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CR- SUBMANIFOLD OF NEARLY HYPERBOLIC COSYMPLECTIC MANIFOLD WITH A QUARTER SYMMETRIC NON METRIC CONNECTION

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ABSTRACT

In present paper, we study some properties of CR-submanifold of a nearly hyperbolic cosymplectic manifold with a quarter symmetric non metric connection, obtain some result on ξ-horizontal and ξ-vertical CR-submanifold of a nearly hyperbolic cosymplectic manifold with a quarter symmetric non metric connection. We also find the integrability conditions of some distributions and study parallel distributions (horizontal & vertical distributions) on CR-submanifold of a nearly hyperbolic cosymplectic manifold with a quarter symmetric non metric connection.

Keywords and Phrases: *CR-submanifold, nearly hyperbolic cosymplectic manifold with a quarter symmetric non metric connection, parallel distribution, and integrability condition.*

INTRODUCTION

The notion of CR-submanifolds of a Kaehler manifold was introduced and studied by A. Bejancu in ([1], [2]). Since then, several paper on Kaehler manifold were published. CR-submanifolds of Sasakian manifold was studied by C.J.Hsu in [3] and M.Kobayashi in [4]. Later, several geometers (see, [5], [6], [7], [8], [9], [10]) enrich the study of CR-submanifolds of almost contact manifolds. On the other hand, almost hyperbolic (f,g,η,ξ)-structure was defined and studied by M.D.Upadhyay and K.K.Dube in [11]. L.Bhatt and K.K.Dube studied CR-submanifolds of a trans-hyperbolic Sasakian manifold in [12]. Ahmad M. and Ali K. study CR-submanifold of a nearly hyperbolic cosymplectic manifold [13]. In this paper, we study some properties of CR- Submanifold of nearly hyperbolic cosymplectic manifold with a quarter symmetric non metric connection.

The paper is organized as follows. In section 2, we give a brief description of nearly hyperbolic cosymplectic manifold with a quarter symmetric non metric connection. In section 3, some properties of CR- Submanifold of nearly hyperbolic cosymplectic manifold with a quarter symmetric non metric connection are investigated. In section 4, some result on parallel distribution on ξ-horizontal and ξ-vertical CR- Submanifold of nearly hyperbolic cosymplectic manifold with a quarter symmetric non metric connections are obtained.

2. PRELIMINARIES

Let $\bar{M}$ be an n-dimensional almost hyperbolic contact metric manifold with the almost hyperbolic contact metric structure $(\emptyset,\xi,\eta.g)$, where a tensor $\emptyset$ of type (1,1) a vector field ξ, called structure vector field and η, the dual 1-form of ξ satisfying the following

(2.1) $\emptyset^2X = X + \eta(X)\boldsymbol{\xi}, \quad g(X,\xi) = \eta(X),$

(2.2) $\eta(\xi) = -I,\ \emptyset(\xi) = 0,\ \eta o\emptyset = 0,$

(2.3) $g(\emptyset X,\emptyset Y) = -g(X,Y) - \eta(X)\eta(Y)$

For any X,Y tangent to $\bar{M}$ [9, 6]. In this case

(2.4) $g(\emptyset X,Y) = -g(X,\emptyset Y).$

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An almost hyperbolic contact metric structure $(\emptyset,\xi,\eta,g)$ on $\bar{M}$ is called hyperbolic cosymplectic manifold [12] if and only if

(2.5) $(\nabla_X\emptyset)Y+(\nabla_Y\emptyset)X=0$ for all X,Y tangent to $\bar{M}$.

A hyperbolic cosymplectic manifold $\bar{M}$ is called nearly hyperbolic cosymplectic manifold, if

(2.6) $\nabla_X\xi=0$ for a Riemannian Connection $\bar{\nabla}$.

Now, Let M be a submanifold immersed in $\bar{M}$. The Riemannian metric induced on M is denoted by the same symbol g. Let $TM\ and\ T^{\perp}M$ be the Lie algebra of vector fields tangential to M and normal to M respectively and ∇ be the induced Levi-Civita connection on N, then the Gauss and Weingarten formulas are given respectively by

(2.7) $\bar{\nabla}_XY=\nabla_XY+h(X,Y)$.

