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CONTROLLABILITY OF NONLINEAR SYSTEMS IN UNIFORMLY CONVEX BANACH SPACE USING KIRK FIXED POINT THEOREM

NASEIF J. AL-JAWARI[1], ABDULRHMAN H. NJEM*[2]

Al-Mustansiriyah University
College of Science, Department of Mathematics, Baghdad, Iraq.

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ABSTRACT

In this paper, sufficient conditions for controllability of nonlinear system in uniformly convex Banach spaces are established. The results are obtained by using strongly continuous semigroup theory and some techniques of nonlinear functional analysis, such as, Kirk fixed point theorem. Moreover example is provided to illustrate the theory.

KeyWords: *Controllability, Uniformly convex Banach space, Semigroup theory, Kirk fixed point theorem.*

1. INTRODUCTION

The theory of semigroup of linear operators lends a convenient setting and offers many advantages for applications. Control theory in infinite-dimensional spaces is a relatively new field and started blooming only after well-developed semigroup theory was at hand. Many scientific and engineering problems can be modeled by partial differential equations, integral equations, or coupled ordinary and partial differential equations, that can be described as differential equations in infinite-dimensional spaces using semigroup. Nonlinear equations, with and without delays, serve as an abstract formulation for many partial equations which arise in problems connected with heat flow in materials with memory, viscoelasticity, and other physical phenomena. So, the study of controllability results for such system in infinite-dimensional spaces is important. For the motivation of abstract system and controllability of linear system, one can refer to the [1,2]. In this paper we discuss the controllability of mild solution of the following nonlinear control problem in arbitrary uniformly convex Banach spaces.

$$\dot{z}(t) + Az(t) = (Bu)(t) + f(t,z(t)) + Q(t,K(t,z(t))), \text{ almost everywhere (a.e.) in } J=[0,b],\ z(0) = z_0, \tag{1.1}$$

where the state $z(.)$ takes values in the uniformly convex Banach space $S=L_P(J,X)$, for $1 < p < \infty$ with X a Banach space, and the control function $u(.)$ is given in $L^2(J,U)$, a uniformly convex Banach space of admissible control functions, with U a Banach space. Here, the linear operator $-A$ generates a strongly continuous semigroup (C_0–semigroup) $T(t)$, $t > 0$, on a uniformly convex Banach space S with norm $\|\ .\ \|_p$, and B is a bounded linear operator from U into S. The nonlinear operators $f\in C(J\times S,S)$, $K\in C(J\times S,S)$ and $Q\in C(J\times S,S)$ are all satisfy Lipschitz condition on the seconed argument.

Controllability of the above system in any Banach space, with different conditions has been studied by several authors. The case where $Q\equiv 0$ in (1.1), Yamamoto [3], studied the controllability for parabolic functions with uniformly bounded nonlinear terms. Al-Moosawy [4] discussed the controllability of the mild solution for the system (1.1) by using Banach fixed point theorem, where $f\equiv 0$, A generates C_0–semigroup on a Banach space and the operators K, Q are satisfying Lipschitz condition on the second argument. The work in [4] extended to study the controllability in quasi-Banach spaces of kind L^p, $0 < p < 1$, in [5] by using a quasi-Banach contraction principle theorem. In [6] The controllability of the system (1.1), where $f\equiv 0$, in any quasi-Banach space by using quasi-Banach contraction principle theorem is presented. The case where the operator Q in (1.1) is an integral operator is established in [7] by using Schauder fixed point. The controllability of the system (1.1), where $T(t)$, $t > 0$ is a compact semigroup on a Banach space and the operators f, K and Q are all uniformly bounded continuous in (1.1) is studied in [8] by using Schauder fixed point theorem. From all the above and since every uniformly convex Banach space is a Banach space but the converse, in general, not true. And since a nonexpansive mapping on a nonempty, closed, bounded and convex subset of a Banach space has no fixed point in general (see, Example 2.2), we find a reasonable justification to accomplish the study of this paper. The purpose of this paper is to study the controllability of nonlinear system (1.1) in uniformly convex Banach spaces by using Kirk fixed point theorem .

