) = -g(X, Y) - \eta(X) \eta(Y) \]

For any X, Y tangent to M [9, 6]. In this case
\[ g(\theta X, Y) = -g(X, \theta Y). \]

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An almost hyperbolic contact metric structure $(\mathcal{F}, \xi, \eta, g)$ on $\overline{M}$ is called hyperbolic cosymplectic manifold [12] if and only if
\[
(\nabla_X \mathcal{F})Y + (\nabla_Y \mathcal{F})X = 0 \quad \text{for all } X, Y \text{ tangent to } \overline{M}.
\]

A hyperbolic cosymplectic manifold $\overline{M}$ is called nearly hyperbolic cosymplectic manifold, if
\[
\nabla_X \xi = 0 \quad \text{for a Riemannian Connection } \nabla.
\]

Now, Let $M$ be a submanifold immersed in $\overline{M}$. The Riemannian metric induced on $M$ is denoted by the same symbol $g$. Let $TM$ and $T^\perp M$ be the Lie algebra of vector fields tangential to $M$ and normal to $M$ respectively and $\nabla$ be the induced Levi-Civita connection on $N$, then the Gauss and Weingarten formulas are given respectively by
\[
\nabla_X Y = \nabla_X Y + h(X, Y).
\]

Now we define a quarter symmetric non-metric connection
\[
\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)\mathcal{F}X
\]

Putting $Y = N$
\[
\overline{\nabla}_X N = \nabla_X N + \eta(N)\mathcal{F}X
\]
\[
\overline{\nabla}_Y N = \nabla_Y N
\]

Adding above two equations
\[
(\overline{\nabla}_X \mathcal{F})Y + \mathcal{F}((\overline{\nabla}_Y \mathcal{F})X + \mathcal{F}(\nabla_X \mathcal{F}Y - \nabla_Y \mathcal{F}X)) + \mathcal{F}(\nabla_Y \mathcal{F}X - \nabla_X \mathcal{F}Y) = (\overline{\nabla}_X \mathcal{F})Y + (\overline{\nabla}_Y \mathcal{F})X
\]

From (2.5) and quarter symmetric non-metric connection
\[
(\overline{\nabla}_X \mathcal{F})Y + (\overline{\nabla}_Y \mathcal{F})X + \mathcal{F}(\nabla_X \mathcal{F}Y - \nabla_Y \mathcal{F}X) + \mathcal{F}(\nabla_Y \mathcal{F}X - \nabla_X \mathcal{F}Y) = 0
\]
\[
(\overline{\nabla}_X \mathcal{F})Y + (\overline{\nabla}_Y \mathcal{F})X = -\eta(Y)\mathcal{F}^2 X - \eta(X)\mathcal{F}^2 Y
\]
\[
(\overline{\nabla}_X \mathcal{F})Y - (\overline{\nabla}_Y \mathcal{F})X = -\eta(Y)(X + \eta(X)\xi) - \eta(X)(Y + \eta(Y)\xi)
\]

Quarter symmetric non-metric connection
\[
\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)\mathcal{F}X
\]

Putting $Y = \xi$
\[
\overline{\nabla}_X \xi = \nabla_X \xi + \eta(\xi)\mathcal{F}X
\]

Putting $Y = \mathcal{F}$
\[
\overline{\nabla}_X \mathcal{F} = \nabla_X \mathcal{F} + \eta(\mathcal{F})\mathcal{F}X
\]
**Definition 1:** An m-dimensional submanifold $M$ of $\bar{M}$ is called a CR-Submanifold of almost nearly hyperbolic contact manifold $\bar{M}$, if there exists a differentiable distribution $D:x \to D_x$ on $M$ satisfying the following conditions:

i. $D$ is invariant, that is $\mathcal{D}D_x \subset D_x$ for each $x \in M$.

ii. The complementary orthogonal distribution $D^\perp$ of $D$ is anti-invariant, that is $\mathfrak{D}D_x \subset D_x$. If $\dim D_x = 0$ (resp., $\dim D_x = 0$), then the CR-Submanifold is called an invariant (resp., anti-invariant) submanifold. The distribution $D$ (resp., $D^\perp$) is called the horizontal (resp., vertical) distribution. Also, the pair $(D, D^\perp)$ is called $\xi - \text{horizontal}$ (resp., $\xi$-vertical) if $\xi_x \subset D_x$ (resp., $\xi_x \subset D^\perp_x$).

