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FIXED POINT OF T_F-CONTRACTIVE SINGLE VALUED MAPPINGS IN COMPLETE G-METRIC SPACE

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ABSTRACT

In this work, we introduced T_F – Contraction in complete G-Metric Spaces and we study some fixed point Theorems of generalized T_F – Contraction mapping in complete G-metric spaces.

Key Words: *G-metric space, T_F- contraction, Graph Closed, Subsequentially convergent.*

1. INTRODUCTION

Some generalizations of the notion of a metric space have been proposed by some authors. Gahler [1, 2] coined the term of 2-metric spaces. This is extended to D-metric space by Dhage (1992) [3, 4]. Dhage proved many fixed point Theorems in D-metric space. In 2006, Mustafa in collaboration with Sims introduced a new notion of generalized metric space called G-metric space [5]. In fact, Mustafa et al. studied many fixed point results for a self mapping in G-metric spaces under certain conditions; see [5, 6, 7, 8 and 9].

In 2010 Moradi *et al*. [10] introduced a new type of fixed point Theorem by defining T_F –Contraction as a new contractive condition in complete metric spaces. In this work, we introduced T_F – Contraction in complete G-Metric Spaces and we study some fixed point Theorems of generalized T_F – Contraction mapping in complete G-metric spaces.

2. DEFINITIONS AND PRELIMINARIES

Definition 2.1 [5]: Let X be a non empty set, and let $G: X \times X \times X \to [0, \infty)$ be a function satisfying the following axioms
$(G1)$ $G(x, y, z) = 0$ if $x = y = z$,
$(G2)$ $G(x, x, y) > 0$ for all $x, y \in X$, with $x \neq y$.v
$(G3)$ $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $y \neq z$.
$(G4)$ $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$, (symmetry in all three variables)
$(G5)$ $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangular inequality)

Then the function G is called a generalized metric, or more specially a G-metric on X, and the pair (X, G) is called a G-metric space.

Example: Let (X, d) be a usual metric space. Then (X, G_s) and (X, G_m) are G-metric spaces, where

$$G_s(x, y, z) = d(x, y) + d(y, z) + d(x, z) \text{ for all } x, y, z \in X$$

and

$$G_m(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\} \text{ for all } x, y, z \in X.$$

Definition 2.2 [5]: Let (X, G) and (X', G') be G-metric spaces and let $f: (X, G) \to (X', G')$ be a function, then f is said to be G-continuous at a point $a \in X$ if given $\varepsilon > 0$ there exist $\delta > 0$ such that $x, y \in X, G(a, x, y) < \delta$ implies that $G'(fa, fx, fy) < \varepsilon$. A function f is G-continuous on X if and only if it is G-continuous at all $a \in X$.

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Definition 2.3 [5]: Let (X, G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X, therefore; we say that $\{x_n\}$ is G -convergent to x if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$; that is ,for any $\varepsilon > 0$, there exist $\mathrm{N} \in N$ such that $G(x, x_n, x_m) < \varepsilon$ for all $n.m \geq$N.We call x is the limit of the sequence $\{x_n\}$ and we write $x_n \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$.

Proposition 2.4 [5]: Let (X, G) and (X', G') be G metric spaces, then a function $f: X \to X$ is said to be G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous, that is, whenever $\{x_n\}$ is G-convergent to x, $\{fx_n\}$ is G-convergent to $f(x)$.

Proposition 2.5 [5]: Let (X, G) be a G-metric space. Then the following statements are equivalent

(a) $\{x_n\}$ is G-convergent to x.
(b) $G(x_n, x_n, x) \to 0$ as $n \to \infty$.
(c) $G(x_n, x, x) \to 0$ as $n \to \infty$.
(d) $G(x_n, x_m, x) \to 0$ as $n \to \infty$.

Proposition 2.6 [5]: Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-cauchy sequence if given $\varepsilon > 0$, there is N$\in N$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \geq$N; that is if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Proposition 2.7 [5]: In a G-metric space (X, G), the following two statements are equivalent.