Now we define a quarter symmetric non-metric connection

$$\bar{\nabla}_XY=\nabla_XY+\eta(Y)\emptyset X$$

Putting $Y=N$

$$\bar{\nabla}_XN=\nabla_XN+\eta(N)\emptyset X$$
$$\bar{\nabla}_XN=\nabla_XN$$

(2.8) $\bar{\nabla}_XN=-A_NX+\nabla_X^{\perp}N$ for any $X,Y\in TM\ and\ N\in T^{\perp}M$, where $\nabla^{\perp}$ is a connection on the normal bundle $T^{\perp}M,h$ is the second fundamental form and A_N is the Weingarten map associated with N as

(2.9) $g(A_NX,Y)=g(h(X,Y),N)$ for any $X\in M\ and\ X\in T_xM$. We write

(2.10) $X=PX+QX$

where $PX\in D\ and\ QX\in D^{\perp}$.

Similarly, for N normal to M, we have

(2.11) $\emptyset N=BN+CN$

where BN $(resp. CN)$ is the tangential component $(resp. normal\ component)$ of $\emptyset N$.

Now we define a quarter symmetric non-metric connection

$$\bar{\nabla}_XY=\nabla_XY+\eta(Y)\emptyset X$$

Putting $Y=\emptyset Y$

$$\bar{\nabla}_X\emptyset Y=\nabla_X\emptyset Y+\eta(\emptyset Y)\phi X$$
$$\bar{\nabla}_X\emptyset Y=\nabla_X\emptyset Y$$
$$(\bar{\nabla}_X\emptyset)Y+\emptyset\left(\bar{\nabla}_XY\right)=(\nabla_X\emptyset)Y+\emptyset(\nabla_XY)$$

Interchanging X and Y

$$\left(\bar{\nabla}_Y\emptyset\right)X+\emptyset\left(\bar{\nabla}_YX\right)=(\nabla_Y\emptyset)X+\emptyset(\nabla_YX)$$

Adding above two equations

$$(\bar{\nabla}_X\emptyset)Y+\left(\bar{\nabla}_Y\emptyset\right)X+\emptyset\left(\bar{\nabla}_XY-\nabla_XY\right)+\emptyset\left(\bar{\nabla}_YX-\nabla_YX\right)=(\nabla_X\emptyset)Y+(\nabla_Y\emptyset)X$$

From (2.5) and quater symmetric non-metric connection

$$(\bar{\nabla}_X\emptyset)Y+\left(\bar{\nabla}_Y\emptyset\right)X+\emptyset(\eta(Y)\emptyset X)+\emptyset(\eta(X)\emptyset Y)=0$$
$$(\bar{\nabla}_X\emptyset)Y+\left(\bar{\nabla}_Y\emptyset\right)X=-\eta(Y)\emptyset^2X-\eta(X)\emptyset^2Y$$
$$(\bar{\nabla}_X\emptyset)Y+\left(\bar{\nabla}_Y\emptyset\right)X=-\eta(Y)(X+\eta(X)\boldsymbol{\xi})-\eta(X)(Y+\eta(Y)\boldsymbol{\xi})$$

(2.12) $(\bar{\nabla}_X\emptyset)Y+\left(\bar{\nabla}_Y\emptyset\right)X=-\eta(Y)X-\eta(X)Y-2\eta(X)\eta(Y)\boldsymbol{\xi}$

Quarter symmetric non-metric connection

$$\bar{\nabla}_XY=\nabla_XY+\eta(Y)\emptyset X$$

Putting $Y=\xi$

$$\bar{\nabla}_X\xi=\nabla_X\xi+\eta(\xi)\emptyset X$$

(2.13) $\bar{\nabla}_X\emptyset\xi=-\emptyset X$

Definition 1: An m-dimensional submanifold M of $\bar{M}$ is called a CR-Submanifold of almost nearly hyperbolic contact manifold $\bar{M}$, if there exists a differentiable distribution $D: x \longrightarrow D_x$ on M satisfying the following conditions:

i. D is invariant, that is $\emptyset D_x \subset D_x$ $for\ each\ x \in M$.

ii. The complementary orthogonal distribution $D^\perp$ of D is anti-invariant, that is $\emptyset D_x^\perp \subset T_x^\perp M$. If $\dim D_x^\perp = 0$ $(resp., \dim D_x = 0)$, then the CR-Submanifold is called an invariant (resp., anti-invariant) submanifold. The distribution D $(resp., D^\perp)$ is called the horizontal (resp., vertical) distribution. Also, the pair $(D, D^\perp)$ is called $\xi - horizontal\ (resp., vertical)\ if\ \xi_X \in D_X$ $(resp., \xi_X \in D_X^\perp)$.