Corresponding Author: Abdulrhman H. Njem[2]*

2. FIXED POINT THEOREMS AND SEMIGROUP THEORY

Fixed point throrems are the basic mathematical tools used in solving nonlinear equations. In this section we present the basic fixed point results and some definitions of one - parameter semigroup of operators.

Definition 2.1[9]: let (X, $\| \cdot \|$) be a normed space. A map T: $X \to X$ is said to be ***Lipschitz continuous*** if there is $\lambda \geq 0$ such that

$$\|T(x_1) - T(x_2)\| \leq \lambda \, \|x_1 - x_2\|, \text{ for all } x_1, x_2 \in X.$$

The smallest λ for which the above inequality holds is the *Lipschitz constant* of T. If $\lambda \leq 1$, T is said to be ***nonexpansive***, *if* $\lambda < 1$, T is said to be *a* ***contraction*** .

Note that each contraction is nonexpansive, while an isometry is nonexpansive but not contractive.

Theorem 2.1[9] (Banach Theorem): Eevry contraction mapping of a Banach space into itself has a unique fixed point .

Theorem 2.2[9] (Schauder Theorem): Every continuous operator that maps a nonempty convex subset of a Banach space into a compact subset of itself has at least one fixed point .

Definition 2.2[10]: A normed space X is said to be ***uniformly convex*** if for every $\epsilon > 0$, there exists some $\delta > 0$ such that for all x and y in X with $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x - y\| \geq \epsilon$, we have $\|x + y\| \leq 2(1 - \delta)$.

Definition 2.3[10]: Let $0 < p < \infty$, then the collection of all measurable function f for which $|f|^p$ is integrable will be denoted by $L_p(\mu)$. for each $f \in L_p(\mu)$, let $\|f\|_p = (\int |f|^p \, d\mu)^{1/p}$, the number $\|f\|_p$ is called the ***L_P*-norm** of f .

Examle 2.1[10]: Every Hilbert space is uniformly convex. The spaces ℓ^p of p – summable scalar seqences and the space L^p of p- integrable functions are uniformly convex for $1< p < \infty$. For properties of uniformly convex Banach space, every uniformly convex Banach space is reflexive .

To have an extension of the Banach theorem to nonexpansive maps, we need to impose some geometric conditions on the domain of the nonexpansive map. See the following

Let X be a Banach space, $C \subset X$ nonempty, closed, bounded and convex, and let $T : C \to C$ be a ***nonexpansive*** map. The problem is whether T admits a fixed point in C. The answer, in general, is false .

Example 2.2[10]: Let $X = c_0$ (the space of all sequences of scalars converging to zero) with the supremum norm. Setting $C = \{y \in X : \|y\| \leq 1\}$, the map $T : C \to C$ defined by $f(x) = (1, x_0, x_1, \ldots)$, for $x = (x_0, x_1, x_2, \ldots) \in C$ is nonexpansive but clearly admits no fixed point in C .

Things are quite different in uniformly convex Banah spaces.

Theorem 2.3[9](Kirk Fixed Point Theorem): Let X be a uniformly convex Banach space and $C \subseteq X$ be a nonempty, closed, bounded and convex. If T:$C \to C$ is a nonexpansive mapping, then T has a fixed point .

Definition 2.4[11]: A family $T(t)$, $0 \leq t < \infty$ of bounded linear operators on a Banach space X is called a (one – parameter) ***semigroup*** on X if it is satisfies the following conditions:

$T(t+s) = T(t)T(s)$, $t,s \geq 0$ *and* $T(0) = I$. (I is the identity operator on X)

Definition 2.5[11]: The infinitesimal generator A of the ***semigroup*** $T(t)$ on Banach space X is defined by $Ax = lim_{t \to 0^+} \frac{T(t)x - x}{t}$, where the limit exists and the domain of A is $D(A) = \{x \in X: lim_{t \to 0^+} \frac{T(t)x - x}{t}$, exists$\}$.

