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ON THE LATTICE OF SUBGROUPS

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ABSTRACT

Let $\mathcal{G}$ be the set of all 2×2 non-singular matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, when a, b, c, d are integers modulo p. Then $\mathcal{G}$ is a group under matrix multiplication modulo p, of order $p(p^2-1)\ (p-1)$. Let G be the subgroup of $\mathcal{G}$ defined by G = $\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \epsilon \mathcal{G}; ad - bc = 1\right\}$. Then G is of order $p(p^2-1)$. Let L(G) be the lattice formed by all subgroups of G. In this paper we give the structure of L(G) in the case when p = 5.

On lattice of subgroups of a group

Let L(G) be the lattice formed by all subgroups of a group G. Study on Lattices of subgroups of groups began in the thirties of the 20th century.

A celebrated theorem of O. Ore[12] in 1938 states that "If L(G) is a distributive lattice, any finite set of elements from G generates a cyclic subgroup and vice-versa." Thereafter, subgroup lattice theory has witnessed many contributions namely O. Ore[12], R. Baer[1], K. Iwasawa[8], A. W. Jones[9], Michio Suzuki[11], [2], [15] etc.,

In 1992 Karen M. Gragg and P. S. Kung [6] have attempted to characterize the finite groups with a consistent lattice of subgroups. In that endeavour, they discovered that the lattice of subnormal subgroups of a finite group is consistent and dually semimodular (lower semimodular). A. Vethamanickam has cited from their theorem and has given a counter example in his thesis [16]. Suzuki's [11] results are mainly concerning L-isomorphic groups. i.e, groups whose lattices of subgroups are isomorphic.

Our original attempt was to study some lattice theoretic properties of L(G) where G is the group of 2×2 matrices whose determinant is 1 modulo p, where p is a prime. In this paper we give the structure of L(G) when p = 5. In section 1, we give the preliminary definitions needed for the development of the paper and a lemma for finding the order of G.

1. PRELIMINARIES

Definition 1.1[13]: A partial order on a non-empty set P is a binary relation $\leq$ on P that is reflexive, antisymmetric and transitive. The pair (P,$\leq$) is called a partially ordered set or poset. Poset (P, $\leq$) is totally ordered if every x, y $\in$ P are comparable, that is $x \leq y$ or $y \leq x$. A non – empty subset S of P is a chain in P if S is totally ordered by $\leq$.

Definition 1.2 [13]: Let (P, $\leq$) be aposet and let S $\subseteq$ P. An upper bound for S is an element x $\in$ P for which $s \leq x \forall s \in S$. The least upper bound of S is called the supremum or join of S. A lower bound of S is an element x$\in$ P for which $x \leq s \ \forall \ s \in S$. The greatest lower bound of S is called the infimum or meet of S. Poset (P$\leq$) is called a lattice if every pair x, y $\in$ P has a supremum and an infimum.

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Definition 1.3[10]: Semimodular lattice A lattice L is called semimodular if whenever a covers $a \wedge b$, then $a \vee b$ covers b, for all a, b∈L.

Definition 1.4 [4]: **Interval** For a, b ∈ L, a≤ b, we define the intervals:

The closed interval [a, b] = {x: a ≤ x≤ b}.

The half – open intervals
(a, b] = {x: a < x≤ b}
[a, b) = {x: a ≤ x< b}

The open interval
(a, b) = {x: a < x < b}

Definition 1.5: Join – irreducible element An element 'a' of a lattice L is called join-irreducible if $x \vee y = a$ implies x = a or y = a.

Definition 1.6: Consistent Lattice A lattice L is said to be consistent if whenever j is a join – irreducible element in L, then for every x ∈ L, $x \vee j$ is join – irreducible in the upper interval [x, 1].

Theorem 1.7 [7]: (Lagrange's Theorem) If G is a finite group and H is a subgroup of G, then order of H is a divisor of order of G.

Theorem 1.8 [7]: If G is a finite group and a ∈ G, then order of a is divisor of order of G.

Theorem 1.9 [7]: (Sylow's theorem) If p is a prime number and p^{α} / o(G), then G has a subgroup of order p^{α}. If p^m / o(G), $p^{m+1} \nmid$ o(G), then G has a subgroup of order p^m.

Definition 1.10 [7]: A subgroup of G of order p^m, where p^m/ o(G) but $p^{m+1} \nmid$ o(G), is called a p-sylow subgroup of G.

Theorem 1.11 [7] (Sylow's Theorem)

The number of p-sylow subgroups in G, for a given prime, is of the form 1+kp. In particular, this number is a divisor of o(G), that is 1+kp / o(G).

Result 1.12 [3]: Let p and q be two primes such that p > q. Then if p ≡ 1(mod q), there exists a group of order pq whose centre is {e}. When $p \not\equiv 1(\bmod q)$, a group of order pq if it exists is cyclic.

Definition 1.13 [7]: A subgroup N of G is said to be a normal subgroup of G if for every g ∈ G and n ∈ N, $g\,ng^{-1} \in N$.

Result 1.14: If H is the only subgroup of order o(H) in the finite group G, then H is a normal subgroup of G.

Theorem 1.15 [5]: Let G be a group of order pq, where p and q are distinct primes and p < q. Then G has only one subgroup of order q. This subgroup of order q is normal in G.

Result 1.16 [7]: If N is a normal subgroup of G and H is any subgroup of G, then NH is a subgroup of G.

Result 1.17[7]: If N and M are two normal subgrops of G, then NM is also a normal subgroup of G.

We first prove the following:

Lemma 1.18: Let $\mathcal{G} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in Z_{p,} ad - bc \neq 0\}$. $\mathcal{G}$ forms a group under matrix multiplication modulo p. Let $G = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{G} : ad - bc = 1\right\}$. Then o(G) = $p(p^2 - 1)$.

Proof: We first prove that o($\mathcal{G}$) = $p(p^2 - 1)(p - 1)$.

For that, we first count the number of ways in which ad – bc = 0.

We separate this counting into two cases.

In first case we count the ways when ad = bc = 0.

In second case, we will count the number of ways in which ad = bc$\neq$ 0.

Case-1: ad = bc = 0.

When a = 0, we can choose d in p ways; and when d = 0, we can choose a in p ways. But in this a = d = 0 has been counted twice. Therefore there are 2p – 1 ways in which ad = 0.

Similarly bc = 0 in 2p–1 ways. Thus there are $(2p-1)^2$ ways in which ad = bc = 0.

Case-2: ad = bc $\neq$ 0

Since ad $\neq$ 0, therefore a $\neq$ 0 and d $\neq$ 0. Similarly, b $\neq$ 0 and c $\neq$ 0. We can choose some value of a $\neq$ 0 in p -1 ways; some value of d $\neq$ 0 in p – 1 ways; some value of b $\neq$ 0 in p – 1 ways; then find the value of c with those chosen values of a, d, b. Since ad = bc(mod p) has unique solution in c for non-zero values of a, d, b thus we get unique value of c. Thus ad = bc $\neq$ 0 can be chosen in $(p-1)^3$ ways.

Finally, the number of ways of choosing a, b, c, d with ad – bc $\neq$ 0 equals number of ways of choosing a, b, c, d without any restriction minus the number of ways of choosing a, b, c, d with ad – bc = 0. Thus the number of ways of choosing a, b, c, d with ad – bc $\neq$ 0 is $p^4 - (2p-1)^2 - (p-1)^3$.

On simplification, we get

$$\begin{aligned} O(\mathcal{G}) &= p^4 - 4p^2 + 4p - 1 - p^3 + 3p^2 - 3p + 1 \\ &= p^4 - p^3 - p^2 + p \\ &= p^2(p^2 - p) - 1(p^2 - p) \\ &= (p^2 - p)(p^2 - 1) \\ &= p(p-1)(p^2 - 1) \end{aligned}$$

Next we claim that $o(G) = p(p^2 - 1)$

We separate our counting of a, b, c, d for which ad – bc = 1 in three separate cases.