3. SOME BASIC LEMMAS

Lemma 3.1: Let M be a CR- submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$ then

(3.1) $$-\eta(Y)PX - \eta(X)PY - 2\eta(X)\eta(Y)P\xi + \emptyset P(\nabla_X Y) + \emptyset P(\nabla_Y X) = P\nabla_X(\emptyset PY) + P\nabla_Y(\emptyset PX) - PA_{\emptyset QY}X - PA_{\emptyset QX}Y$$

(3.2) $$-\eta(Y)QX - \eta(X)QY - 2\eta(X)\eta(Y)Q\xi + 2Bh(X,Y) = Q\nabla_X(\emptyset PY) + Q\nabla_Y(\emptyset PX) - QA_{\emptyset QY}X - QA_{\emptyset QX}Y$$

(3.3) $$\emptyset Q(\nabla_X Y) + \emptyset Q(\nabla_Y X) + 2Ch(X,Y) = h(X, \emptyset PY) + h(Y, \emptyset PX) + \nabla_X^\perp \emptyset QY + \nabla_Y^\perp \emptyset QX\ for\ any\ X, Y \in TM.$$

Proof: Using (2.4), (2.5), (2.6), we get

$$Y = PY + QY.$$
$$\emptyset Y = \emptyset PY + \emptyset QY.$$

Differentiating covariantly

Left side: $\bar{\nabla}_X \emptyset Y = (\bar{\nabla}_X \emptyset)Y + \emptyset(\bar{\nabla}_X Y)$
$= (\bar{\nabla}_X \emptyset)Y + \emptyset(\nabla_X Y + h(X,Y))$
$= (\bar{\nabla}_X \emptyset)Y + \emptyset \nabla_X Y + \emptyset h(X,Y)$

Right side: $\bar{\nabla}_X(\emptyset PY + \emptyset QY) = \bar{\nabla}_X(\emptyset PY) + \bar{\nabla}_X(\emptyset QY)$.
$\bar{\nabla}_X(\emptyset PY + \emptyset QY) = \nabla_X(\emptyset PY) + h(X, \emptyset PY) - A_{\emptyset QY}X + \nabla_X^\perp \emptyset QY$.

From Left and Right side

$$(\bar{\nabla}_X \emptyset)Y + \emptyset(\nabla_X Y) + \emptyset h(X,Y) = \nabla_X(\emptyset PY) + h(X, \emptyset PY) - A_{\emptyset QY}X + \nabla_X^\perp \emptyset QY.$$

Interchanging X & Y,

$$(\bar{\nabla}_Y \emptyset)X + \emptyset(\nabla_Y X) + \emptyset h(Y,X) = \nabla_Y(\emptyset PX) + h(Y, \emptyset PX) - A_{\emptyset QX}Y + \nabla_Y^\perp \emptyset QX.$$

Adding above two equations

$$(\bar{\nabla}_X \emptyset)Y + (\bar{\nabla}_Y \emptyset)X + \emptyset(\nabla_X Y) + \emptyset(\nabla_Y X) + 2\emptyset h(X,Y) = \nabla_X(\emptyset PY) + \nabla_Y(\emptyset PX) + h(X, \emptyset PY) + h(Y, \emptyset PX) - A_{\emptyset QY}X - A_{\emptyset QX}Y + \nabla_X^\perp \emptyset QY + \nabla_Y^\perp \emptyset QX$$

Using (2.12), we have

$$-\eta(Y)X - \eta(X)Y - 2\eta(X)\eta(Y)\boldsymbol{\xi} + \emptyset(\nabla_X Y) + \emptyset(\nabla_Y X) + 2\emptyset h(X,Y)$$
$$= \nabla_X(\emptyset PY) + \nabla_Y(\emptyset PX) + h(X, \emptyset PY) + h(Y, \emptyset PX) - A_{\emptyset QY}X - A_{\emptyset QX}Y + \nabla_X^\perp \emptyset QY + \nabla_Y^\perp \emptyset QX$$
$$-\eta(Y)PX - \eta(Y)QX - \eta(X)PX - \eta(X)QY - 2\eta(X)\eta(Y)P\xi - 2\eta(X)\eta(Y)Q\xi + \emptyset P(\nabla_X Y) + \emptyset Q(\nabla_X Y)$$
$$+P\emptyset(\nabla_Y X) + \emptyset Q(\nabla_Y X) + 2Bh(X,Y) + 2Ch(X,Y) = P\nabla_X(\emptyset PY) + Q\nabla_X(\emptyset PY) + P\nabla_Y(\emptyset PX) + Q\nabla_Y(\emptyset PX)$$
$$+h(X, \emptyset PY) + h(Y, \emptyset PX) - PA_{\emptyset QY}X - QA_{\emptyset QY}X - PA_{\emptyset QX}Y - QA_{\emptyset QX}Y + \nabla_X^\perp \emptyset QY + \nabla_Y^\perp \emptyset QX$$