Definition 2.6[11]: A *semigroup* $T(t)$, $0 \leq t < \infty$ of boubded linear operators on Banach space X is said to be ***strongly continuous semigroup*** **(or Co-semigroup)** if: $\|T(t)x - x\|_X \to 0$ *as* $t \to 0^+$ for all $x \in X$.

3. CONTROLLABILITY OF NONLINEAR CONTROL PROBLEMS

In this section we will study the controllability of mild solution to the problem (1.1) in uniformly convex Banach space by using ***C_0* – *semigroup*** and ***"Kirk Fixed Point Theorem"*** .

3.1 Problem Formulation (I)

Let $S=L_P(J,X)$, for $1 < p < \infty$ with X a Banach space, be uniformly convex Banach space and U be a Banach space, with norms $\|.\|_p$ and $|\,.\,|$, respectively. Consider the following nonlinear control problem in infinite dimensional state space :

$$\dot{z}(t) + Az(t) = (Bu)(t) + f(t,z(t)) + Q(t,K(t,z(t))), \text{ (a.e.) in } J=[0,b],\ z(0)=z_0, \tag{3.1}$$

where $B: U \rightarrow S$ is a linear bounded operator, $\|B\|_p \le c$, where c is a constant, and the control function $u \in L^2(J,U)$ a uniformly convex Banach space of admissible control functions. Let A: $D(A) \subset S \rightarrow S$ be a linear operator, and let $S_0 = \{ z : z \in C(J,X) \subset S, z(0)=z_0 , \|z(t)\|_p \le r$, for $t \in J \}$, where r is a positive constant, be a subset of S .

In order that problem **(3.1)** makes sense throughout the paper we shall assume the following basic hypothesis:
(H_1) The linear operator $-A$ generates a C_0-semigroup $T(t)$, $t \ge 0$, such that $\|T(t)\|_p \le M$, where $M > 0$ is a constant.

(H_2) The nonlinear operators $f : J\times S \rightarrow S$, and K: $J\times S \rightarrow S$ are continuous and satisfy lipshitz condition on the seconed argument:
$\|f(t,z_1) - f(t,z_2)\|_p \le M_1\|z_1 - z_2\|_p$ and $\|K(t,z_1) - K(t,z_2)\|_p \le M_4\|z_1 - z_2\|_p$,
where M_1, M_4 are positive constant, and $z_1, z_2 \in S_0$. Also let
$M_3 = \max_{t\in J}\|f(t,0)\|_p$, and $k_1 = \max_{t\in J}\|z(t)\|_p$.

(H_3) The nonlinear operator Q: $J\times S \rightarrow S$ is continuous and there exist a constants M_2, M_5, such that for all $z_1 , z_2 \in S_0$, we have :
$\|Q(t, K(t,z_1)) - Q(t,K(t,z_2))\|_p \le M_2\|K(t,z_1) - K(t,z_2)\|_p \le M_2 M_4\|z_1 - z_2\|_p$, and let $M_5 = \max_{t\in J}\|Q(t,K(t,0))\|_p$.

(H_4) The linear operator W from $L^2(J,U)$ into S, defined by

$$Wu = \int_0^b T(t-s)Bu(s)ds,$$

induces a bounded inverse operator $\widetilde{W}^{-1}$ defined on $L^2(J,U)/\ker(W)$, and there exist positive constant $k_2 > 0$ such that $\|\widetilde{W}^{-1}\|_p \le k_2$**.**

The construction of $\widetilde{W}^{-1}$ is outlined as follows [12,13] .

Let $Y = L^2[J,U]/\ker(W)$. Since $\ker(W)$ is closed, Y is Banach space under the norm
$\|[u]\|_Y = \inf_{u\in[u]}\|u\|_{L^2[J,U]} = \inf_{W\hat{u}=0}\|u + \hat{u}\|_{L^2[J,U]}$,
where $[u]$ are the equivalence classes of u.

Define $\widetilde{W}$: $Y \rightarrow X$ by $\widetilde{W}[u] = Wu$, $u \in [u]$.

Then, $\widetilde{W}$ is one-to-one and $\|\widetilde{W}[u]\|_x \le \|W\|\,\|[u]\|_Y$.