Case 1: ad = 0: This restrict bc = -1. The number of ways of choosing a, d for which ad = 0 is: 2p – 1. On the other hand, the number of ways choosing b, c for which bc = -1 is: p – 1. Thus when ad = 0, we can choose a, b, c, d in (2p – 1) (p – 1) ways.

Case 2: bc = 0: Analogous to previous case, we get number of ways of choosing a, b, c, d as (2p – 1) (p – 1).

Case 3: ad $\neq$ 0 and bc $\neq$ 0: In this case we get number of ways of choosing a, b, c, d as (p – 1) (p – 1). Thus total number of ways of choosing a, b, c, d for which ad – bc = 1 is 2(2p – 1) (p – 1) + (p – 2) times $(p-1)^2$. On simplifying, we get

$$\begin{aligned} O(G) &= (p-1)[2(2p-1) + (p-1)(p-2)] \\ &= (p-1)[4p - 2 + p^2 - 3p + 2] \\ &= (p-1)(p^2 + p) \\ &= p(p+1)(p-1) \\ &= p(p^2 - 1) \end{aligned}$$

Thus we have proved $o(G) = p(p^2 - 1)$.

2. In this section, we arrange the elements of G according to their orders.

Let $\mathcal{G}$ be the set of all 2 × 2 non-singular matrices over Z_5. Then $\mathcal{G}$ is a group under matrix multiplication modulo 5 and

$$\begin{aligned} o(\mathcal{G}) &= p(p^2 - 1)(p - 1) \\ &= 5(5^2 - 1)(5 - 1) \\ &= 5 \times 24 \times 4 \\ &= 480. \end{aligned}$$

Let G be the subgroup of $\mathcal{G}$ defined by $G = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \epsilon \mathcal{G}; ad - bc = 1\right\}$.

Then $o(G) = p(p^2 - 1) = 5 \times 24 = 120$.

The elements of G according to their orders are as follows.

2.1Element of order 1(one element)

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

2.2. Element of order 2 (one element)

$$\begin{pmatrix}4 & 0\\0 & 4\end{pmatrix}$$

2.3. Elements of order 3 (20 elements)

$\begin{pmatrix}2 & 4\\2 & 2\end{pmatrix},\begin{pmatrix}2 & 1\\3 & 2\end{pmatrix};\begin{pmatrix}2 & 3\\1 & 2\end{pmatrix},\begin{pmatrix}2 & 2\\4 & 2\end{pmatrix};\begin{pmatrix}3 & 1\\2 & 1\end{pmatrix},\begin{pmatrix}1 & 4\\3 & 3\end{pmatrix};\begin{pmatrix}3 & 2\\1 & 1\end{pmatrix},\begin{pmatrix}1 & 3\\4 & 3\end{pmatrix};$
$\begin{pmatrix}1 & 1\\2 & 3\end{pmatrix},\begin{pmatrix}3 & 4\\3 & 1\end{pmatrix};\begin{pmatrix}1 & 2\\1 & 3\end{pmatrix},\begin{pmatrix}3 & 3\\4 & 1\end{pmatrix};\begin{pmatrix}0 & 2\\2 & 4\end{pmatrix},\begin{pmatrix}4 & 3\\3 & 0\end{pmatrix};\begin{pmatrix}4 & 2\\2 & 0\end{pmatrix},\begin{pmatrix}0 & 3\\3 & 4\end{pmatrix};$
$\begin{pmatrix}0 & 1\\4 & 4\end{pmatrix},\begin{pmatrix}4 & 4\\1 & 0\end{pmatrix};\begin{pmatrix}4 & 1\\4 & 0\end{pmatrix},\begin{pmatrix}0 & 4\\1 & 4\end{pmatrix}.$

2.4. Elements of order 4: (30 elements)

$\begin{pmatrix}0 & 2\\2 & 0\end{pmatrix},\begin{pmatrix}0 & 3\\3 & 0\end{pmatrix};\begin{pmatrix}0 & 1\\4 & 0\end{pmatrix},\begin{pmatrix}0 & 4\\1 & 0\end{pmatrix};\begin{pmatrix}2 & 0\\0 & 3\end{pmatrix},\begin{pmatrix}3 & 0\\0 & 2\end{pmatrix};\begin{pmatrix}2 & 2\\0 & 3\end{pmatrix},\begin{pmatrix}3 & 3\\0 & 2\end{pmatrix};$
$\begin{pmatrix}2 & 0\\2 & 3\end{pmatrix},\begin{pmatrix}3 & 0\\3 & 2\end{pmatrix};\begin{pmatrix}3 & 0\\2 & 2\end{pmatrix},\begin{pmatrix}2 & 0\\3 & 3\end{pmatrix};\begin{pmatrix}3 & 2\\0 & 2\end{pmatrix},\begin{pmatrix}2 & 3\\0 & 3\end{pmatrix};\begin{pmatrix}2 & 0\\1 & 3\end{pmatrix},\begin{pmatrix}3 & 0\\4 & 2\end{pmatrix};$
$\begin{pmatrix}2 & 1\\0 & 3\end{pmatrix},\begin{pmatrix}3 & 4\\0 & 2\end{pmatrix};\begin{pmatrix}3 & 1\\0 & 2\end{pmatrix},\begin{pmatrix}2 & 4\\0 & 3\end{pmatrix};\begin{pmatrix}3 & 0\\1 & 2\end{pmatrix},\begin{pmatrix}2 & 0\\4 & 3\end{pmatrix};\begin{pmatrix}1 & 1\\3 & 4\end{pmatrix},\begin{pmatrix}4 & 4\\2 & 1\end{pmatrix};$
$\begin{pmatrix}1 & 3\\1 & 4\end{pmatrix},\begin{pmatrix}4 & 2\\4 & 1\end{pmatrix};\begin{pmatrix}4 & 3\\1 & 1\end{pmatrix},\begin{pmatrix}1 & 2\\4 & 4\end{pmatrix};\begin{pmatrix}4 & 1\\3 & 1\end{pmatrix},\begin{pmatrix}1 & 4\\2 & 4\end{pmatrix}.$

2.5 Elements of order 5: (24 elements)

$\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix},\begin{pmatrix}1 & 0\\4 & 1\end{pmatrix};\begin{pmatrix}1 & 0\\3 & 1\end{pmatrix},\begin{pmatrix}1 & 0\\2 & 1\end{pmatrix};\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix},\begin{pmatrix}1 & 4\\0 & 1\end{pmatrix};\begin{pmatrix}1 & 3\\0 & 1\end{pmatrix},\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix};$
$\begin{pmatrix}2 & 3\\3 & 0\end{pmatrix},\begin{pmatrix}0 & 2\\2 & 2\end{pmatrix};\begin{pmatrix}3 & 1\\1 & 4\end{pmatrix},\begin{pmatrix}4 & 4\\4 & 3\end{pmatrix};\begin{pmatrix}0 & 3\\3 & 2\end{pmatrix},\begin{pmatrix}2 & 2\\2 & 0\end{pmatrix};\begin{pmatrix}4 & 1\\1 & 3\end{pmatrix},\begin{pmatrix}3 & 4\\4 & 4\end{pmatrix};$
$\begin{pmatrix}3 & 3\\2 & 4\end{pmatrix},\begin{pmatrix}4 & 2\\3 & 3\end{pmatrix};\begin{pmatrix}0 & 1\\4 & 2\end{pmatrix},\begin{pmatrix}2 & 4\\1 & 0\end{pmatrix};\begin{pmatrix}3 & 2\\3 & 4\end{pmatrix},\begin{pmatrix}4 & 3\\2 & 3\end{pmatrix};\begin{pmatrix}0 & 4\\1 & 2\end{pmatrix},\begin{pmatrix}2 & 1\\4 & 0\end{pmatrix}.$