Comparing horizontal, vertical and normal components, we get

Tangential Component:

$$-\eta(Y)PX - \eta(X)PY - 2\eta(X)\eta(Y)P\xi + \emptyset P(\nabla_X Y) + \emptyset P(\nabla_Y X)$$
$$= P\nabla_X(\emptyset PY) + P\nabla_Y(\emptyset PX) - PA_{\emptyset QY}X - PA_{\emptyset QX}Y$$

Vertical Component:

$$-\eta(Y)QX - \eta(X)QY - 2\eta(X)\eta(Y)Q\xi + 2Bh(X,Y) = Q\nabla_X(\emptyset PY) + Q\nabla_Y(\emptyset PX) - QA_{\emptyset QY}X - QA_{\emptyset QX}Y$$

Normal Component:

$$\emptyset Q(\nabla_X Y) + \emptyset Q(\nabla_Y X) + 2Ch(X,Y) = h(X, \emptyset PY) + h(Y, \emptyset PX) + \nabla_X^\perp \emptyset QY + \nabla_Y^\perp \emptyset QX$$

Hence the Lemma is proved. ▯

Lemma 3.2: Let M be a CR- submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$ then

$$2(\bar{\nabla}_X \emptyset)Y = -\eta(Y)X - \eta(X)Y - 2\eta(X)\eta(Y)\xi + \nabla_X \emptyset Y - \nabla_Y \emptyset X + h(X, \emptyset Y) - h(Y, \emptyset X) - \emptyset[X, Y]$$

$$2\left(\bar{\nabla}_Y \emptyset\right)X = -\eta(Y)X - \eta(X)Y - 2\eta(X)\eta(Y)\xi - \nabla_X \emptyset Y + \nabla_Y \emptyset X + h(Y, \emptyset X) - h(X, \emptyset Y) + \emptyset[X, Y]$$

$for\ any\ X, Y \in D.$

Proof: From Gauss formula (2.7), we have

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y).$$
$$\bar{\nabla}_X \emptyset Y = \nabla_X \emptyset Y + h(X, \emptyset Y).$$
$$\bar{\nabla}_Y \emptyset X = \nabla_Y \emptyset X + h(\emptyset X, Y).$$

(3.6) $\bar{\nabla}_X \emptyset Y - \bar{\nabla}_Y \emptyset X = \nabla_X \emptyset Y + h(X, \emptyset Y) - \nabla_Y \emptyset X - h(Y, \emptyset X).$

Also, we have

$$\bar{\nabla}_X \emptyset Y = (\nabla_X \emptyset)Y + \emptyset \bar{\nabla}_X Y.$$
$$\bar{\nabla}_Y \emptyset X = (\nabla_Y \emptyset)X + \emptyset \bar{\nabla}_Y X.$$

Subtracting above,

$$\bar{\nabla}_X \emptyset Y - \bar{\nabla}_Y \emptyset X = (\bar{\nabla}_X \emptyset)Y - (\bar{\nabla}_Y \emptyset)X + \emptyset(\bar{\nabla}_X Y - \bar{\nabla}_Y X)$$

(3.7) $\bar{\nabla}_X \emptyset Y - \bar{\nabla}_Y \emptyset X = (\bar{\nabla}_X \emptyset)Y - (\bar{\nabla}_Y \emptyset)X + \emptyset[X, Y].$

From (3.6) and (3.7), we get

$$(\bar{\nabla}_X \emptyset)Y - (\bar{\nabla}_Y \emptyset)X + \emptyset[X, Y] = \nabla_X \emptyset Y + h(X, \emptyset Y) - \nabla_Y \emptyset X - h(Y, \emptyset X).$$

(3.8) $(\bar{\nabla}_X \emptyset)Y - (\bar{\nabla}_Y \emptyset)X = \nabla_X \emptyset Y + h(X, \emptyset Y) - \nabla_Y \emptyset X - h(Y, \emptyset X) - \emptyset[X, Y].$

Adding (3.8) and (2.12), we obtain

$$2(\bar{\nabla}_X \emptyset)Y = -\eta(Y)X - \eta(X)Y - 2\eta(X)\eta(Y)\boldsymbol{\xi} + \nabla_X \emptyset Y + h(X, \emptyset Y) - \nabla_Y \emptyset X - h(Y, \emptyset X) - \emptyset[X, Y]$$