(1) The sequence $\{x_n\}$ is G-cauchy.
(2) For every $\varepsilon > 0$, there exist $N \in N$ such that $G(x_n, x_m, x_m) < \varepsilon$ for all $n, m \geq N$.

Definition 2.9 [5]: A G-metric space (X, G) is said to be G-complete (or a complete G-metric pace) if every G-cauchy sequence in (X, G) is G-convergent in (X, G).

Proposition 2.10 [5]: Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.

Definition 2.11 [5]: A G-metric space $(X, G)$is called a symmetric G-metric space if

$$G(x, y, y) = G(y, x, x) \text{ for all } x, y \in X.$$

Proposition 2.12 [5]: Every G-metric space (X, G) defines a metric space (X, d_G) by

$$d_G(x, y) = G(x, y, y) + G(y, x, x) \text{ for all } x, y \in X.$$

Note that, if (X, G) is a symmetric space G-metric space, then

$$d_G(x, y) = 2\, G(x, y, y) \text{ for all } x, y \in X$$

However, if (X, G) is not asymmetric space, then it holds by the G-metric properties that

$$\frac{3}{2} G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y) \text{ for all } x, y \in X.$$

In general, these inequalities cannot be improved.

Proposition 2.13 [5]: A G-metric space (X, G) is G-complete if and only if (X, d_G) is a complete metric space.

Proposition 2.14 [5]: Let (X, G) be a G-metric space. Then for any $x, y, z, a \in X$, it follows that

(1) If $G(x, y, z) = 0$ then $x = y = z$.
(2) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$.
(3) $G(x, y, y) \leq 2\, G(y, x, x)$.
(4) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$.
(5) $G(x, y, z) \leq \frac{2}{3}\{G(x, a, a) + G(y, a, a) + G(z, a, a)$.

It is well known that the first important result on fixed point theory is Banach Contraction Principle. Due to the importance, there exist many extension of it.

Theorem 2.15 [10]: A mapping $T: X \to X$, where (X, d) is a metric space, is said to be a contraction if there exist $k \in [0,1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq k\, d(x, y) \tag{2.1}$$

If the metric space (X, d) is a complete then the mapping satisfying (1) has a unique fixed point.

Theorem 2.16: A mapping$T: X \to X$, where (X, G) is a G-metric space, is said to be a contraction if there exist $k \in [0,1)$ such that for all $x, y, z \in X$,

$$G(Tx, Ty, Tz) \leq k\, G(x, y, z). \tag{2.2}$$

If the metric space (X, G) is complete then the mapping satisfying (2) has a unique fixed point.

Definition 2.17 [10]: Let (X, d) be a metric space, let $f, T: X \to X$ be two self mappings and let $F: [0, \infty) \to [0, \infty), F$ is nondecreasing, continuous from right and $F^{-1}(0) = 0.$ a mapping f is said to be $\boldsymbol{T_F}$**-contraction** if there exist $\alpha \in [0,1)$ such that for all $x, y \in X$,

$$F\big(d(Tfx, Tfy)\big) \leq \alpha\, F\big(d(Tx, Ty)\big). \tag{2.3}$$

Remarks:

(1) By taking $Tx \equiv x$ and $F(x) \equiv x$ then T_F-contraction and contraction are equivalent.

(2) By taking $Fx \equiv x$ we can define T-contraction and by taking $Tx \equiv x$ we can define I_F-contraction (I is identity function).

Definition 2.18 [10]: Let (X, d) a metric space. A mapping $T: X \to X$ is said to be sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ is also convergence. T is said to be subsequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ has a convergent subsequence.

Definition 2.19 [10]: Let (X, d) be a metric space. A mapping $T: X \to X$ is said to be **graph closed** if for every sequence $\{x_n\}$ such that $\lim_{n\to\infty} Tx_n = a$ then for some $\in X, Tb = a$.

