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ON PSEUDO CHEBYSHEV MATRIX POLYNOMIALS OF TWO VARIABLES

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The main aim of this paper defined a new polynomial, say, pseudo Chebyshev matrix polynomials. We start from pseudo Hermite matrix polynomials to introduce families the definition of the pseudo Chebyshev matrix polynomials and to study their properties. Some formulas related to an explicit representation, a matrix differential recurrence relation are deduced.

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1. INTRODUCTION:

This class of functions providing a fairly natural generalization of the ordinary exponential, hyperbolic and trigonometric functions [4], offers the possibility of exploring, from a more general and unifying point of view, the theory of special functions including generalized cases. The concepts and Ricci [12] have opened a wider scenario on the possibility of employing larger classes of pseudo type functions and the initial effort has been made in [2,3], where families of pseudo Laguerre and pseudo Hermite polynomials have been introduced. The Hermite and Chebyshev matrix polynomials have been introduced and studied in [1, 6, 7, 8, 9, 10] for matrices in $\mathbb{C}^{N\times N}$ whose eigenvalues are all situated in the right open half-plane.

If D_0 is the complex plane cut along the negative real axis and $\log(z)$ denotes the principal logarithm of z. If A is a matrix in $\mathbb{C}^{N\times N}$, its two-norm denoted by

$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

where for a vector y in $\mathbb{C}^N$, $\|y\|_2 = (y^T y)^{\frac{1}{2}}$ denotes the usual Euclidean norm of y. The set of all the eigenvalues of A is denoted by $\sigma(A)$. If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z.

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which are defined in an open set Ω of the complex plane, and A is a matrix in $\mathbb{C}^{N\times N}$ with $\sigma(A)\subset\Omega$, then from the properties of the matrix functional calculus [6], it follows that

$$f(A)g(A)=g(A)f(A). \tag{1.1}$$

If A is a matrix with $\sigma(A)\subset D_0$, then $\sqrt{A}=\exp(\frac{1}{2}\log(A))$ denotes the image by $z^{\frac{1}{2}}=\sqrt{z}=\exp(\frac{1}{2}\log(z))$ of the matrix functional calculus acting on the matrix A. If A is a positive stable matrix in $\mathbb{C}^{N\times N}$ [1, 6, 8]

$$\operatorname{Re}(z)>0,\quad \text{for all } z\in\sigma(A). \tag{1.2}$$

The operator D_x^{-n} defines the inverse of the derivative and once acting on unity yields [2, 3, 11]

$$D_x^{-n}(1)=\frac{x^n}{n!}. \tag{1.3}$$

The following two identities are a fairly direct consequence of the previous considerations, it is indeed easily checked that

$$D_x^{-i}(D_x^{-n})(1)=\frac{x^{n+i}}{(n+i)!}. \tag{1.4}$$

The definition of two variables pseudo Hermite matrix polynomials by introducing in the following a new family of functions [11]

$$k_n(x,y,A;r,0)=n!\sum_{k=0}^{n}\frac{(-y)^{n-k}(x\sqrt{rA})^{rk}}{(rk)!(n-k)!} \tag{1.5}$$

are easily shown to satisfy the differential equation

$$\left[y\frac{\partial^r}{\partial x^r}-\frac{x}{r}\left(\sqrt{rA}\right)^r+n\left(\sqrt{rA}\right)^r\right]k_n(x,y,A;r,0)=0 \tag{1.6}$$

In the forthcoming section of the paper, we will discuss the properties of these new families of polynomials and we will analyze possible developments and applications of the theory pseudo.

2. PSEUDO CHEBYSHEV MATRIX POLYNOMIALS:

The pseudo Chebyshev matrix polynomials of the second kind are defined by the series

$$U_n(x,y,A;r,0)=\sum_{k=0}^{n}\frac{(nr-n+k)!(-y)^{n-k}(x\sqrt{rA})^{rk}}{(rk)!(n-k)!} \tag{2.1}$$

by means of the integral transform

$$U_n(x,y,A;r,0)=\frac{1}{n!}\int_0^\infty \exp(-t)t^{nr}k_n(x,\frac{y}{t},A;r,0)dt \tag{2.2}$$

where A is satisfying the condition (1.2). It is clear that

$U_{-1}(x,y,A;r,0)=0$ and $U_0(x,y,A;r,0)=I$.