Subtracting (3.8) from (2.12), we obtain

$$2\left(\bar{\nabla}_Y \emptyset\right)X = -\eta(Y)X - \eta(X)Y - 2\eta(X)\eta(Y)\boldsymbol{\xi} - \nabla_X \emptyset Y - h(X, \emptyset Y) + \nabla_Y \emptyset X + h(Y, \emptyset X) + \emptyset[X, Y]$$

Hence the Lemma is proved. ▯

Corollary 3.3: If M be a ξ −vertical CR-submanifold of a CR- submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$ with quarter symmetric metric connection. Then

$$2(\bar{\nabla}_X \emptyset)Y = \nabla_X \emptyset Y + h(X, \emptyset Y) - \nabla_Y \emptyset X - h(Y, \emptyset X) - \emptyset[X, Y]$$

and

$$2\left(\bar{\nabla}_Y \emptyset\right)X = \nabla_Y \emptyset X - \nabla_X \emptyset Y - h(X, \emptyset Y) + h(Y, \emptyset X) + \emptyset[X, Y] \quad for\ any\ X, Y \in D.$$

Lemma 3.4: Let M be a CR- submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$ then

$$2(\bar{\nabla}_X \emptyset)Y = -\eta(Y)X - \eta(X)Y - 2\eta(X)\eta(Y)\xi + A_{\emptyset X}Y - A_{\emptyset Y}X + \nabla_X^{\perp} \emptyset Y - \nabla_Y^{\perp} \emptyset X - \emptyset[X, Y]$$

and

$$2\left(\bar{\nabla}_Y \emptyset\right)X = -\eta(Y)X - \eta(X)Y - 2\eta(X)\eta(Y)\xi - A_{\emptyset X}Y + A_{\emptyset Y}X - \nabla_X^{\perp} \emptyset Y + \nabla_Y^{\perp} \emptyset X + \emptyset[X, Y] \quad for\ any\ X, Y \in \mathrm{D}^{\perp}.$$

Proof: From Weingarten formula (2.8), we have

$$\bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N$$

Putting $N = \emptyset Y$

$$\bar{\nabla}_X \emptyset Y = -A_{\emptyset Y}X + \nabla_X^{\perp} \emptyset Y.$$
$$\bar{\nabla}_Y \emptyset X = -A_{\emptyset X}Y + \nabla_Y^{\perp} \emptyset X.$$

(3.10) $\bar{\nabla}_X \emptyset Y - \bar{\nabla}_Y \emptyset X = A_{\emptyset X}Y - A_{\emptyset Y}X + \nabla_X^{\perp} \emptyset Y - \nabla_Y^{\perp} \emptyset X.$

Also,

(3.11) $\bar{\nabla}_X \emptyset Y - \bar{\nabla}_Y \emptyset X = (\bar{\nabla}_X \emptyset)Y - (\bar{\nabla}_Y \emptyset)X + \emptyset[X, Y].$

From (3.10) and (3.11), we get

(3.12) $(\bar{\nabla}_X \emptyset)Y - (\bar{\nabla}_Y \emptyset)X = A_{\emptyset X}Y - A_{\emptyset Y}X + \nabla_X^{\perp} \emptyset Y - \nabla_Y^{\perp} \emptyset X - \emptyset[X, Y].$

(2.12) $(\bar{\nabla}_X \emptyset)Y + \left(\bar{\nabla}_Y \emptyset\right)X = -\eta(Y)X - \eta(X)Y - 2\eta(X)\eta(Y)\boldsymbol{\xi}$

Adding (3.12) and (2.12), we obtain

$$2(\overline{\nabla}_X\emptyset)Y = -\eta(Y)X - \eta(X)Y - 2\eta(X)\eta(Y)\boldsymbol{\xi} + A_{\emptyset X}Y - A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y^{\perp}\emptyset X - \emptyset[X,Y]$$

Subtracting (3.12) from (2.12), we obtain

$$2(\overline{\nabla}_Y\emptyset)X = -\eta(Y)X - \eta(X)Y - 2\eta(X)\eta(Y)\boldsymbol{\xi} - A_{\emptyset X}Y + A_{\emptyset Y}X - \nabla_X^{\perp}\emptyset Y + \nabla_Y^{\perp}\emptyset X + \emptyset[X,Y]$$

Hence the Lemma is proved.