Example: the identity function on X is graph closed.

Before presenting the main results in this paper we introduce following concepts, which will be used in our result.

Definition 2.20: ($\boldsymbol{T_F}$-contraction in $\boldsymbol{G}$-metric space): Let (X, G) be a metric space, let $f, T: X \to X$ be two self mappings and let $F: [0, \infty) \to [0, \infty), F$ is nondecreasing, continuous from right and $F^{-1}(0) = 0.$ a mapping f is said to be $\boldsymbol{T_F}$**-contraction** if there exist $\alpha \in [0,1)$ such that for all x, y, z

$$F\big(G(Tfx, Tfy, Tfz)\big) \leq \alpha\, F\big(G(Tx, Ty, Tz)\big). \tag{2.4}$$

Remarks:

(1) By taking $Tx \equiv x$ and $F(x) \equiv x$ then T_F-contraction and contraction are equivalent.

(2) By taking $Fx \equiv x$ we can define T-contraction and by taking $Tx \equiv x$ we can define I_F-contraction (I is identity function).

Definition 2.21: Let (X, G) a metric space. A mapping $T: X \to X$ is said to be sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ is also convergence. T is said to be Subsequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ has a convergent subsequence.

Definition 2.22: Let (X, G) be a metric space. A mapping $T: X \to X$ is said to be **graph closed** if for every sequence $\{x_n\}$ such that $\lim_{n\to\infty} Tx_n = a$ then for some $\in X, Tb = a$.

Example: the identity function on X is graph closed.

Theorem 2.23: [10] Let (X, d) be a metric space, let $f, T: X \to X$ be two self mappings such that T is one to one and graph closed (subseuentially convergent and continuous) and f is T_F-contraction, that is there exist $\alpha \in [0,1)$ such that for all $x, y \in X$,

$$F\big(d(Tfx, Tfy)\big) \leq \alpha\, F\big(d(Tx, Ty)\big). \tag{2.5}$$

Where $F: [0, \infty) \to [0, \infty), F$ is nondecreasing, continuous from right and $F^{-1}(0) = 0$, then f has a unique fixed point in X, also for every $x \in X$, the sequence of iterates $\{Tf^n x\}$ converges to the fixed point.

Motivated by the above result, we address the same question on G-metric space. We establish fixed point result in the third part of the paper.

3. MAIN RESULT

Theorem 3.1: Let (X,G) be a complete G- metric space, let $f,T:X\to X$ be two self mappings such that T is one to one and graph closed (subseuentially convergent and continuous) and f is T_F-contraction, that is there exist $\alpha\in[0,1)$ such that for all $x,y,z\in X$,

$$F\big(G(Tfx,Tfy,Tfz)\big)\le\alpha\,F\big(G(Tx,Ty,Tz)\big). \tag{3.1}$$

Where $F:[0,\infty)\to[0,\infty), F$ is nondecreasing, continuous from right and $F^{-1}(0)=0$, then f has a unique fixed point in X, also for every $x_0\in X$, the sequence of iterates $\{f^n x_0\}$ converges to the fixed point.

Proof: Let $x_0\in X$ be an arbitrary point and $x_n=fx_{n-1}=f^n x_0$ (3.2)

Now,

$$F\big(G(Tx_n,Tx_{n+1},Tx_{n+1})\big)=F(G(Tfx_{n-1},Tfx_n,Tfx_n)$$
$$F\big(G(Tx_n,Tx_{n+1},Tx_{n+1})\big)\le\alpha\,F\big(G(Tx_{n-1},Tx_n,Tx_n)\big),$$
$$F\big(G(Tx_n,Tx_{n+1},Tx_{n+1})\big)\le\alpha.\alpha\,F\big(G(Tx_{n-2},Tx_{n-1},Tx_{n-1})\big),$$
$$F\big(G(Tx_n,Tx_{n+1},Tx_{n+1})\big)\le\alpha.\alpha.\alpha\,F\big(G(Tx_{n-3},Tx_{n-2},Tx_{n-2})\big), \tag{3.3}$$