### 3. SOME BASIC LEMMAS

**Lemma 3.1:** Let $M$ be a CR- submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$ then

\begin{align*}
&\text{(3.1)} \quad -\eta(Y)PX - \eta(X)PY - 2\eta(X)\eta(Y)p\xi + \mathcal{D}(\nabla X Y) + \mathfrak{D}(\nabla Y X) = PA_{\varphi QY}X - PA_{\varphi QY}Y \\
&\text{(3.2)} \quad -\eta(Y)QX - \eta(X)QY - 2\eta(X)\eta(Y)\eta \xi + 2Bh(X, Y) = Q\nabla X (\mathfrak{D} Y) + Q\nabla Y (\mathfrak{D} X) - QA_{\varphi QY}X - QA_{\varphi QY}Y \\
&\text{(3.3)} \quad \mathcal{D}Q(\nabla X Y) + \mathfrak{D}(\nabla Y X) + 2Ch(X, Y) = h(X, \mathfrak{D} Y) + h(Y, \mathfrak{D} X) + \mathcal{D}Q\mathfrak{D} QY + \mathcal{D}Q\mathfrak{D} QX \text{ for any } X, Y \in TM.
\end{align*}

**Proof:** Using (2.4), (2.5), (2.6), we get

\begin{align*}
Y &= PY + QY, \\
\mathfrak{D}Y &= \mathfrak{D}PY + \mathfrak{D}QY.
\end{align*}

Differentiating covariantly

Left side: \(\nabla X \mathfrak{D}Y = (\nabla X \mathfrak{D})Y + \mathfrak{D}(\nabla X Y)\)

\begin{align*}
&\nabla X \mathfrak{D}Y = \nabla X (\mathfrak{D}Y) + \mathfrak{D}(\nabla X Y) \\
&= (\nabla X \mathfrak{D})Y + \mathfrak{D}(\nabla X Y) + \mathfrak{D}h(X, Y)
\end{align*}

Right side:

\begin{align*}
\nabla X (\mathfrak{D} Y) + \mathfrak{D}(\nabla X Y) &= \nabla X (\mathfrak{D} Y) + \mathfrak{D}(\nabla X Y) + \mathfrak{D}h(X, Y) \\
&= \nabla X (\mathfrak{D} Y) + \mathfrak{D}(\nabla X Y) + \mathfrak{D}h(X, Y)
\end{align*}

From Left and Right side

\((\nabla X \mathfrak{D})Y + \mathfrak{D}(\nabla X Y) + \mathfrak{D}h(X, Y) = \nabla X (\mathfrak{D} Y) + \mathfrak{D}(\nabla X Y) + \mathfrak{D}h(X, Y) - A_{\varphi QY}X + \nabla^\perp_{\xi} \mathfrak{D}QY.
\)

Interchanging $X \& Y$,

\((\nabla Y \mathfrak{D})X + \mathfrak{D}(\nabla Y X) + \mathfrak{D}h(Y, X) = \nabla Y (\mathfrak{D} X) + \mathfrak{D}(\nabla Y X) + \mathfrak{D}h(Y, X) - A_{\varphi QY}Y + \nabla^\perp_{\xi} \mathfrak{D}QX.
\)

Adding above two equations

\begin{align*}
(\nabla X \mathfrak{D})Y + (\nabla Y \mathfrak{D})X + \mathfrak{D}(\nabla X Y) + \mathfrak{D}(\nabla Y X) + 2\mathfrak{D}h(X, Y) &= \nabla X (\mathfrak{D} Y) + \nabla Y (\mathfrak{D} X) + \mathfrak{D}h(X, Y) + \mathfrak{D}h(Y, X) - A_{\varphi QY}X - A_{\varphi QY}Y + \nabla^\perp_{\xi} \mathfrak{D}QX + \mathfrak{D}QX.
\end{align*}