Corollary 3.5: If M be a $\xi -$ horizontal CR-submanifold of of a CR- submanifold of a nearly hyperbolic cosymplectic manifold $\overline{M}$ with quarter symmetric metric connection. Then

$$2(\overline{\nabla}_X\emptyset)Y = A_{\emptyset X}Y - A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y^{\perp}\emptyset X - \emptyset[X,Y].$$

and

$$2(\overline{\nabla}_Y\emptyset)X = A_{\emptyset Y}X - A_{\emptyset X}Y + \nabla_Y^{\perp}\emptyset X - \nabla_X^{\perp}\emptyset Y + \emptyset[X,Y].\ for\ any\ X,Y \in \mathrm{D}^{\perp}.$$

Lemma 3.6: Let M be a CR- submanifold of a nearly hyperbolic cosymplectic manifold $\overline{M}$ then

$$2(\overline{\nabla}_X\emptyset)Y = -\eta(Y)X - \eta(X)Y - 2\eta(X)\eta(Y)\xi - A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y\emptyset X - h(Y,\emptyset X) - \emptyset[X,Y]$$

$$2(\overline{\nabla}_Y\emptyset)X = -\eta(Y)X - \eta(X)Y - 2\eta(X)\eta(Y)\xi + A_{\emptyset Y}X - \nabla_X^{\perp}\emptyset Y + \nabla_Y\emptyset X + h(Y,\emptyset X) + \emptyset[X,Y]$$

$for\ any\ X \in D\ and\ Y \in D^{\perp}$.

Proof: Using Gauss and Weingarten formula for $X \in D\ and\ Y \in D^{\perp}$ respectively, we have

$$\overline{\nabla}_X\emptyset Y = -A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y.$$

and $\quad \overline{\nabla}_Y\emptyset X = \nabla_Y\emptyset X + h(Y,\emptyset X).$

(3.14) $\quad \overline{\nabla}_X\emptyset Y - \overline{\nabla}_Y\emptyset X = -A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y\emptyset X - h(Y,\emptyset X).$

Also, we have

(3.15) $\quad \overline{\nabla}_X\emptyset Y - \overline{\nabla}_Y\emptyset X = (\overline{\nabla}_X\emptyset)Y - (\overline{\nabla}_Y\emptyset)X + \emptyset[X,Y].$

By virtue of (3.14) and (3.15), we get

(3.16) $\quad (\overline{\nabla}_X\emptyset)Y - (\overline{\nabla}_Y\emptyset)X = -A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y\emptyset X - h(Y,\emptyset X) - \emptyset[X,Y].$

(2.12) $\quad (\overline{\nabla}_X\emptyset)Y + (\overline{\nabla}_Y\emptyset)X = -\eta(Y)X - \eta(X)Y - 2\eta(X)\eta(Y)\boldsymbol{\xi}$

Adding (3.16) and (2.12), we obtain

$$2(\overline{\nabla}_X\emptyset)Y = -\eta(Y)X - \eta(X)Y - 2\eta(X)\eta(Y)\boldsymbol{\xi} - A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y\emptyset X - h(Y,\emptyset X) - \emptyset[X,Y]$$

Subtracting (3.16) from (2.12), we obtain

$$2(\overline{\nabla}_Y\emptyset)X = -\eta(Y)X - \eta(X)Y - 2\eta(X)\eta(Y)\boldsymbol{\xi} + A_{\emptyset Y}X - \nabla_X^{\perp}\emptyset Y + \nabla_Y\emptyset X + h(Y,\emptyset X) + \emptyset[X,Y]$$

Hence the Lemma is proved. ▯

4. PARALLEL DISTRIBUTION

Definition 2: The horizontal (resp., vertical) distribution $D(resp.,D^{\perp})$ is said to be parallel [13] with respect to the connection on $M\ if\ \nabla_X Y \in D\ (resp.,\nabla_Z W \in D^{\perp})$ for any vector field $X,Y \in D\ (resp.,W,Z \in D^{\perp})$.

Theorem 4.1: Let M be a $\xi - vertical$ CR-submanifold of a nearly hyperbolic cosymplectic manifold $\overline{M}$. If the horizontal distribution D is parallel, Then

(4.1) $\quad h(X,\emptyset Y) = h(Y,\emptyset X).\quad for\ any\ X,Y \in D$

Proof: Using parallelism of horizontal distribution D, we have

(4.2) $\quad \nabla_X(\emptyset Y) \in D\quad and\ \nabla_Y\emptyset X \in D\quad for\ any\ X,Y \in D.$

From Vertical component,

$$-\eta(Y)QX - \eta(X)QY - 2\eta(X)\eta(Y)Q\xi + 2Bh(X,Y) = Q\nabla_X(\emptyset PY) + Q\nabla_Y(\emptyset PX) - QA_{\emptyset QY}X - QA_{\emptyset QX}Y$$