2.6. Elements of order 6 : (20 elements)

$\begin{pmatrix}3 & 1\\3 & 3\end{pmatrix},\begin{pmatrix}3 & 4\\2 & 3\end{pmatrix};\begin{pmatrix}3 & 3\\1 & 3\end{pmatrix},\begin{pmatrix}3 & 2\\4 & 3\end{pmatrix};\begin{pmatrix}4 & 1\\2 & 2\end{pmatrix},\begin{pmatrix}2 & 4\\3 & 4\end{pmatrix};\begin{pmatrix}4 & 2\\1 & 2\end{pmatrix},\begin{pmatrix}2 & 3\\4 & 4\end{pmatrix};$
$\begin{pmatrix}2 & 1\\2 & 4\end{pmatrix},\begin{pmatrix}4 & 4\\3 & 2\end{pmatrix};\begin{pmatrix}2 & 2\\1 & 4\end{pmatrix},\begin{pmatrix}4 & 3\\4 & 2\end{pmatrix};\begin{pmatrix}1 & 2\\2 & 0\end{pmatrix},\begin{pmatrix}0 & 3\\3 & 1\end{pmatrix};\begin{pmatrix}0 & 2\\2 & 1\end{pmatrix},\begin{pmatrix}1 & 3\\3 & 0\end{pmatrix};$
$\begin{pmatrix}0 & 4\\1 & 1\end{pmatrix},\begin{pmatrix}1 & 1\\4 & 0\end{pmatrix};\begin{pmatrix}0 & 1\\4 & 1\end{pmatrix},\begin{pmatrix}1 & 4\\1 & 0\end{pmatrix}.$

2.7. Elements of order 10: (24 elements)

$\begin{pmatrix}4 & 0\\1 & 4\end{pmatrix},\begin{pmatrix}4 & 0\\4 & 4\end{pmatrix};\begin{pmatrix}4 & 0\\2 & 4\end{pmatrix},\begin{pmatrix}4 & 0\\3 & 4\end{pmatrix};\begin{pmatrix}4 & 1\\0 & 4\end{pmatrix},\begin{pmatrix}4 & 4\\0 & 4\end{pmatrix};\begin{pmatrix}4 & 2\\0 & 4\end{pmatrix},\begin{pmatrix}4 & 3\\0 & 4\end{pmatrix};$
$\begin{pmatrix}3 & 2\\2 & 0\end{pmatrix},\begin{pmatrix}0 & 3\\3 & 3\end{pmatrix};\begin{pmatrix}1 & 1\\1 & 2\end{pmatrix},\begin{pmatrix}2 & 4\\4 & 1\end{pmatrix};\begin{pmatrix}0 & 2\\2 & 3\end{pmatrix},\begin{pmatrix}3 & 3\\3 & 0\end{pmatrix};\begin{pmatrix}2 & 1\\1 & 1\end{pmatrix},\begin{pmatrix}1 & 4\\4 & 2\end{pmatrix}$
$\begin{pmatrix}2 & 2\\3 & 1\end{pmatrix},\begin{pmatrix}1 & 3\\2 & 2\end{pmatrix};\begin{pmatrix}3 & 1\\4 & 0\end{pmatrix},\begin{pmatrix}0 & 4\\1 & 3\end{pmatrix};\begin{pmatrix}2 & 3\\2 & 1\end{pmatrix},\begin{pmatrix}1 & 2\\3 & 2\end{pmatrix};\begin{pmatrix}3 & 4\\1 & 0\end{pmatrix},\begin{pmatrix}0 & 1\\4 & 3\end{pmatrix}.$

3. In this section, we find all the subgroups of G of different orders.

According to Lagrange's theorem, we need to check only among the divisors of 120 for the orders of the subgroups.

3.1. Subgroups of order 2

Let H be an arbitrary subgroup of G of order 2. Since $\begin{pmatrix}4 & 0\\0 & 4\end{pmatrix}$ is the only element of G of order 2,
$H_1 = \left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}4 & 0\\0 & 4\end{pmatrix}\right\}$ is the only subgroup of order 2.

3.2. Subgroups of order 3

Since 3 is a prime number, then any subgroup of order 3 is cyclic and hence it is generated by an element of order 3. Thus all the subgroups of order 3 are obtained:

$K_1 = \left\{\begin{pmatrix}2 & 4\\2 & 2\end{pmatrix},\begin{pmatrix}2 & 1\\3 & 2\end{pmatrix},e\right\}; K_2 = \left\{\begin{pmatrix}2 & 3\\1 & 2\end{pmatrix},\begin{pmatrix}2 & 2\\4 & 2\end{pmatrix},e\right\}; K_3 = \left\{\begin{pmatrix}3 & 1\\2 & 1\end{pmatrix},\begin{pmatrix}1 & 4\\3 & 3\end{pmatrix},e\right\};$
$K_4 = \left\{\begin{pmatrix}3 & 2\\1 & 1\end{pmatrix},\begin{pmatrix}1 & 3\\4 & 3\end{pmatrix},e\right\}; K_5 = \left\{\begin{pmatrix}1 & 1\\2 & 3\end{pmatrix},\begin{pmatrix}3 & 4\\3 & 1\end{pmatrix},e\right\}; K_6 = \left\{\begin{pmatrix}1 & 2\\1 & 3\end{pmatrix},\begin{pmatrix}3 & 3\\4 & 1\end{pmatrix},e\right\};$
$K_7 = \left\{\begin{pmatrix}0 & 2\\2 & 4\end{pmatrix},\begin{pmatrix}4 & 3\\3 & 0\end{pmatrix},e\right\}; K_8 = \left\{\begin{pmatrix}4 & 2\\2 & 0\end{pmatrix},\begin{pmatrix}0 & 3\\3 & 4\end{pmatrix},e\right\}; K_9 = \left\{\begin{pmatrix}0 & 1\\4 & 4\end{pmatrix},\begin{pmatrix}4 & 4\\1 & 0\end{pmatrix},e\right\};$
$K_{10} = \left\{\begin{pmatrix}4 & 1\\4 & 0\end{pmatrix},\begin{pmatrix}0 & 4\\1 & 4\end{pmatrix},e\right\}.$

Since 3 /o(G), $3^2 \nmid$ o(G) , G has a 3-sylow subgroup of order 3.

The number of 3-sylow subgroups is of the form 1 + 3k and 1 + 3k /o(G). That is, 1 + 3k / $2^3 \times 3 \times 5$.

Therefore, 1+3k / $2^3 \times 5$. The possible values for k are 0, 1, 3.

Therefore, the maximum number of 3-sylow subgroups of order 3 is 10 when k = 3.

So, these are the only subgroups of order 3.

3.3. Subgroups of order 4

Let L be an arbitrary subgroup of G of order 4. Then the elements of L must have order 1, 2 or 4. If L contains an element of order 4, then L is generated by an element of order 4.

Thus all the subgroups of G of order 4 are obtained:

$L_1 = \left\{\begin{pmatrix}0 & 2\\ 2 & 0\end{pmatrix}, \begin{pmatrix}0 & 3\\ 3 & 0\end{pmatrix}, \begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}, e\right\}; L_2 = \left\{\begin{pmatrix}0 & 1\\ 4 & 0\end{pmatrix}, \begin{pmatrix}0 & 4\\ 1 & 0\end{pmatrix}, \begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}, e\right\}$

$L_3 = \left\{\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}, \begin{pmatrix}3 & 0\\ 0 & 2\end{pmatrix}, \begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}, e\right\}; L_4 = \left\{\begin{pmatrix}2 & 2\\ 0 & 3\end{pmatrix}, \begin{pmatrix}3 & 3\\ 0 & 2\end{pmatrix}, \begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}, e\right\}$

$L_5 = \left\{\begin{pmatrix}2 & 0\\ 1 & 3\end{pmatrix}, \begin{pmatrix}3 & 0\\ 4 & 2\end{pmatrix}, \begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}, e\right\}; L_6 = \left\{\begin{pmatrix}1 & 1\\ 3 & 4\end{pmatrix}, \begin{pmatrix}4 & 4\\ 2 & 1\end{pmatrix}, \begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}, e\right\}$