Using (2.12), we have

\begin{align*}
-\eta(Y)X - \eta(X)Y &= 2\eta(X)\eta(Y)\xi + \mathfrak{D}(\nabla X Y) + \mathfrak{D}(\nabla Y X) + 2\mathfrak{D}h(X, Y) \\
&= \nabla X (\mathfrak{D} Y) + \nabla Y (\mathfrak{D} X) + \mathfrak{D}h(X, Y) - A_{\varphi QY}X - A_{\varphi QY}Y + \nabla^\perp_{\xi} \mathfrak{D}QX + \mathfrak{D}QX
\end{align*}

Comparing horizontal, vertical and normal components, we get

**Tangential Component:**

\begin{align*}
-\eta(Y)PX - \eta(X)PY - 2\eta(X)\eta(Y)\xi + \mathfrak{D}(\nabla X Y) + \mathfrak{D}(\nabla Y X) &= PA_{\varphi QY}X + PA_{\varphi QY}Y 
\end{align*}

**Vertical Component:**

\begin{align*}
-\eta(Y)QX - \eta(X)QY - 2\eta(X)\eta(Y)\xi + 2Bh(X, Y) &= Q\nabla X (\mathfrak{D} Y) + Q\nabla Y (\mathfrak{D} X) - QA_{\varphi QY}X - QA_{\varphi QY}Y
\end{align*}

**Normal Component:**

\begin{align*}
\mathcal{D}Q(\nabla X Y) + \mathfrak{D}(\nabla Y X) + 2Ch(X, Y) &= h(X, \mathfrak{D} Y) + h(Y, \mathfrak{D} X) + \mathcal{D}Q\mathfrak{D} QY + \mathcal{D}Q\mathfrak{D} QX
\end{align*}

Hence the Lemma is proved. ☐
Lemma 3.2: Let $M$ be a CR-submanifold of a nearly hyperbolic cosymplectic manifold $\tilde{M}$ then

\[ 2(\overline{\nabla}_Y \phi)Y = -\eta(Y)X - \eta(X)Y - 2\eta(X)\eta(Y)\xi + \nabla_X \phi Y - \nabla_Y \phi X + h(Y, \phi X) - h(Y, \phi X) - \phi[X, Y] \]

\[ 2(\overline{\nabla}_Y \phi)X = -\eta(Y)X - \eta(X)Y - 2\eta(X)\eta(Y)\xi - \nabla_X \phi Y + \nabla_Y \phi X + h(Y, \phi X) - h(X, \phi Y) + \phi[X, Y] \]

for any $X, Y \in D$.

**Proof:** From Gauss formula (2.7), we have

\[ \overline{\nabla}_Y Y = \nabla_Y Y + h(X, Y). \]
\[ \overline{\nabla}_Y \phi Y = \nabla_Y \phi Y + h(X, \phi Y). \]
\[ \overline{\nabla}_Y \phi X = \nabla_Y \phi X + h(\phi X, Y). \]

(3.6) \[ \overline{\nabla}_Y \phi Y - \nabla_Y \phi X = \nabla_Y \phi Y + h(X, \phi Y) - \nabla_Y \phi X - h(Y, \phi X) - \phi[X, Y]. \]

Also, we have

\[ \overline{\nabla}_Y \phi Y = (\nabla_Y \phi)Y + \phi \overline{\nabla}_Y Y. \]
\[ \overline{\nabla}_Y \phi X = (\nabla_Y \phi)X + \phi \overline{\nabla}_Y X. \]

Subtracting above,

\[ \overline{\nabla}_Y \phi Y - \nabla_Y \phi X = (\nabla_Y \phi)Y - (\nabla_Y \phi)X + \phi( \overline{\nabla}_Y Y - \nabla_Y X) \]

(3.7) \[ \overline{\nabla}_Y \phi Y - \nabla_Y \phi X = (\nabla_Y \phi)Y - (\nabla_Y \phi)X + \phi[X, Y]. \]

From (3.6) and (3.7), we get

\[ (\overline{\nabla}_Y \phi)Y - (\nabla_Y \phi)X + \phi[X, Y] = \nabla_Y \phi Y + h(X, \phi Y) - \nabla_Y \phi X - h(Y, \phi X). \]