As Q is a projection operator on $D^{\perp}$, We have

(4.3) $\quad Bh(X,Y) = 0.$

We know,

$$\emptyset N = BN + CN$$

Putting $N = h(X,Y)$

$$\emptyset h(X,Y) = Bh(X,Y) + Ch(X,Y)$$

From (4.3)

(4.4) $$2\emptyset h(X,Y) = 2Ch(X,Y)$$

From normal component,

$$\emptyset Q(\nabla_X Y) + \emptyset Q(\nabla_Y X) + 2Ch(X,Y) = h(X,\emptyset PY) + h(Y,\emptyset PX) + \nabla_X^{\perp}\emptyset QY + \nabla_Y^{\perp}\emptyset QX.$$

$$2Ch(X,Y) = h(X,\emptyset PY) + h(Y,\emptyset PX)$$

(4.5) $$2Ch(X,Y) = h(X,\emptyset Y) + h(Y,\emptyset X)\ ,\ for\ any\ X,Y \in D.$$

Applying (4.5) in (4.4)

(4.6) $$2\emptyset h(X,Y) = h(X,\emptyset Y) + h(Y,\emptyset X)$$

Replacing $X\ by\ \emptyset X$

$$2\emptyset h(\emptyset X,Y) = h(\emptyset X,\emptyset Y) + h(Y,\ \emptyset^2 X)$$

$$2\emptyset h(\emptyset X,Y) = h(\emptyset X,\emptyset Y) + h(Y,X + \eta(X)\boldsymbol{\xi})$$

$$2\emptyset h(\emptyset X,Y) = h(\emptyset X,\emptyset Y) + h(Y,X) + h(Y,\eta(X)\boldsymbol{\xi})$$

(4.7) $$2\emptyset h(\emptyset X,Y) = h(\emptyset X,\emptyset Y) + h(Y,X)$$

Now, replacing $Y \longrightarrow \emptyset Y$ in (4.6), we get

$$h(X,\ \emptyset^2 Y) + h(\emptyset Y,\emptyset X) = 2\emptyset h(X,\emptyset Y).$$

$$h(X,Y + \eta(Y)\boldsymbol{\xi}) + h(\emptyset Y,\emptyset X) = 2\emptyset h(X,\emptyset Y).$$

(4.8) $$h(X,Y) + h(\emptyset Y,\emptyset X) = 2\emptyset h(X,\emptyset Y).$$

Thus from (4.7) and (4.8), we find

$$2\emptyset h(\emptyset X,Y) = 2\emptyset h(X,\emptyset Y).$$

Operating $\emptyset$ on both sides, we get

$$h(X,\emptyset Y) = h(Y,\emptyset X).$$

Hence the Theorem is proved. ▯

Theorem 4.2: Let M be a CR-submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$. If the distribution $D^{\perp}$ is parallel with respect to the connection on M, then

(4.9) $$A_{\emptyset Y}X + A_{\emptyset X}Y \in D^{\perp}\ .\ for\ any\ X,Y \in D^{\perp}.$$

Proof: Let, $X,Y \in D^{\perp}$.then using Weingarten Formula .

We have,

$$\bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N$$

Putting $N = \emptyset Y$

$$\bar{\nabla}_X \emptyset Y = -A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y$$

$$(\bar{\nabla}_X\emptyset)Y + \emptyset(\bar{\nabla}_X Y) = -A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y$$

Using Gauss formula

$$(\bar{\nabla}_X\emptyset)Y = -A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \emptyset(\nabla_X Y + h(X,Y))$$

(4.11) $$(\bar{\nabla}_X\emptyset)Y = -A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \emptyset\nabla_X Y - \emptyset h(X,Y)$$

Interchanging X and Y

(4.12) $$(\bar{\nabla}_Y\emptyset)X = -A_{\emptyset X}Y + \nabla_Y^{\perp}\emptyset X - \emptyset\nabla_Y X - \emptyset h(Y,X)$$

Adding (4.11) and (4.12), we get

(4.13) $$(\bar{\nabla}_X\emptyset)Y + (\bar{\nabla}_Y\emptyset)X = -A_{\emptyset Y}X - A_{\emptyset X}Y + \nabla_X^{\perp}\emptyset Y + \nabla_Y^{\perp}\emptyset X - \emptyset\nabla_X Y - \emptyset\nabla_Y X - 2\emptyset h(X,Y)$$