After repeated applications of R.H.S of the above equation, we obtain

$$F\big(G(Tx_n,Tx_{n+1},Tx_{n+1})\big)\le\alpha^n\,F\big(G(Tx_0,Tx_1,Tx_1)\big) \tag{3.4}$$

Again using (3.4) for all $m,n\in N$, taking $m>n$, we have

$$F\big(G(Tx_n,Tx_m,Tx_m)\big)=F\big(G(Tf^n x_0,Tf^m x_0,Tf^m x_0)\big),$$
$$F\big(G(Tx_n,Tx_m,Tx_m)\big)\le\alpha^n\,F(G(Tx_0,Tf^{m-n}x_0,Tf^{m-n}x_0) \tag{3.5}$$

Letting $m,n\to\infty$ in (3.5), we obtain

$F\big(G(Tx_n,Tx_m,Tx_m)\big)\to 0^+$ as $m,n\to\infty$.

So, we have $G(Tx_n,Tx_m,Tx_m)\to 0^+$ as $m,n\to\infty$.

Thus we hold that $\{Tx_n\}$ is a Cauchy Sequence in complete metric space (X,G).

By taking in view the completeness of X, we obtain that there exist $v\in X$ such that

$$\lim_{n\to\infty}Tx_n=v \tag{3.6}$$

Note that T is subsequentially convergent, then $\{x_n\}$ has a convergent subsequence, so there is $u\in X$ such that

$$\lim_{k\to\infty}x_{n(k)}=u \tag{3.7}$$

Also, T is continuous and $x_{n(k)}\to u$, therefore

$$\lim_{k\to\infty}Tx_{n(k)}=Tu \tag{3.8}$$

Note that $\{Tx_{n(k)}\}$ is a subsequence of $\{Tx_n\}$, so $Tu=v$. (3.9)

Now we will show that $u\in X$ is a fixed point of f. Indeed, we have

$$F\big(G(Tu,Tfu,Tfu)\big)\le F\big[G\big(Tu,Tx_{n(k)},Tx_{n(k)}\big)+G\big(Tx_{n(k)},Tfu,Tfu\big)\big],$$
$$F\big(G(Tu,Tfu,Tfu)\big)\le F\big[G\big(Tu,Tx_{n(k)},Tx_{n(k)}\big)+G\big(Tf^{n(k)}x_0,Tfu,Tfu\big)\big],$$
$$F\big(G(Tu,Tfu,Tfu)\big)\le F[G\big(Tu,Tx_{n(k)},Tx_{n(k)}\big)+G(Tf^{n(k)}x_0,Tf^{n(k)+1}x_0,Tf^{n(k)+1}x_0)\ +G(Tf^{n(k)+1}x_0,Tfu,Tfu)],$$
$$F\big(G(Tu,Tfu,Tfu)\big)\le F[G\big(Tu,Tx_{n(k)},Tx_{n(k)}\big)+G(Tfx_{n(k)-1},Tfx_{n(k)+1},Tfx_{n(k)+1})+G(Tfx_{n(k)},Tfu,Tfu)], \tag{3.10}$$

Letting $k\to\infty$ in (3.10), we have

$$F\big(G(Tu,Tfu,Tfu)\big)\le F[G(Tu,Tu,Tu)+G(Tfu,Tfu,Tfu)+G(Tfu,Tfu,Tfu)],$$
$$F\big(G(Tu,Tfu,Tfu)\big)\le F[0+0+0], F\big(G(Tu,Tfu,Tfu)\big)\le F(0), F\big(G(Tu,Tfu,Tfu)\big)\le 0 \tag{3.11}$$

Last inequality (3.11) is contradiction unless $G(Tu,Tfu,Tfu)=0$.