(3.8) \[ (\overline{\nabla}_Y \phi)Y - (\nabla_Y \phi)X = \nabla_Y \phi Y + h(X, \phi Y) - \nabla_Y \phi X - h(Y, \phi X) - \phi[X, Y]. \]

Adding (3.8) and (2.12), we obtain

\[ 2(\overline{\nabla}_Y \phi)Y = -\eta(Y)X - \eta(X)Y - 2\eta(X)\eta(Y)\xi + \nabla_X \phi Y + h(Y, \phi Y) - \nabla_Y \phi X - h(Y, \phi X) - \phi[X, Y] \]

Subtracting (3.8) from (2.12), we obtain

\[ 2(\overline{\nabla}_Y \phi)X = -\eta(Y)X - \eta(X)Y - 2\eta(X)\eta(Y)\xi - \nabla_X \phi Y - h(X, \phi Y) + \nabla_Y \phi X + h(Y, \phi X) + \phi[X, Y] \]

Hence the Lemma is proved. □

**Corollary 3.3:** If $M$ be a $\xi$-vertical CR-submanifold of a CR-submanifold of a nearly hyperbolic cosymplectic manifold $\tilde{M}$ with quarter symmetric metric connection. Then

\[ 2(\overline{\nabla}_Y \phi)Y = \nabla_Y \phi Y + h(X, \phi Y) - \nabla_Y \phi X - h(Y, \phi X) - \phi[X, Y] \]

and

\[ 2(\overline{\nabla}_Y \phi)X = \nabla_Y \phi X - \nabla_X \phi Y - h(X, \phi Y) + h(Y, \phi X) + \phi[X, Y] \] for any $X, Y \in D$.

**Lemma 3.4:** Let $M$ be a CR-submanifold of a nearly hyperbolic cosymplectic manifold $\tilde{M}$ then

\[ 2(\overline{\nabla}_Y \phi)Y = -\eta(Y)X - \eta(X)Y - 2\eta(X)\eta(Y)\xi + A_{\phi Y} Y - A_{\phi Y} X + V_{\phi Y} Y - V_{\phi Y} X - \phi[X, Y] \]

and

\[ 2(\overline{\nabla}_Y \phi)X = -\eta(Y)X - \eta(X)Y - 2\eta(X)\eta(Y)\xi - A_{\phi Y} Y + A_{\phi Y} X - V_{\phi Y} Y + V_{\phi Y} X + \phi[X, Y] \] for any $X, Y \in D^\perp$.

**Proof:** From Weingarten formula (2.8), we have

\[ \overline{\nabla}_Y N = -A_{\phi Y} X + V_{\phi Y} X \]

Putting

\[ N = \phi Y \]
\[ \overline{\nabla}_Y \phi Y = -A_{\phi Y} Y + V_{\phi Y} Y. \]
\[ \overline{\nabla}_Y \phi X = -A_{\phi Y} X + V_{\phi Y} X. \]

(3.10) \[ \overline{\nabla}_Y \phi Y - \nabla_Y \phi X = A_{\phi Y} Y - A_{\phi Y} X + V_{\phi Y} Y - V_{\phi Y} X. \]

Also,

(3.11) \[ \overline{\nabla}_Y \phi Y - \nabla_Y \phi X = (\overline{\nabla}_Y \phi)Y - (\overline{\nabla}_Y \phi)X + \phi[X, Y]. \]

From (3.10) and (3.11), we get

(3.12) \[ (\overline{\nabla}_Y \phi)Y - (\overline{\nabla}_Y \phi)X = A_{\phi Y} Y - A_{\phi Y} X + V_{\phi Y} Y - V_{\phi Y} X - \phi[X, Y]. \]

(2.12) \[ (\overline{\nabla}_Y \phi)Y + (\overline{\nabla}_Y \phi)X = -\eta(Y)X - \eta(X)Y - 2\eta(X)\eta(Y)\xi \]
Corollary 3.5: If M be a horizontal CR-submanifold of of a CR-submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$ with quarter symmetric metric connection. Then

$$2(\overline{\nabla}_X\phi) = -A_{\eta X}X - A_{\eta Y}Y + \nabla^Y_\phi Y - \nabla^Y_\phi X - \phi[X,Y].$$

and

$$2(\overline{\nabla}_Y\phi) = A_{\eta Y}X - A_{\eta X}Y + \nabla^Y_\phi X - \nabla^Y_\phi Y + \phi[X,Y].$$

for any $X, Y \in D^1$.