$L_7 = \left\{\begin{pmatrix}2 & 0\\ 2 & 3\end{pmatrix}, \begin{pmatrix}3 & 0\\ 3 & 2\end{pmatrix}, \begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}, e\right\}; L_8 = \left\{\begin{pmatrix}2 & 1\\ 0 & 3\end{pmatrix}, \begin{pmatrix}3 & 4\\ 0 & 2\end{pmatrix}, \begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}, e\right\}$

$L_9 = \left\{\begin{pmatrix}1 & 3\\ 1 & 4\end{pmatrix}, \begin{pmatrix}4 & 2\\ 4 & 1\end{pmatrix}, \begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}, e\right\}; L_{10} = \left\{\begin{pmatrix}3 & 2\\ 0 & 2\end{pmatrix}, \begin{pmatrix}2 & 3\\ 0 & 3\end{pmatrix}, \begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}, e\right\}$

$L_{11} = \left\{\begin{pmatrix}3 & 0\\ 1 & 2\end{pmatrix}, \begin{pmatrix}2 & 0\\ 4 & 3\end{pmatrix}, \begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}, e\right\}; L_{12} = \left\{\begin{pmatrix}4 & 1\\ 3 & 1\end{pmatrix}, \begin{pmatrix}1 & 4\\ 2 & 4\end{pmatrix}, \begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}, e\right\}$

$L_{13} = \left\{\begin{pmatrix}3 & 0\\ 2 & 2\end{pmatrix}, \begin{pmatrix}2 & 0\\ 3 & 3\end{pmatrix}, \begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}, e\right\}; L_{14} = \left\{\begin{pmatrix}3 & 1\\ 0 & 2\end{pmatrix}, \begin{pmatrix}2 & 4\\ 0 & 3\end{pmatrix}, \begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}, e\right\}$

$L_{15} = \left\{\begin{pmatrix}4 & 3\\ 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 2\\ 4 & 4\end{pmatrix}, \begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}, e\right\}.$

Since the cube of an element of order 4 is another element of order 4, and we have 30 elements of order 4 and only one element of order 2, the fifteen pairs of elements of order 4 give the fifteen 4 – element subgroups.

Here $H_1 \subset L_i$ for all i.

3.4. Subgroups of order 5:

Since |G| = $2^3 \times 3 \times 5$, 5 / o(G) but $5^2 \nmid$ o(G). Therefore, G has a 5-sylow subgroup of order 5. The number of 5-sylow subgroups of G of order 5 is of the form 1 + 5k and 1 + 5k / o(G). That is, 1 + 5k / 24, hence the possible values of k are 0, 1.

Therefore the maximum number of 5-sylow subgroups of order 5 is 6 when k = 1 and the following are the only subgroups of G of order 5.

The subgroups are

$M_1 = \left\{\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 4 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 3 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}, e\right\}$

$M_2 = \left\{\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 4\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}, e\right\}$

$M_3 = \left\{\begin{pmatrix}2 & 3\\ 3 & 0\end{pmatrix}, \begin{pmatrix}0 & 2\\ 2 & 2\end{pmatrix}, \begin{pmatrix}3 & 1\\ 1 & 4\end{pmatrix}, \begin{pmatrix}4 & 4\\ 4 & 3\end{pmatrix}, e\right\}$

$M_4 = \left\{\begin{pmatrix}0 & 3\\ 3 & 2\end{pmatrix}, \begin{pmatrix}2 & 2\\ 2 & 0\end{pmatrix}, \begin{pmatrix}4 & 1\\ 1 & 3\end{pmatrix}, \begin{pmatrix}3 & 4\\ 4 & 4\end{pmatrix}, e\right\}$

$M_5 = \left\{\begin{pmatrix}3 & 3\\ 2 & 4\end{pmatrix}, \begin{pmatrix}4 & 2\\ 3 & 3\end{pmatrix}, \begin{pmatrix}0 & 1\\ 4 & 2\end{pmatrix}, \begin{pmatrix}2 & 4\\ 1 & 0\end{pmatrix}, e\right\}$

$M_6 = \left\{\begin{pmatrix}3 & 2\\ 3 & 4\end{pmatrix}, \begin{pmatrix}4 & 3\\ 2 & 3\end{pmatrix}, \begin{pmatrix}0 & 4\\ 1 & 2\end{pmatrix}, \begin{pmatrix}2 & 1\\ 4 & 0\end{pmatrix}, e\right\}$

3.5. Subgroups of order 6

Let N be an arbitrary subgroup of G of order 6. Since |N| = 2 × 3, by theorem 1.15 N has exactly one subgroup of order 3. Also, if N contains an element of order 6, then N is generated by an element of order 6. The subgroup of order 6 are

$N_1 = \left\{\begin{pmatrix}2 & 4\\ 2 & 2\end{pmatrix}, \begin{pmatrix}2 & 1\\ 3 & 2\end{pmatrix}, \begin{pmatrix}3 & 1\\ 3 & 3\end{pmatrix}, \begin{pmatrix}3 & 4\\ 2 & 3\end{pmatrix}, \begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}, e\right\}$

$N_2 = \left\{\begin{pmatrix}2 & 3\\ 1 & 2\end{pmatrix}, \begin{pmatrix}2 & 2\\ 4 & 2\end{pmatrix}, \begin{pmatrix}3 & 3\\ 1 & 3\end{pmatrix}, \begin{pmatrix}3 & 2\\ 4 & 3\end{pmatrix}, \begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}, e\right\}$

$N_3 = \left\{\begin{pmatrix}3 & 1\\ 2 & 1\end{pmatrix}, \begin{pmatrix}1 & 4\\ 3 & 3\end{pmatrix}, \begin{pmatrix}4 & 1\\ 2 & 2\end{pmatrix}, \begin{pmatrix}2 & 4\\ 3 & 4\end{pmatrix}, \begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}, e\right\}$

$N_4 = \left\{\begin{pmatrix}3&2\\1&1\end{pmatrix},\begin{pmatrix}1&3\\4&3\end{pmatrix},\begin{pmatrix}4&2\\1&2\end{pmatrix},\begin{pmatrix}2&3\\4&4\end{pmatrix},\begin{pmatrix}4&0\\0&4\end{pmatrix},e\right\}$
$N_5 = \left\{\begin{pmatrix}1&1\\2&3\end{pmatrix},\begin{pmatrix}3&4\\3&1\end{pmatrix},\begin{pmatrix}2&1\\2&4\end{pmatrix},\begin{pmatrix}4&4\\3&2\end{pmatrix},\begin{pmatrix}4&0\\0&4\end{pmatrix},e\right\}$
$N_6 = \left\{\begin{pmatrix}1&2\\1&3\end{pmatrix},\begin{pmatrix}3&3\\4&1\end{pmatrix},\begin{pmatrix}2&2\\1&4\end{pmatrix},\begin{pmatrix}4&3\\4&2\end{pmatrix},\begin{pmatrix}4&0\\0&4\end{pmatrix},e\right\}$
$N_7 = \left\{\begin{pmatrix}0&2\\2&4\end{pmatrix},\begin{pmatrix}4&3\\3&0\end{pmatrix},\begin{pmatrix}1&2\\2&0\end{pmatrix},\begin{pmatrix}0&3\\3&1\end{pmatrix},\begin{pmatrix}4&0\\0&4\end{pmatrix},e\right\}$
$N_8 = \left\{\begin{pmatrix}4&2\\2&0\end{pmatrix},\begin{pmatrix}0&3\\3&4\end{pmatrix},\begin{pmatrix}0&2\\2&1\end{pmatrix},\begin{pmatrix}1&3\\3&0\end{pmatrix},\begin{pmatrix}4&0\\0&4\end{pmatrix},e\right\}$
$N_9 = \left\{\begin{pmatrix}0&1\\4&4\end{pmatrix},\begin{pmatrix}4&4\\1&0\end{pmatrix},\begin{pmatrix}0&4\\1&1\end{pmatrix},\begin{pmatrix}1&1\\4&0\end{pmatrix},\begin{pmatrix}4&0\\0&4\end{pmatrix},e\right\}$
$N_{10} = \left\{\begin{pmatrix}4&1\\4&0\end{pmatrix},\begin{pmatrix}0&4\\1&4\end{pmatrix},\begin{pmatrix}1&4\\1&0\end{pmatrix},\begin{pmatrix}0&1\\4&1\end{pmatrix},\begin{pmatrix}4&0\\0&4\end{pmatrix},e\right\}$

Since each subgroup of order 6 contains two elements of order 6 and we have only 20 elements of order 6 and ten subgroups of order 3, there will be no other subgroups of order 6 except the above ten.
Here $K_i \subset N_i$ for all i.