Thus, we obtained $Tu=Tfu$. Also, T is one to one, we obtain $u=fu$. (3.12)

Thus, we provide $u\in X$ is a fixed point of f.

Now, we show that the fixed point is unique.

Assume u' is another fixed point of f, then we have $fu' = u'$. (3.13)
$F\left(G(Tu, Tu', Tu')\right) = F\left(G(Tfu, Tfu', Tfu')\right)$.
$F\left(G(Tu, Tu', Tu')\right) \leq \alpha\, F\left(G(Tu, Tu', Tu')\right)$
$(1-\alpha)\, F\left(G(Tu, Tu', Tu')\right) \leq 0.$

This implies $F\left(G(Tu, Tu', Tu')\right) \leq 0$.

This is a contradiction unless $G(Tu, Tu', Tu') = 0$,

Therefore $Tu = Tu'$ and T is one to one, so we obtain $u = u'$.

Therefore u is a unique fixed point of f.

Also, if we take T is sequentially convergent, by replacing $\{n\}$ with $\{n(k)\}$ in (3.7), we obtain

$$\lim_{n\to\infty} x_n = u \tag{3.14}$$

Thus the equation (3.14) shows that $\{x_n\}$ converges to the fixed point of f.

Thus $x_n = f^n x_0$ converges the fixed point of f [From (3.2)].

If T is sequentially convergent then for every $x_0 \in X$ the sequence of iterates $\{f^n x_0\}$ converges to the fixed point.

Thus the proof is completed.

Example (1): Let $X = [0, \infty)$ and $d(x, y) = |x - y|$.

Define $G(x, y, z) = |x - y| + |y - z| + |z - x|$, (3.15)
then (X, G) is a complete G-metric space.

Consider two mappings $T, f\colon X \to X$ by $Tx = \frac{1}{x} + 1$ (3.16)
and $f(x) = 2x$. (3.17)
Where T is one to one, subsequentially convergent and continuous.

Define $F\colon [0,\infty) \to [0,\infty), F(x) = x$, then $F(x)$ is nondecreasing and continuous from the right and $F^{-1}(0) = 0$.

Now,
$F\left(G(Tfx, Tfy, Tfz)\right) = G(Tfx, Tfy, Tfz)$, since $(F(x) = x)$
$F\left(G(Tfx, Tfy, Tfz)\right) = G\left(T(2x), T(2y), T(2z)\right)$, [From (3.16)]
$F\left(G(Tfx, Tfy, Tfz)\right) = G\left(\frac{1}{2x} + 1, \frac{1}{2y} + 1, \frac{1}{2z} + 1\right)$, [From (3.17)]
$F\left(G(Tfx, Tfy, Tfz)\right) = \left|\frac{1}{2x} + 1 - \frac{1}{2y} - 1\right| + \left|\frac{1}{2y} + 1 - \frac{1}{2z} - 1\right| + \left|\frac{1}{2z} + 1 - \frac{1}{2x} - 1\right|$, [From (3.15)]
$F\left(G(Tfx, Tfy, Tfz)\right) = \left|\frac{1}{2x} - \frac{1}{2y}\right| + \left|\frac{1}{2y} - \frac{1}{2z}\right| + \left|\frac{1}{2z} - \frac{1}{2x}\right|$,
$F\left(G(Tfx, Tfy, Tfz)\right) = \frac{1}{2}\left[\left|\frac{1}{x} - \frac{1}{y}\right| + \left|\frac{1}{y} - \frac{1}{z}\right| + \left|\frac{1}{z} - \frac{1}{x}\right|\right]$, (3.18)
$F\left(G(Tx, Ty, Tz)\right) = G(Tx, Ty, Tz)$, (Since $(F(x) = x)$
$F\left(G(Tx, Ty, Tz)\right) = G\left(\frac{1}{x} + 1, \frac{1}{y} + 1, \frac{1}{z} + 1\right)$, [From (3.16)]
$F\left(G(Tx, Ty, Tz)\right) = \left|\frac{1}{x} + 1 - \frac{1}{y} - 1\right| + \left|\frac{1}{y} + 1 - \frac{1}{z} - 1\right| + \left|\frac{1}{z} + 1 - \frac{1}{x} - 1\right|$, [From (3.17)]
$F\left(G(Tx, Ty, Tz)\right) = \left|\frac{1}{x} - \frac{1}{y}\right| + \left|\frac{1}{y} - \frac{1}{z}\right| + \left|\frac{1}{z} - \frac{1}{x}\right|$ (3.19)