Lemma 3.6: Let $M$ be a CR-submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$ then

$$2(\overline{\nabla}_X\phi) = -\eta(Y)X - \eta(X)Y - 2\eta(X)\eta(Y)\xi - A_{\eta Y}X + \nabla^Y_\phi Y - \eta(X) - \phi[X,Y].$$

Also, we have

$$2(\overline{\nabla}_Y\phi) = -\eta(Y)X - \eta(X)Y - 2\eta(X)\eta(Y)\xi + A_{\eta Y}X - \nabla^Y_\phi Y + \eta(Y) - \phi[X,Y].$$

By virtue of (3.14) and (3.15), we get

$$(\overline{\nabla}_X\phi)(Y) = A_{\eta Y}X + \nabla^Y_\phi Y - \eta(Y) - \phi[X,Y].$$

(2.12) $$(\overline{\nabla}_Y\phi)(Y) = -\eta(Y)X - \eta(X)Y - 2\eta(X)\eta(Y)\xi$$

Adding (3.16) and (2.12), we obtain

$$2(\overline{\nabla}_X\phi) = -\eta(Y)X - \eta(X)Y - 2\eta(X)\eta(Y)\xi - A_{\eta Y}X + \nabla^Y_\phi Y - \eta(Y) - \phi[X,Y].$$

Subtracting (3.16) from (2.12), we obtain

$$2(\overline{\nabla}_Y\phi) = -\eta(Y)X - \eta(X)Y - 2\eta(X)\eta(Y)\xi + A_{\eta Y}X - \nabla^Y_\phi Y + \eta(Y) - \phi[X,Y].$$

Hence the Lemma is proved. □

4. PARALLEL DISTRIBUTION

Definition 2: The horizontal (resp., vertical) distribution $D(\operatorname{resp.}, D^1)$ is said to be parallel [13] with respect to the connection on $M$ if $\nabla_XY \in D$ (resp., $\nabla_XW \in D^1$) for any vector field $X, Y \in D$ (resp., $W, Z \in D^1$).

Theorem 4.1: Let $M$ be a $\xi$-vertical CR-submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$. If the horizontal distribution $D$ is parallel, Then

$$h(X, \phi Y) = h(Y, \phi X).$$

for any $X, Y \in D$

Proof: Using parallelism of horizontal distribution $D$, we have

$$(\overline{\nabla}_X(\phi Y)) = D^1 \text{ and } \overline{\nabla}_X\phi X \in D$$

for any $X, Y \in D$.

From Vertical component,

$$-\eta(Y)QX - \eta(X)QY - 2\eta(X)\eta(Y)\xi + 2Bh(X, Y) = Q\nabla_X(\phi PY) + Q\nabla_Y(\phi PX) - QA_{\phi Y}X - QA_{\phi X}Y$$

As $Q$ is a projection operator on $D^1$, We have

$$Bh(X, Y) = 0.$$
Putting \( N = h(X, Y) \)
\( \emptyset h(X, Y) = B h(X, Y) + Ch(X, Y) \)

From (4.3)
\[
2 \emptyset h(X, Y) = 2C h(X, Y)
\]

From normal component,
\[
\emptyset Q(\nabla_X Y) + \emptyset Q(\nabla_Y X) + 2Ch(X, Y) = h(X, \emptyset P Y) + h(Y, \emptyset P X) + \nabla^\perp \emptyset Q X.
\]
\[
2 Ch(X, Y) = h(X, \emptyset P Y) + h(Y, \emptyset P X)
\]

(4.5) \( 2Ch(X, Y) = h(X, \emptyset Y) + h(Y, \emptyset X) \), \( \text{for any } X, Y \in D \).