From (2.13) and (4.13)

$$-\eta(X)Y - \eta(Y)X - 2\eta(X)\eta(Y)\xi = -A_{\emptyset Y}X - A_{\emptyset X}Y + \nabla_X^{\perp}\emptyset Y + \nabla_Y^{\perp}\emptyset X - \emptyset\nabla_X Y - \emptyset\nabla_Y X - 2\emptyset h(X,Y)$$

Taking inner product w.r.to $Z \in D$

$$-\eta(X)g(Y,Z) - \eta(Y)g(X,Z) - 2\eta(X)\eta(Y)g(\xi,Z) = -g(A_{\emptyset Y}X,Z) - g(A_{\emptyset X}Y,Z) + g(\nabla_X^{\perp}\emptyset Y,Z)$$
$$+g(\nabla_Y^{\perp}\emptyset X,Z) - g(\emptyset\nabla_X Y,Z) - g(\emptyset\nabla_Y X\ ,Z) - 2\emptyset g(h(X,Y),Z)$$

$$g(\mathrm{A}_{\emptyset Y}X + \mathrm{A}_{\emptyset X}Y, Z) = 0$$

This implies that

$$(\mathrm{A}_{\emptyset Y}X + \mathrm{A}_{\emptyset X}Y) \in D^{\perp}$$

$for\ any\ X, Y \in D^{\perp}$.

Hence theorem is proved.

Definition 4.3: A CR-submanifold is said to be mixed-totally geodesic if $h(X, Z) = 0, for\ all\ X \in D\ and\ Z \in D^{\perp}$.

Lemma 4.4: Let M be a CR-submanifold of a nearly trans-hyperbolic Cosymplectic manifold $\bar{M}$. Then M is mixed totally geodesic if and only if $A_N X \in D$ for all $X \in D$.

Definition 4.5: A Normal vector field $N \neq 0$ is called $D - parallel$ normal section if $\nabla_X^{\perp} N = 0$, for all $X \in D$.

Theorem 4.6: Let M be a mixed totally geodesic CR-submanifold of a nearly trans- hyperbolic Sasakian manifold $\bar{M}$. Then the normal section $N \in \emptyset D^{\perp}$ is $D\ parallel$ if and only if $\nabla_X \emptyset N \in D\ for\ all\ X \in D$.

Proof: Let $N \in \emptyset D^{\perp}$, then from (3.2) we have

$$-\eta(Y)QX - \eta(X)QY - 2\eta(X)\eta(Y)Q\xi + 2Bh(X,Y) = Q\nabla_X(\emptyset PY) + Q\nabla_Y(\emptyset PX) - QA_{\emptyset QY}X - QA_{\emptyset QX}Y$$

As Q is a projection operator on $D^{\perp}$, then

(4.15) $\quad 2Bh(X,Y) = Q\nabla_Y(\emptyset X) - QA_{\emptyset Y}X.$

Using definition of mixed geodesic CR-submanifold,

$$h(X,Y) = 0,\ \ if\ \ X \in D\ and\ Z \in D^{\perp}$$

(4.16) $\quad Q\nabla_Y(\emptyset X) = QA_{\emptyset Y}X.$

As $A_{\emptyset Y}X \in D$, for $X \in D$.

Therefore, $\quad QA_{\emptyset Y}X = 0$

(4.17) $\quad Q\nabla_Y(\emptyset X) = 0$

By normal component

$$\emptyset Q(\nabla_X Y) + \emptyset Q(\nabla_Y X) + 2Ch(X,Y) = h(X, \emptyset PY) + h(Y, \emptyset PX) + \nabla_X^{\perp}\emptyset QY + \nabla_Y^{\perp}\emptyset QX$$

As Q is a projection operator on $D^{\perp}$, then

$$\emptyset Q(\nabla_X Y) = \nabla_X^{\perp}\emptyset QY$$
$$\emptyset Q(\nabla_X Y) = \nabla_X^{\perp}\emptyset Y$$

Putting , $Y = \emptyset N$

$$(\emptyset Q)\nabla_X \emptyset N = \nabla_X^{\perp}\emptyset^2 N$$
$$(\emptyset Q)\nabla_X \emptyset N = \nabla_X^{\perp}(N + \eta(N)\xi)$$

(4.20) $\quad (\emptyset Q)\nabla_X \emptyset N = \nabla_X^{\perp} N$

Then by Definition of Parallelism of N, We have

$$(\emptyset Q)\nabla_X \emptyset N = 0$$
$$Q\nabla_X \emptyset N = 0$$

Consequently, we get

$$\nabla_X(\emptyset N) \in D, \quad for\ all\ X \in D.$$

Converse part is easy consequence of (4.20)

This completes the Proof.

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