Substitute (3.19) in (3.18) we obtain,

$$F\left(G(Tfx, Tfy, Tfz)\right) = \frac{1}{2} F\left(G(Tx, Ty, Tz)\right),$$

Compare above equation with (3.1), there exist $\alpha = \frac{1}{2} \in [0,1)$ such that

$$F\big(G(Tfx, Tfy, Tfz)\big) \le \alpha\, F\big(G(Tx, Ty, Tz)\big).$$

So f is T_F-contraction and the conditions of Theorem 3.1 hold.

Therefore f has a unique fixed point, that is $\mathbf{0}$.

Example 2: Let $X = \{0\} \cup \{\frac{1}{n} / n \in N\}$ endowed with the Euclidian metric that is

$$d(x,y) = |x - y|.$$

Define $G(x,y,z) = |x-y| + |y-z| + |z-x|$, (3.20)
then (X, G) is a complete G-metric space.

Consider two mappings $T, f: X \to X$ by
$f(0) = 0$ and $f\left(\frac{1}{n}\right) = \frac{1}{n+1}$ for all $n \ge 1$, (3.21)
$T(0) = 0$ and $T\left(\frac{1}{n}\right) = \frac{1}{n^n}$ for all $n \ge 1$. (3.22)
Where T is one to one, subsequentially convergent and continuous.

Define $F: [0,\infty) \to [0,\infty), F(x) = x,$ then $F(x)$ is nondecreasing and continuous from the right and $F^{-1}(0) = 0$.

For $l, m, n \in N$,

Now,

$$F\left(G\left(Tf\left(\frac{1}{l}\right), Tf\left(\frac{1}{m}\right), Tf\left(\frac{1}{n}\right)\right)\right) = G\left(Tf\left(\frac{1}{l}\right), Tf\left(\frac{1}{m}\right), Tf\left(\frac{1}{n}\right)\right), \text{ (Since } (F(x) = x)).$$

$$F\left(G\left(Tf\left(\frac{1}{l}\right), Tf\left(\frac{1}{m}\right), Tf\left(\frac{1}{n}\right)\right)\right) = \left|Tf\left(\frac{1}{l}\right) - Tf\left(\frac{1}{m}\right)\right| + \left|Tf\left(\frac{1}{m}\right) - Tf\left(\frac{1}{n}\right)\right| + \left|Tf\left(\frac{1}{n}\right) - Tf\left(\frac{1}{l}\right)\right| \text{ [From (3.20)]}$$

$$F\left(G\left(Tf\left(\frac{1}{l}\right), Tf\left(\frac{1}{m}\right), Tf\left(\frac{1}{n}\right)\right)\right) = \left|\frac{1}{(l+1)^{l+1}} - \frac{1}{(m+1)^{m+1}}\right| + \left|\frac{1}{(m+1)^{m+1}} - \frac{1}{(n+1)^{n+1}}\right| + \left|\frac{1}{(n+1)^{n+1}} - \frac{1}{(l+1)^{l+1}}\right|. \text{ [From (3.21 \& 3.22)]}$$