It has already been shown that most of the properties of the $U_n(x, y, A; r, 0)$, linked to the ordinary case by

$$U_n(x, y, A; r, 0) = y^n U_n(\frac{x}{\sqrt[r]{y}}, A; r, 0) \text{ and } U_n(x, 1, A; r, 0) = U_n(x, A; r, 0) \tag{2.3}$$

can be directly inferred from those of the pseudo Hermite matrix polynomials and from the integral representation given in (2.2). We obtain another representation for the pseudo Chebyshev matrix polynomials as follows by the series

$$U_n(x, y, A; r, 0) = \sum_{k=0}^{n} \frac{(nr - n + k)!(\sqrt{rA})^{rk} (-y)^{n-k} D_x^{-rk}(1)}{(n-k)!}. \tag{2.4}$$

Before getting into the main body of the paper, let us recall some important properties of pseudo Chebyshev matrix polynomials of the multiplication theorems, which will be used in the forthcoming papers.

Theorem 2.1: Multiplication Theorem

$$U_n(x, \alpha y, A; r, 0) = \alpha^n U_n(\frac{x}{\sqrt[r]{\alpha}}, y, A; r, 0) \tag{2.5}$$

and

$$U_n(\alpha x, \alpha^r y, A; r, 0) = \alpha^{nr} U_n(x, y, A; r, 0) \tag{2.6}$$

where α is constant.

Proof: By using (2.1), we consider the series in the form

$$\alpha^n U_n(\frac{x}{\sqrt[r]{\alpha}}, y, A; r, 0) = \alpha^n \sum_{k=0}^{n} \frac{(nr - n + k)!(-y)^{n-k}}{(rk)!(n-k)!} (\frac{x\sqrt{rA}}{\sqrt[r]{\alpha}})^{rk}$$
$$= \sum_{k=0}^{n} \frac{(nr - n + k)!(-\alpha y)^{n-k}}{(rk)!(n-k)!} (x\sqrt{rA})^{rk} = U_n(x, \alpha y, A; r, 0).$$

From (2.1) yields the Chebyshev matrix polynomials as given in the following

$$U_n(\alpha x, \alpha^r y, A; r, 0) = \sum_{k=0}^{n} \frac{(nr - n + k)!(-\alpha^r y)^{n-k}}{(rk)!(n-k)!} (x\alpha\sqrt{rA})^{rk}$$
$$= \alpha^{nr} U_n(x, y, A; r, 0).$$

Therefore, the expressions (2.5) and (2.6) are established. The differential recurrence relations are carried out on the pseudo Chebyshev matrix polynomials in the following.

Corollary 2.1: The pseudo Chebyshev matrix polynomials satisfying the following partial differential equations

$$n\, r\, U_n(x, y, A; r, 0) = x \frac{\partial}{\partial x} U_n(x, y, A; r, 0) + r\, y \frac{\partial}{\partial y} U_n(x, y, A; r, 0) \tag{2.7}$$

and

$$\left[x^2 \frac{\partial^2}{\partial x^2} + \left[1 - (2n-1) r\right] x \frac{\partial}{\partial x} + r^2 y^2 \frac{\partial^2}{\partial y^2} + n\, r^2 (n-1) \right] U_n(x, y, A; r, 0) = 0 \tag{2.8}$$

Proof: Differentiating the identity (2.1) with respect to x , y yields

$$\frac{\partial}{\partial x} U_n(x, y, A; r, 0) = \sqrt{rA} \sum_{k=0}^{n} \frac{(nr - n + k)!(rk)(x\sqrt{rA})^{rk-1}(-y)^{n-k}}{(rk)!(n-k)!} \quad \frac{\partial^2}{\partial x^2} U_n(x, y, A; r, 0) \tag{2.9}$$

$$= \left(\sqrt{rA}\right)^2 \sum_{k=0}^{n} \frac{(nr - n + k)!(rk)\left(r\, k - 1\right)(x\sqrt{rA})^{rk-2}(-y)^{n-k}}{(rk)!(n-k)!} \tag{2.10}$$

$$\frac{\partial}{\partial y}U_n(x,y,A;r,0)=-\sum_{k=0}^{n}\frac{(nr-n+k)!(x\sqrt{rA})^{rk}(n-k)(-y)^{n-k-1}}{(rk)!(n-k)!} \tag{2.11}$$

and

$$\frac{\partial^2}{\partial y^2}U_n(x,y,A;r,0)=\sum_{k=0}^{n}\frac{(nr-n+k)!(n-k)(n-k-1)(x\sqrt{rA})^{rk}(-y)^{n-k-1}}{(rk)!(n-k)!} \tag{2.12}$$

Multiply of (2.9) by x and multiply of (2.11) by $r\,y$, we obtain differential recurrence relation (2.7) follows directly. By using (2.1), (2.9), (2.10) and (2.12), we get directly the equation (2.8). In the following result, the pseudo Chebyshev matrix polynomials appear as finite series solutions of the r - th order matrix differential equation.