3.6. Subgroups of order 8
Since $|G| = 2^3 \times 3 \times 5$, 2^3 / o(G) but $2^4 \nmid$ o(G).Therefore G has a 2-sylow subgroup of order 8. The number of 2-sylow subgroups is of the form 1 + 2k and 1 + 2k / o(G).

That is, $1+2k / 2^3 \times 3 \times 5$. Therefore 1+2k / 15. The possible values of k are 0, 1, 2.

Hence the maximum number of subgroups of order 8 is 5 when k = 2.

Since G has no element of order 8, the elements of subgroups of order 8 must have order 1, 2 or 4.

The five subgroups of order 8 are
$P_1 = \left\{\begin{pmatrix}0&2\\2&0\end{pmatrix},\begin{pmatrix}0&3\\3&0\end{pmatrix},\begin{pmatrix}0&1\\4&0\end{pmatrix},\begin{pmatrix}0&4\\1&0\end{pmatrix},\begin{pmatrix}2&0\\0&3\end{pmatrix},\begin{pmatrix}3&0\\0&2\end{pmatrix},\begin{pmatrix}4&0\\0&4\end{pmatrix},e\right\}$
$P_2 = \left\{\begin{pmatrix}2&2\\0&3\end{pmatrix},\begin{pmatrix}3&3\\0&2\end{pmatrix},\begin{pmatrix}2&0\\1&3\end{pmatrix},\begin{pmatrix}3&0\\4&2\end{pmatrix},\begin{pmatrix}1&1\\3&4\end{pmatrix},\begin{pmatrix}4&4\\2&1\end{pmatrix},\begin{pmatrix}4&0\\0&4\end{pmatrix},e\right\}$
$P_3 = \left\{\begin{pmatrix}2&0\\2&3\end{pmatrix},\begin{pmatrix}3&0\\3&2\end{pmatrix},\begin{pmatrix}2&1\\0&3\end{pmatrix},\begin{pmatrix}3&4\\0&2\end{pmatrix},\begin{pmatrix}1&3\\1&4\end{pmatrix},\begin{pmatrix}4&2\\4&1\end{pmatrix},\begin{pmatrix}4&0\\0&4\end{pmatrix},e\right\}$
$P_4 = \left\{\begin{pmatrix}3&2\\0&2\end{pmatrix},\begin{pmatrix}2&3\\0&3\end{pmatrix},\begin{pmatrix}3&0\\1&2\end{pmatrix},\begin{pmatrix}2&0\\4&3\end{pmatrix},\begin{pmatrix}4&1\\3&1\end{pmatrix},\begin{pmatrix}1&4\\2&4\end{pmatrix},\begin{pmatrix}4&0\\0&4\end{pmatrix},e\right\}$
$P_5 = \left\{\begin{pmatrix}3&0\\2&2\end{pmatrix},\begin{pmatrix}2&0\\3&3\end{pmatrix},\begin{pmatrix}3&1\\0&2\end{pmatrix},\begin{pmatrix}2&4\\0&3\end{pmatrix},\begin{pmatrix}4&3\\1&1\end{pmatrix},\begin{pmatrix}1&2\\4&4\end{pmatrix},\begin{pmatrix}4&0\\0&4\end{pmatrix},e\right\}$

Here $L_1, L_2, L_3 \subset P_1$; $L_4, L_5, L_6 \subset P_2$; $L_7, L_8, L_9 \subset P_3$; $L_{10}, L_{11}, L_{12} \subset P_4$; $L_{13}, L_{14}, L_{15} \subset P_5$

3, 7. Subgroups of order 10
Let Q be an arbitrary subgroup of order 10. Since $|Q| = 2 \times 5$, by theorem 1.15Q has exactly one subgroup of order 5. Also, if Q contains an element of order 10, then Q is generated by an element of order 10.

The subgroups of order 10 are
$Q_1 = \left\{\begin{pmatrix}1&0\\1&1\end{pmatrix},\begin{pmatrix}1&0\\4&1\end{pmatrix},\begin{pmatrix}1&0\\3&1\end{pmatrix},\begin{pmatrix}1&0\\2&1\end{pmatrix},\begin{pmatrix}4&0\\1&4\end{pmatrix},\begin{pmatrix}4&0\\4&4\end{pmatrix},\begin{pmatrix}4&0\\2&4\end{pmatrix},\begin{pmatrix}4&0\\3&4\end{pmatrix},\begin{pmatrix}4&0\\0&4\end{pmatrix},e\right\}$
$Q_2 = \left\{\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}1&4\\0&1\end{pmatrix},\begin{pmatrix}1&3\\0&1\end{pmatrix},\begin{pmatrix}1&2\\0&1\end{pmatrix},\begin{pmatrix}4&1\\0&4\end{pmatrix},\begin{pmatrix}4&4\\0&4\end{pmatrix},\begin{pmatrix}4&2\\0&4\end{pmatrix},\begin{pmatrix}4&3\\0&4\end{pmatrix},\begin{pmatrix}4&0\\0&4\end{pmatrix},e\right\}$
$Q_3 = \left\{\begin{pmatrix}2&3\\3&0\end{pmatrix},\begin{pmatrix}0&2\\2&2\end{pmatrix},\begin{pmatrix}3&1\\1&4\end{pmatrix},\begin{pmatrix}4&4\\4&3\end{pmatrix},\begin{pmatrix}3&2\\2&0\end{pmatrix},\begin{pmatrix}0&3\\3&3\end{pmatrix},\begin{pmatrix}1&1\\1&2\end{pmatrix},\begin{pmatrix}2&4\\4&1\end{pmatrix},\begin{pmatrix}4&0\\0&4\end{pmatrix},e\right\}$
$Q_4 = \left\{\begin{pmatrix}0&3\\3&2\end{pmatrix},\begin{pmatrix}2&2\\2&0\end{pmatrix},\begin{pmatrix}4&1\\1&3\end{pmatrix},\begin{pmatrix}3&4\\4&4\end{pmatrix},\begin{pmatrix}0&2\\2&3\end{pmatrix},\begin{pmatrix}3&3\\3&0\end{pmatrix},\begin{pmatrix}2&1\\1&1\end{pmatrix},\begin{pmatrix}1&4\\4&2\end{pmatrix},\begin{pmatrix}4&0\\0&4\end{pmatrix},e\right\}$
$Q_5 = \left\{\begin{pmatrix}3&3\\2&4\end{pmatrix},\begin{pmatrix}4&2\\3&3\end{pmatrix},\begin{pmatrix}0&1\\4&2\end{pmatrix},\begin{pmatrix}2&4\\1&0\end{pmatrix},\begin{pmatrix}2&2\\3&1\end{pmatrix},\begin{pmatrix}1&3\\2&2\end{pmatrix},\begin{pmatrix}3&1\\4&0\end{pmatrix},\begin{pmatrix}0&4\\1&3\end{pmatrix},\begin{pmatrix}4&0\\0&4\end{pmatrix},e\right\}$
$Q_6 = \left\{\begin{pmatrix}3&2\\3&4\end{pmatrix},\begin{pmatrix}4&3\\2&3\end{pmatrix},\begin{pmatrix}0&4\\1&2\end{pmatrix},\begin{pmatrix}2&1\\4&0\end{pmatrix},\begin{pmatrix}2&3\\2&1\end{pmatrix},\begin{pmatrix}1&2\\3&2\end{pmatrix},\begin{pmatrix}3&4\\1&0\end{pmatrix},\begin{pmatrix}0&1\\4&3\end{pmatrix},\begin{pmatrix}4&0\\0&4\end{pmatrix},e\right\}$

Since each subgroup of order 10 contains exactly four elements of order 10 and we have only 24 elements of order 10 and six subgroups of order 5, there will be no other subgroups of order 10 except the above six. Here $M_i \subset Q_i$ for all i.