We have $\frac{1}{(n+1)^{n+1}} \le \frac{1}{3}\left(\frac{1}{n^n}\right)$ $for\ all\ n \ge 1.$ (3.23)

$$F\left(G\left(Tf\left(\frac{1}{l}\right), Tf\left(\frac{1}{m}\right), Tf\left(\frac{1}{n}\right)\right)\right) \le \left|\frac{1}{3}\left(\frac{1}{l^l}\right) - \frac{1}{3}\left(\frac{1}{m^m}\right)\right| + \left|\frac{1}{3}\left(\frac{1}{m^m}\right) - \frac{1}{3}\left(\frac{1}{n^n}\right)\right| + \left|\frac{1}{3}\left(\frac{1}{n^n}\right) - \frac{1}{3}\left(\frac{1}{l^l}\right)\right|, \text{ [From (3.23)]}$$

$$F\left(G\left(Tf\left(\frac{1}{l}\right), Tf\left(\frac{1}{m}\right), Tf\left(\frac{1}{n}\right)\right)\right) \le \frac{1}{3}\left\{\left|\frac{1}{l^l} - \frac{1}{m^m}\right| + \left|\frac{1}{m^m} - \frac{1}{n^n}\right| + \left|\frac{1}{n^n} - \frac{1}{l^l}\right|\right\} \tag{3.24}$$

$$F\left(G\left(T\left(\frac{1}{l}\right), T\left(\frac{1}{m}\right), T\left(\frac{1}{n}\right)\right)\right) = G\left(T\left(\frac{1}{l}\right), T\left(\frac{1}{m}\right), T\left(\frac{1}{n}\right)\right),$$

$$F\left(G\left(T\left(\frac{1}{l}\right), T\left(\frac{1}{m}\right), T\left(\frac{1}{n}\right)\right)\right) = \left|T\left(\frac{1}{l}\right) - T\left(\frac{1}{m}\right)\right| + \left|T\left(\frac{1}{m}\right) - T\left(\frac{1}{n}\right)\right| + \left|T\left(\frac{1}{n}\right) - T\left(\frac{1}{l}\right)\right|,$$

$$F\left(G\left(T\left(\frac{1}{l}\right), T\left(\frac{1}{m}\right), T\left(\frac{1}{n}\right)\right)\right) = \left|\frac{1}{l^l} - \frac{1}{m^m}\right| + \left|\frac{1}{m^m} - \frac{1}{n^n}\right| + \left|\frac{1}{n^n} - \frac{1}{l^l}\right| \tag{3.25}$$

Substitute (3.25) in (3.24), we obtain

$$F\left(G\left(Tf\left(\frac{1}{l}\right), Tf\left(\frac{1}{m}\right), Tf\left(\frac{1}{n}\right)\right)\right) \le \left(\frac{1}{3}\right) F\left(G\left(T\left(\frac{1}{l}\right), T\left(\frac{1}{m}\right), T\left(\frac{1}{n}\right)\right)\right) \tag{3.26}$$

Compare (3.26) with (3.1), there exist $\alpha = \frac{1}{3} \in [0,1)$ such that

$$F\big(G(Tfx, Tfy, Tfz)\big) \le \alpha\, F\big(G(Tx, Ty, Tz)\big).$$

So f is T_F-contraction and the conditions of Theorem 3.1 hold.

Therefore f has a unique fixed point, that is $\mathbf{0}$.

Theorem 3.2: Let (X, G) be a complete G- metric space, let $f, T: X \to X$ be two self mappings such that T is one to one and graph closed (subseuentially convergent and continuous) and f is T-contraction, that is there exist $\alpha \in [0,1)$ such that for all $x, y, z \in X$,

$$G(Tfx, Tfy, Tfz) \leq \alpha\, G(Tx, Ty, Tz). \tag{3.27}$$

Then f has a unique fixed point in X, also for every $x_0 \in X$, the sequence of iterates $\{f^n x_0\}$ converges to the fixed point.

Proof: By taking $F(x) = x$ in Theorem 3.1, the condition (3.1) reduces to the condition (3.27) and proof follows the Theorem 3.1

Corollary: If $F(x) = Tx = x$ in Theorem 3.1 then we obtain Theorem 2.16.

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