Applying (4.5) in (4.4)
\[
2 \emptyset h(X, Y) = h(X, \emptyset Y) + h(Y, \emptyset X)
\]

Replacing \( X \) by \( \emptyset X \)
\[
2 \emptyset h(\emptyset X, Y) = h(\emptyset X, \emptyset Y) + h(Y, \emptyset^2 X)
\]
\[
2 \emptyset h(\emptyset X, Y) = h(\emptyset X, \emptyset Y) + h(Y, X + \eta(X) \xi)
\]
\[
2 \emptyset h(\emptyset X, Y) = h(\emptyset X, \emptyset Y) + h(Y, X) + h(Y, \eta(X) \xi)
\]

(4.7) \( 2 \emptyset h(\emptyset X, Y) = h(\emptyset X, \emptyset Y) + h(Y, X) \)

Now, replacing \( Y \rightarrow \emptyset Y \) in (4.6), we get
\[
h(X, \emptyset^2 Y) + h(\emptyset Y, \emptyset X) = 2 \emptyset h(X, \emptyset Y).
\]
\[
h(X, Y + \eta(Y) \xi) + h(\emptyset Y, \emptyset X) = 2 \emptyset h(X, \emptyset Y).
\]

(4.8) \( h(X, Y) + h(\emptyset Y, \emptyset X) = 2 \emptyset h(X, \emptyset Y) \).

Thus from (4.7) and (4.8), we find
\[
2 \emptyset h(\emptyset X, Y) = 2 \emptyset h(X, \emptyset Y).
\]

Operating \( \emptyset \) on both sides, we get
\[
h(X, \emptyset Y) = h(Y, \emptyset X).
\]

Hence the Theorem is proved. \( \Box \)

**Theorem 4.2:** Let \( M \) be a CR-submanifold of a nearly hyperbolic cosymplectic manifold \( \bar{M} \). If the distribution \( D^\perp \) is parallel with respect to the connection on \( M \), then
\[
A_{\emptyset Y} X + A_{\emptyset X} Y \in D^\perp, \text{ for any } X, Y \in D^\perp.
\]

**Proof:** Let \( X, Y \in D^\perp \) then using Weingarten Formula.

We have,
\[
\nabla_X N = -A_{\emptyset X} + \nabla^\perp_X N
\]

Putting \( N = \emptyset Y \)
\[
\nabla_X \emptyset Y = -A_{\emptyset Y} X + \nabla^\perp_X \emptyset Y
\]
\[
(\nabla_X \emptyset) Y + \emptyset (\nabla_X Y) = -A_{\emptyset Y} X + \nabla^\perp_X \emptyset Y
\]

Using Gauss Formula
\[
(\nabla_X \emptyset) Y = -A_{\emptyset Y} X + \nabla^\perp_X \emptyset Y - \emptyset (\nabla_X Y + h(X, Y))
\]

(4.11) \( (\nabla_X \emptyset) Y = -A_{\emptyset Y} X + \nabla^\perp_X \emptyset Y - \emptyset (\nabla_X Y) - \emptyset h(X, Y) \)

Interchanging \( X \) and \( Y \)
\[
(\nabla_Y \emptyset) = -A_{\emptyset X} Y + \nabla^\perp_Y \emptyset Y - \emptyset (\nabla_Y X - \emptyset h(Y, X))
\]

(4.12) \( (\nabla_Y \emptyset) = -A_{\emptyset X} Y + \nabla^\perp_Y \emptyset Y - \emptyset (\nabla_Y X) - \emptyset h(Y, X) \)

Adding (4.11) and (4.12), we get
\[
(\nabla_X \emptyset) Y + (\nabla_Y \emptyset) X = -A_{\emptyset Y} X - A_{\emptyset X} Y + \nabla^\perp_X \emptyset Y + \nabla^\perp_Y \emptyset X - \emptyset (\nabla_X Y) - \emptyset (\nabla_Y X) - 2 \emptyset h(X, Y)
\]

From (2.13) and (4.13)
\[
- \eta(X) Y - \eta(Y) X - 2 \eta(X) \eta(Y) \xi = -A_{\emptyset Y} X - A_{\emptyset X} Y + \nabla^\perp_X \emptyset Y + \nabla^\perp_Y \emptyset X - \emptyset (\nabla_X Y) - \emptyset (\nabla_Y X) - 2 \emptyset h(X, Y)
\]