3.8. Subgroups of order 12
Let R be an arbitrary subgroup of order 12. Since $|R| = 2^2 \times 3$, the number of subgroups of R of order 4 is 1 + 2k and 1 + 2k / 3. Therefore, the possible values of k are 0, 1.

Hence the number of subgroups of R of order 4 is either 1 or 3. Similarly, the number of subgroups of R of order 3 is 1 + 3k and 1 + 3k / 4. Therefore, the possible values of k are 0, 1.

Hence the number of subgroups of R of order 3 is either 1 or 4.There are four possibilities.

(i) The number of subgroups of order 4 is 3 and of order 3 is 4.
(ii) The number of subgroups of order 4 is 1 and of order 3 is 1.
(iii) The number of subgroups of order 4 is 1 and of order 3 is 4.
(iv) The number of subgroups of order 4 is 3 and of order 3 is 1.

Case-(i): cannot occur, because three subgroups of order 4 and four subgroups of order 3 contain more than 12 elements.

Case-(ii): Let the one subgroup of order 4 be L and the one subgroup of order 3 be K. Then L and K are normal in R. Hence R = KL must be abelian which is not true by checking all possibilities of K and L. Thus we get a conclusion that this case can not occur.

Case-(iii): Let $\mathcal{A}$ be a collection of four subgroups of order 3 and let L be a subgroup of order 4. Since L is the only subgroup in R of order 4, L is normal in R. Therefore, $r^{-1}lr \in L$ for all $r \in R$, $l \in L$. By taking a subgroup of order 4 at a time, combining this with four subgroups of order 3, we conclude this condition is not held. This means that this case can not occur.

Case-(iv) Taking a subgroup of order 3 at a time, combining this with three subgroups of order 4, we are able to determine the following ten subgroups of order 12 by trial.

$$R_1 = \left\{\begin{matrix} \begin{pmatrix} 2 & 4 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 4 & 1 \end{pmatrix} \\ \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 4 & 4 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 3 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, e \end{matrix}\right\}$$

$$R_2 = \left\{\begin{matrix} \begin{pmatrix} 2 & 2 \\ 4 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 4 \\ 2 & 1 \end{pmatrix}, \\ \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, e \end{matrix}\right\}$$

$$R_3 = \left\{\begin{matrix} \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 3 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 4 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 0 & 3 \end{pmatrix}, \\ \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 4 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, e \end{matrix}\right\}$$

$$R_4 = \left\{\begin{matrix} \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 4 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 3 & 3 \end{pmatrix}, \\ \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 4 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, e \end{matrix}\right\}$$

$$R_5 = \left\{\begin{matrix} \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 0 & 2 \end{pmatrix}, \\ \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 4 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, e \end{matrix}\right\}$$

$$R_6 = \left\{\begin{matrix} \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 3 & 2 \end{pmatrix}, \\ \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 4 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 4 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, e \end{matrix}\right\}$$

$$R_7 = \left\{\begin{matrix} \begin{pmatrix} 0 & 2 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix}, \\ \begin{pmatrix} 2 & 4 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, e \end{matrix}\right\}$$

$$R_8 = \left\{\begin{matrix} \begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 0 & 2 \end{pmatrix}, \\ \begin{pmatrix} 3 & 0 \\ 4 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, e \end{matrix}\right\}$$

$$R_9 = \left\{\begin{matrix} \begin{pmatrix} 0 & 1 \\ 4 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 4 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 0 & 2 \end{pmatrix}, \\ \begin{pmatrix} 3 & 0 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 3 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 4 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 4 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, e \end{matrix}\right\}$$

$$R_{10} = \left\{\begin{matrix} \begin{pmatrix} 0 & 4 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 3 & 2 \end{pmatrix}, \\ \begin{pmatrix} 3 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, e \end{matrix}\right\}$$

Here, $L_3, L_9, L_{15}, N_1 \subset R_1$; $L_3, L_6, L_{12}, N_2 \subset R_2$; $L_5, L_{10}, L_{15}, N_3 \subset R_3$; $L_8, L_{13}, L_{12}, N_4 \subset R_4$; $L_4, L_9, L_{11}, N_5 \subset R_5$; $L_6, L_7, L_{14}, N_6 \subset R_6$; $L_2, L_{11}, L_{14}, N_7 \subset R_7$; $L_2, L_5, L_8, N_8 \subset R_8$; $L_1, L_4, L_{13}, N_9 \subset R_9$; $L_1, L_7, L_{10}, N_{10} \subset R_{10}$;

3.8. Subgroups of order 15

Since $15 = 3 \times 5$ and $5 \not\equiv 1 \bmod 3$, a group of order 15 is cyclic [3]. Therefore, it must be generated by an element of order 15. But in G there is no element of order 15. Hence a subgroup of G of order 15 does not exist.

3.10.Subgroups of order 20

Let T be an arbitrary subgroup of order 20. Since $|T| = 2^2 \times 3$, the number of subgroups of order 4 in T is $1 + 2k$ and $1 + 2k / 5$. The possible values of k are 0, 2. Hence the number of subgroups of T of order 4 is either 1 or 5.

Similarly, the number of subgroups of T of order 5 is $1 + 5k$ and $1 + 5k / 4$. The possible value of k is 0 only. Hence the number of subgroups of T of order 5 is 1. There are two possibilities.

(i) The number of subgroups of order 4 is 1 and of order 5 is 1.
(ii) The number of subgroups of order 4 is 5 and of order 5 is 1.

Case-(i): Let the one subgroup of order 4 in T be L and the one subgroup of order 5 in T be M. Then L and M are normal in T. Hence T = LM must be abelian, but it is not true by checking all possibilities of L and M. Thus we get a conclusion that this case cannot occur.

Case-(ii): Taking a subgroup of order 5 at a time, combining this with five subgroups of order 4, we are able to determine the following six subgroups of order 20 by trial.

Clearly since each subgroup of order 20 contains a subgroup of order 5, each contains a subgroup of order 10.

$$T_1 = \left\{\begin{array}{c} \begin{pmatrix}2&0\\0&3\end{pmatrix},\begin{pmatrix}3&0\\0&2\end{pmatrix},\begin{pmatrix}2&0\\1&3\end{pmatrix},\begin{pmatrix}3&0\\4&2\end{pmatrix},\begin{pmatrix}2&0\\2&3\end{pmatrix},\begin{pmatrix}3&0\\3&2\end{pmatrix},\begin{pmatrix}2&0\\4&3\end{pmatrix}, \\ \begin{pmatrix}3&0\\1&2\end{pmatrix},\begin{pmatrix}2&0\\3&3\end{pmatrix},\begin{pmatrix}3&0\\2&2\end{pmatrix},\begin{pmatrix}1&0\\1&1\end{pmatrix},\begin{pmatrix}1&0\\4&1\end{pmatrix},\begin{pmatrix}1&0\\3&1\end{pmatrix},\begin{pmatrix}1&0\\2&1\end{pmatrix}, \\ \begin{pmatrix}4&0\\1&4\end{pmatrix},\begin{pmatrix}4&0\\4&4\end{pmatrix},\begin{pmatrix}4&0\\2&4\end{pmatrix},\begin{pmatrix}4&0\\3&4\end{pmatrix},\begin{pmatrix}4&0\\0&4\end{pmatrix},e \end{array}\right\}$$