Corollary 2.2: The pseudo Chebyshev matrix polynomials are easy to solution of the matrix differential equation of the r -th order in the form

$$\left[\frac{r^2 y}{\left(\sqrt{rA}\right)^r}\frac{\partial^r}{\partial x^r}-x^2\frac{\partial^2}{\partial x^2}-\left[r+nr^2-2nr+1\right]x\frac{\partial}{\partial x}\right.$$
$$\left.+n\,r^2\left(nr-n+1\right)\right]U_n(x,y,A;r,0)=0 \tag{2.13}$$

The above relations will be used, along with the generalized pseudo Chebyshev matrix polynomials can be shown to satisfy the property, to derive new properties of the family generated by (2.1) yields as given in the following paper. It goes by itself that we can introduce the Chebyshev matrix polynomials

$$\begin{aligned} U_n(x,y,A;r,i) &= (\sqrt{rA})^i D_x^{-i}U_n(x,y,A;r,0)\\ &=\sum_{k=0}^{n}\frac{(nr-n+k)!(-y)^{n-k}(x\sqrt{rA})^{rk+i}}{(n-k)!(rk+i)!} \end{aligned} \tag{2.14}$$

which in terms of generalized pseudo Chebyshev matrix polynomials

$$U_n(x,y,A;r,i)=\frac{n!}{(r\,n+i)!}U_{m+i,r}(x,y,A) \tag{2.15}$$

where $U_{n,m}(x,y,A)$ are generalized Chebyshev matrix polynomials of two variables. We believe interesting to consider a further example relevant to the family of matrix polynomials

$$U_{n,m}(x,y,A;r,i)=n!\sum_{k=0}^{[\frac{n}{m}]}\frac{(nr-n+mk)!(-y)^{n-mk}(x\sqrt{rA})^{rk+i}}{(rk+i)!(n-mk)!};\quad m<r. \tag{2.16}$$

REFERENCES:

[1] R. S. Batahan, A new extension of Hermite matrix polynomials and its applications, *Linear Algebra Appl.*, 419 (2006), 82-92.

[2] G. Dattoli, Pseudo Laguerre and pseudo Hermite polynomials, *Rend. Mat. Acc. Lincei*, 12 (2001), 75-84.

[3] G. Dattoli, C. Cesarano and D. Sacchetti, Pseudo Bessel functions and applications, *Georgian Math. J.*, 9 (2002), 473-480.

[4] G. Dattoli, S. Lorenzutta and D. Sacchetti, Arbitrary-order coherent states and pseudo-hyperbolic functions, *Nuovo Cimento Soc. Ital. Fis. B*, 12 (2001), 719-726.

[5] G. Dattoli and A. Torre, *Theory and applications of special functions*, Arachne Editrice, Rome, 1996.

[6] E. Defez and L. Jódar, Some applications of Hermite matrix polynomials series expansions, *J. Comput. Appl. Math.*, 99 (1998), 105-117.

[7] E. Defez and L. Jódar, Chebyshev matrix polynomials and second order matrix differential equations, *Utilitas Math.*, 61, (2002), 107-123.

[8] L. Jódar and E. Defez, On Hermite matrix polynomials and Hermite matrix functions, *J. Approx. Theory Appl.*, 14 (1998), 36-48.

[9] M. S. Metwally, M. T. Mohamed and A. Shehata, Generalizations of two-index two-variable Hermite matrix polynomials, *Demonstratio Mathematica*, 42 (2009), 687-701.

[10] M. S. Metwally, M. T. Mohamed and A. Shehata, On Hermite-Hermite matrix polynomials, *Math. Bohemica*, 133 (2008), 421-434.

[11] M. S. Metwally, M. T. Mohamed and A. Shehata, On Pseudo Hermite matrix polynomials of two variables, *Banach J. Math. Anal.*, 4 (2010), 147-156.

[12] P. E. Ricci, *Le funzioni pseudo iperboliche pseudo trigonometriche*, Pubblicazione dell'Istituto di Matematica Applicata, n. 192, 1978.

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