Taking inner product w.r.t. \( Z \in D \)
\[
- \eta(X) g(Y, Z) - \eta(Y) g(X, Z) - 2 \eta(X) \eta(Y) g(\xi, Z) = -g(A_{\emptyset Y} X, Z) - g(A_{\emptyset X} Y, Z) + g(\nabla^\perp_X \emptyset Y, Z)
\]
\[
+ g(\nabla^\perp_Y \emptyset X, Z) - g(\emptyset (\nabla_X Y), Z) - g(\emptyset (\nabla_Y X), Z) - 2 \emptyset g(\emptyset h(X, Y), Z)
\]
\[ g(A_{gY}X + A_{gX}Y, Z) = 0 \]

This implies that
\[ (A_{gY}X + A_{gX}Y) \in D^\perp \]

for any \( X, Y \in D^\perp \).

Hence theorem is proved.

**Definition 4.3:** A CR-submanifold is said to be mixed-totally geodesic if
\[ h(X, Y) = 0, \text{ if } X \in D \text{ and } Z \in D^\perp \]

**Lemma 4.4:** Let \( M \) be a CR-submanifold of a nearly trans-hyperbolic Cosymplectic manifold \( \tilde{M} \). Then \( M \) is mixed totally geodesic if and only if \( A_{gX}X \in D \) for all \( X \in D \).

**Definition 4.5:** A Normal vector field \( N \neq 0 \) is called \( D - \text{parallel} \) normal section if
\[ \nabla_X N = 0, \text{ for all } X \in D \]

**Theorem 4.6:** Let \( M \) be a mixed totally geodesic CR-submanifold of a nearly trans-hyperbolic Sasakian manifold \( \tilde{M} \). Then the normal section \( N \in \partial D^\perp \) is \( D - \text{parallel} \) if and only if \( \nabla_X \emptyset N \in D \) for all \( X \in D \).

**Proof:** Let \( N \in \partial D^\perp \), then from (3.2) we have
\[ -\eta(Y)QX - \eta(X)QY - 2\eta(X)\eta(Y)Q\xi + 2Bh(X, Y) = Q\nabla_X (\emptyset PY) + Q\nabla_Y (\emptyset PX) - QA_{gY}X - QA_{gX}Y \]

As \( Q \) is a projection operator on \( D^\perp \), then
\[ 2Bh(X, Y) = Q\nabla_Y (\emptyset X) - QA_{gY}X. \]

Using definition of mixed geodesic CR-submanifold,
\[ h(X, Y) = 0, \text{ if } X \in D \text{ and } Z \in D^\perp \]

\[ Q\nabla_Y (\emptyset X) = QA_{gY}X. \]

As \( A_{gY}X \in D \), for \( X \in D \).

Therefore, \( QA_{gY}X = 0 \)
\[ Q\nabla_Y (\emptyset X) = 0 \]

By normal component
\[ \emptyset Q(\nabla_X Y) + \emptyset Q(\nabla_Y X) + 2Ch(X, Y) = h(X, \emptyset PY) + h(Y, \emptyset PX) + \nabla_X \emptyset QY + \nabla_Y \emptyset QX \]

As \( Q \) is a projection operator on \( D^\perp \), then
\[ \emptyset Q(\nabla_X Y) = \nabla_X \emptyset QY \]
\[ \emptyset Q(\nabla_Y X) = \nabla_Y \emptyset QY \]

Putting \( Y = \emptyset N \)
\[ (\emptyset Q)\nabla_X \emptyset N = \nabla_X \emptyset Q \]
\[ (\emptyset Q)\nabla_Y \emptyset N = \nabla_Y \emptyset Q \]

(4.15)
\[ (\emptyset Q)\nabla_X \emptyset N = \nabla_X \emptyset \]

Then by Definition of Parallelism of \( N \), We have
\[ (\emptyset Q)\nabla_X \emptyset N = 0 \]
\[ Q\nabla_X \emptyset N = 0 \]

Consequently, we get
\[ \nabla_X (\emptyset N) \in D, \text{ for all } X \in D \]

Converse part is easy consequence of (4.20)

This completes the Proof.

**REFERENCES**


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