$$T_2 = \left\{\begin{array}{c} \begin{pmatrix}2&0\\0&3\end{pmatrix},\begin{pmatrix}3&0\\0&2\end{pmatrix},\begin{pmatrix}2&1\\0&3\end{pmatrix},\begin{pmatrix}3&4\\0&2\end{pmatrix},\begin{pmatrix}2&2\\0&3\end{pmatrix},\begin{pmatrix}3&3\\0&2\end{pmatrix},\begin{pmatrix}2&4\\0&3\end{pmatrix}, \\ \begin{pmatrix}3&1\\0&2\end{pmatrix},\begin{pmatrix}2&3\\0&3\end{pmatrix},\begin{pmatrix}3&2\\0&2\end{pmatrix},\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}1&4\\0&1\end{pmatrix},\begin{pmatrix}1&3\\0&1\end{pmatrix},\begin{pmatrix}1&2\\0&1\end{pmatrix}, \\ \begin{pmatrix}4&1\\0&4\end{pmatrix},\begin{pmatrix}4&4\\0&4\end{pmatrix},\begin{pmatrix}4&2\\0&4\end{pmatrix},\begin{pmatrix}4&3\\0&4\end{pmatrix},\begin{pmatrix}4&0\\0&4\end{pmatrix},e \end{array}\right\}$$

$$T_3 = \left\{\begin{array}{c} \begin{pmatrix}0&1\\4&0\end{pmatrix},\begin{pmatrix}0&4\\1&0\end{pmatrix},\begin{pmatrix}2&0\\2&3\end{pmatrix},\begin{pmatrix}3&0\\3&2\end{pmatrix},\begin{pmatrix}2&2\\0&3\end{pmatrix},\begin{pmatrix}3&3\\0&2\end{pmatrix},\begin{pmatrix}4&1\\3&1\end{pmatrix}, \\ \begin{pmatrix}1&4\\2&4\end{pmatrix},\begin{pmatrix}4&3\\1&1\end{pmatrix},\begin{pmatrix}1&2\\4&4\end{pmatrix},\begin{pmatrix}2&3\\3&0\end{pmatrix},\begin{pmatrix}0&2\\2&2\end{pmatrix},\begin{pmatrix}3&1\\1&4\end{pmatrix},\begin{pmatrix}4&4\\4&3\end{pmatrix}, \\ \begin{pmatrix}3&2\\2&0\end{pmatrix},\begin{pmatrix}0&3\\3&3\end{pmatrix},\begin{pmatrix}2&4\\4&1\end{pmatrix},\begin{pmatrix}1&1\\1&2\end{pmatrix},\begin{pmatrix}4&0\\0&4\end{pmatrix},e \end{array}\right\}$$

$$T_4 = \left\{\begin{array}{c} \begin{pmatrix}0&1\\4&0\end{pmatrix},\begin{pmatrix}0&4\\1&0\end{pmatrix},\begin{pmatrix}3&2\\0&2\end{pmatrix},\begin{pmatrix}2&3\\0&3\end{pmatrix},\begin{pmatrix}3&0\\2&2\end{pmatrix},\begin{pmatrix}2&0\\3&3\end{pmatrix},\begin{pmatrix}1&3\\1&4\end{pmatrix}, \\ \begin{pmatrix}4&2\\4&1\end{pmatrix},\begin{pmatrix}1&1\\3&4\end{pmatrix},\begin{pmatrix}4&4\\2&1\end{pmatrix},\begin{pmatrix}0&3\\3&2\end{pmatrix},\begin{pmatrix}2&2\\2&0\end{pmatrix},\begin{pmatrix}4&1\\1&3\end{pmatrix},\begin{pmatrix}3&4\\4&4\end{pmatrix}, \\ \begin{pmatrix}0&2\\2&3\end{pmatrix},\begin{pmatrix}3&3\\3&0\end{pmatrix},\begin{pmatrix}1&4\\4&2\end{pmatrix},\begin{pmatrix}2&1\\1&1\end{pmatrix},\begin{pmatrix}4&0\\0&4\end{pmatrix},e \end{array}\right\}$$

$$T_5 = \left\{\begin{array}{c} \begin{pmatrix}0&2\\2&0\end{pmatrix},\begin{pmatrix}0&3\\3&0\end{pmatrix},\begin{pmatrix}3&4\\0&2\end{pmatrix},\begin{pmatrix}2&1\\0&3\end{pmatrix},\begin{pmatrix}3&0\\1&2\end{pmatrix},\begin{pmatrix}2&0\\4&3\end{pmatrix},\begin{pmatrix}1&1\\3&4\end{pmatrix}, \\ \begin{pmatrix}4&4\\2&1\end{pmatrix},\begin{pmatrix}4&3\\1&1\end{pmatrix},\begin{pmatrix}1&2\\4&4\end{pmatrix},\begin{pmatrix}3&3\\2&4\end{pmatrix},\begin{pmatrix}4&2\\3&3\end{pmatrix},\begin{pmatrix}0&1\\4&2\end{pmatrix},\begin{pmatrix}2&4\\1&0\end{pmatrix}, \\ \begin{pmatrix}2&2\\3&1\end{pmatrix},\begin{pmatrix}1&3\\2&2\end{pmatrix},\begin{pmatrix}3&1\\4&0\end{pmatrix},\begin{pmatrix}0&4\\1&3\end{pmatrix},\begin{pmatrix}4&0\\0&4\end{pmatrix},e \end{array}\right\}$$

$$T_6 = \left\{\begin{array}{c} \begin{pmatrix}0&2\\2&0\end{pmatrix},\begin{pmatrix}0&3\\3&0\end{pmatrix},\begin{pmatrix}3&0\\4&2\end{pmatrix},\begin{pmatrix}2&0\\1&3\end{pmatrix},\begin{pmatrix}3&1\\0&2\end{pmatrix},\begin{pmatrix}2&4\\0&3\end{pmatrix},\begin{pmatrix}1&3\\1&4\end{pmatrix}, \\ \begin{pmatrix}4&2\\4&1\end{pmatrix},\begin{pmatrix}4&1\\3&1\end{pmatrix},\begin{pmatrix}1&4\\2&4\end{pmatrix},\begin{pmatrix}3&2\\3&4\end{pmatrix},\begin{pmatrix}4&3\\2&3\end{pmatrix},\begin{pmatrix}0&4\\1&2\end{pmatrix},\begin{pmatrix}2&1\\4&0\end{pmatrix}, \\ \begin{pmatrix}2&3\\2&1\end{pmatrix},\begin{pmatrix}1&2\\3&2\end{pmatrix},\begin{pmatrix}3&4\\1&0\end{pmatrix},\begin{pmatrix}0&1\\4&3\end{pmatrix},\begin{pmatrix}4&0\\0&4\end{pmatrix},e \end{array}\right\}$$

Here, $L_3, L_5, L_7, L_{11}, L_{13}, Q_1 \subset T_1$; $L_3, L_8, L_4, L_{14}, L_{10}, Q_2 \subset T_2$;
$L_2, L_7, L_4, L_{12}, L_{15}, Q_3 \subset T_3$; $L_2, L_{10}, L_{13}, L_9, L_6, Q_4 \subset T_4$;
$L_1, L_8, L_{11}, L_6, L_{15}, Q_5 \subset T_5$; $L_1, L_5, L_{14}, L_9, L_{12}, Q_6 \subset T_6$;

3.11. Subgroups of order 24

Let S be an arbitrary subgroup of order 24. Since $|S| = 2^3 \times 3$, the number of 2-sylow subgroups of order 8 in S is $1 + 2k$ and $1 + 2k / 3$. The possible values for k are 0, 1. Hence, the number of subgroups of S of order 8 is either 1 or 3.

Similarly, the number of subgroups of order 3 in S is 1+3k and 1+3k / 2^3. The possible values of k are 0, 1. Hence, the number of subgroups of order 3 in S is either 1 or 4.

There are four possibilities.

(i) The number of subgroups of order 8 is 1 and of order 3 is 1.
(ii) The number of subgroups of order 8 is 3 and of order 3 is 1.
(iii) The number of subgroups of order 8 is 3 and of order 3 is 4.
(iv) The number of subgroups of order 8 is 1 and of order 3 is 4.

Case-(i): Let the one subgroup of order 8 in S be P and the one subgroup of order 3 in S be K. Then K and P are normal in S. Hence S = KP must be abelian; but we find that it is not true by checking all possibilities of K and P. Thus we get a conclusion that this case can not occur.

Case-(ii): Let$\mathcal{A}$ be a collection of all the three subgroups of order 8. Let K be a subgroup of order 3. Then K is normal in S since K is the only subgroup in S of order 3. Therefore, $sks^{-1} \in K$ for all $s \in S$ and $k \in K$. By taking a subgroup of order 3 at a time, combining this with three subgroups of order 8, we conclude this condition is not held. This means that this case cannot occur.

Case-(iii): cannot occur, because three subgroups of order 8 and four subgroups of order 3 contain more than 24 elements.

Case-(iv): Taking a subgroup of order 8 at a time, combining this with four subgroups order 3, we are able to determine the following five subgroups of order 24 by trial. Clearly since each subgroup contains four subgroups of order 3, each contains four subgroups of order 6.

$$S_1 = \left\{ \begin{matrix} \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 3 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}, \\ \begin{pmatrix} 3 & 3 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, \\ \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, e, \quad \begin{pmatrix} 4 & 1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 4 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 4 \end{pmatrix}, \\ \begin{pmatrix} 4 & 4 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 4 & 2 \end{pmatrix} \end{matrix} \right\}$$

$$S_2 = \left\{ \begin{matrix} \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 4 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 4 & 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & 4 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 4 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 4 \\ 2 & 1 \end{pmatrix}, \\ \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, e, \quad \begin{pmatrix} 3 & 1 \\ 3 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 4 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & 3 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 4 & 1 \end{pmatrix} \end{matrix} \right\}$$

$$S_3 = \left\{ \begin{matrix} \begin{pmatrix} 2 & 2 \\ 4 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 3 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 4 \\ 1 & 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & 1 \\ 4 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 4 & 1 \end{pmatrix}, \\ \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, e, \quad \begin{pmatrix} 3 & 3 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & 3 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 4 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 4 & 0 \end{pmatrix} \end{matrix} \right\}$$

$$S_4 = \left\{ \begin{matrix} \begin{pmatrix} 2 & 4 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 4 & 4 \end{pmatrix}, \\ \begin{pmatrix} 4 & 4 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 2 & 4 \end{pmatrix}, \\ \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, e, \quad \begin{pmatrix} 3 & 1 \\ 3 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 4 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}, \\ \begin{pmatrix} 1 & 3 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 4 \\ 1 & 1 \end{pmatrix} \end{matrix} \right\}$$

$$S_5 = \left\{ \begin{array}{l} \begin{pmatrix} 2 & 2 \\ 4 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 4 \\ 1 & 4 \end{pmatrix}, \\ \begin{pmatrix} 4 & 1 \\ 4 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 3 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 4 & 4 \end{pmatrix}, \\ \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, e, \quad \begin{pmatrix} 3 & 3 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 4 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}, \\ \qquad \begin{pmatrix} 1 & 3 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 4 & 1 \end{pmatrix} \end{array} \right\}$$

Here, $N_3, N_4, N_5, N_6, P_1 \subset S_1$; $N_1, N_4, N_7, N_{10}, P_2 \subset S_2$
$N_2, N_3, N_7, N_9, P_3 \subset S_3$; $N_1, N_6, N_8, N_9, P_4 \subset S_4$
$N_2, N_5, N_8, N_{10}, P_5 \subset S_5$.

3.12. Subgroups of order 30:
Let U be an arbitrary subgroup of order 30. Since $|U| = 2 \times 3 \times 5$, by multiplying a subgroup of order 3 and a subgroup of order 10 or by multiplying a subgroup of order 5 and a subgroup of order 6, that is by finding K_iQ_j or M_iN_j for all i, j, we get in each case an element of order 4 which can not exist in a subgroup of order 30. Hence, a subgroup of order 30 can not exist.

3.13. Subgroups of order 40:
Let V be an arbitrary subgroup of order 40. Since $|V| = 2^3 \times 5 = 8 \times 5 = 4 \times 10$, by multiplying a subgroup of order 5 and a subgroup of order 8, that is, by finding M_iP_j for all i,j, We get in each case elements of order 3 or 6 which cannot exist in a subgroup of order 40.

Also, by multiplying a subgroup of order 4 and a subgroup of order 10, that is by finding L_iQ_j for all i, j, we get in each element of order 3 or 6 which cannot exist in a subgroup of order 40.

Hence a subgroup of order 40 can not exist.

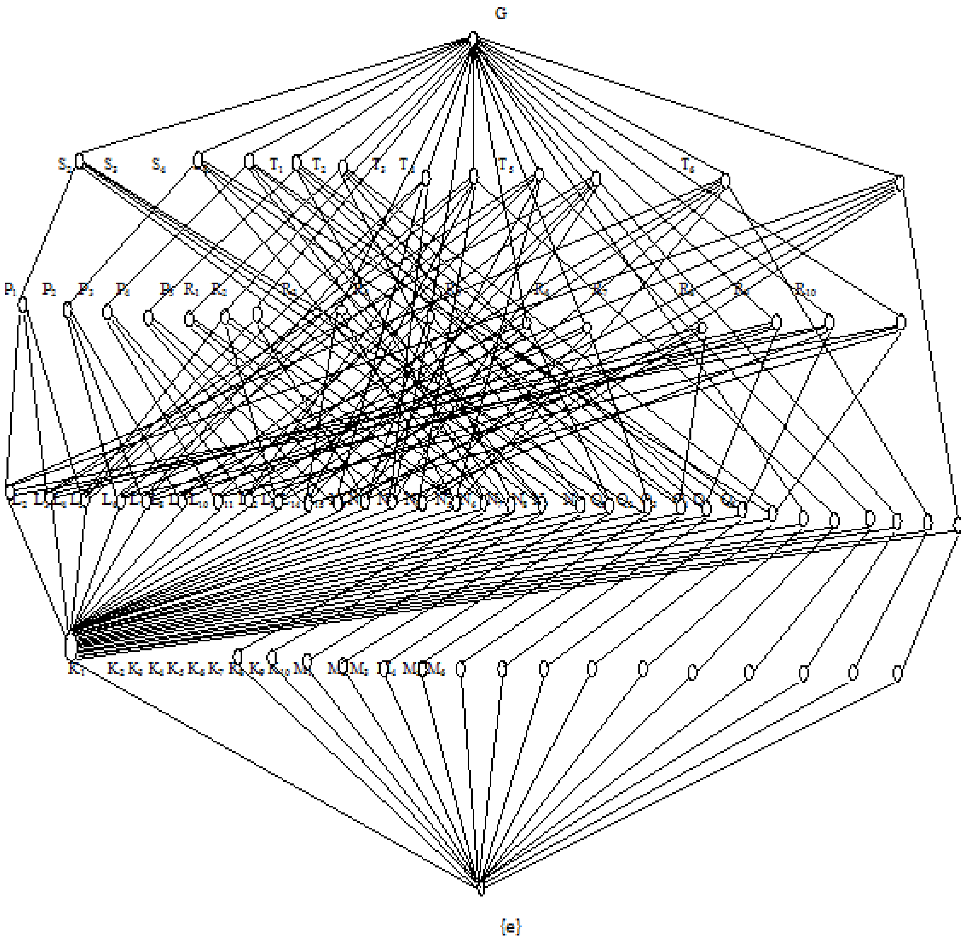
3.14. Subgroups of order 60

Let W be an arbitrary subgroup of order 60. Since $|W| = 2^2 \times 3 \times 5$, by multiplying a subgroup of order 3 and a subgroup of order 20, that is, by finding K_iT_j for all i, j, We get a subset of more than 60 elements.

Similarly by multiplying a subgroup of order 5 and a subgroup of order 12, that is, by finding M_iR_j for all i, j, we get a subset of more than 60 elements.

4. THE STRUCTURE OF L(G)

According to the above results, We have the diagram of the lattice of subgroups of $M_2(Z_5)$ whose elements have determinant value 1



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