Also, V = Range(W) is a Banach space with the norm $\|v\|_v = \|\widetilde{W}^{-1}v\|_Y$.

To see this, note that this norm is equivalent to the graph norm on $D(\widetilde{W}^{-1})$=Range($\widetilde{W}$). $\widetilde{W}$ is bounded, and since $D(\widetilde{W}) = Y$ is closed, $\widetilde{W}^{-1}$ is closed. So, the above norm makes Range (W) = V, a Banach space. Moreover,
$\|Wu\|_v = \|\widetilde{W}^{-1}Wu\|_Y = \|\widetilde{W}^{-1}W[u]\| = \|[u]\| = \inf_{u\in[u]}\|u\| \le \|u\|$;

So, $W \in \mathcal{L}(L^2[J,U], V)$.

Since $L^2[J,U]$ is reflexive and $\ker(W)$ is weakly closed, the infimum is actually attained. Therefore, for any $v \in V$, a control $u \in L^2[J,U]$ can be chosen such that $u = \widetilde{W}^{-1}v$.

3.1.1 Controllability Result of the Problem Formulation (I):

Now we want to define and find the mild solution of problem (3.1). By condition (H_1), $T(t)$, $t > 0$ is the C_0-semigroup generated by the linear operator $-A$, let $z(.) \in S$, be the solution of (3.1), then we have $T(t)z$ is differentiable [11], that implies the S-value function $H(s) = T(t-s)z(s)$ is differentiable for $0 < s < t$; and

$$\frac{dH}{ds} = T(t-s)\frac{d}{ds}z(s) + z(s)\frac{d}{ds}T(t-s)$$

$$\frac{dH}{ds} = T(t-s)[-Az(s)+(Bu)(s)+f(s,z(s))+Q(s,K(s,z(s)))] + z(s)[AT(t-s)]$$

$$\frac{dH}{ds} = T(t-s)(Bu)(s) + T(t-s)\, f(s,(z(s)) + T(t-s)Q(s,K(s,z(s))) \tag{3.2}$$

Integrating **(3.2)** from 0 to t, yields
$H(t)-H(0)=\int_0^t T(t-s)(Bu)(s)ds + \int_0^t T(t-s)\, f(s,z(s))ds + \int_0^t T(t-s)\, Q(s,K(s,z(s)))ds$

Since $H(s) = T(t-s)z(s)$, then
$T(t-t)z(t) - T(t-0)z(0)=\int_0^t T(t-s)(Bu)(s)ds+\int_0^t T(t-s)f(s,z(s))ds+\int_0^t T(t-s)Q(s,K(s,z(s)))ds$. Then

$$z(t) = T(t)z_0 + \int_0^t T(t-s)\,(Bu)(s)ds + \int_0^t T(t-s)\, f(s,z(s))ds + \int_0^t T(t-s)\, Q(s,K(s,z(s)))ds \tag{3.3}$$

So according to the results above, the following definition has been presented.

Definition 3.1: A continuous function $z \in S$ given by (3.3) will be called a *mild solution* to the problem (3.1).

Definition 3.2: The system (3.1) is said to be controllable on the interval J if, for every z_0 , $z_1 \in S$, there exists a control $u \in L^2(J,U)$ such that the mild solution $z(t)$ of (3.3) satisfying $z(b) = z_1$.

Theorem 3.1: Let the hypothesis (H_1) – (H_4) are satisfied for the nonlinear control problem (3.1).
$\dot{z}(t) + Az(t) = (Bu)(t) + f(t,z(t)) + Q(t,K(t,z(t)))$, a.e. in $J =[0,b]$, *with* $z(0)= z_0$. Assume further that

(H_5) $(M\|z_0\|_p+ h_1+h_2+h_3+h_4+bMck_2[\|z_1\|_p+M\|z_0\|_p+h_1+h_2+h_3+h_4]) \le r$
where $h_1=bMM_1k_1$, $h_2=bMM_3$, $h_3=bMM_2M_4k_1$, $h_4=bMM_5$ **.**

(H_6) $\lambda = bMM_1+bMM_2M_4+b^2M^2M_1ck_2+b^2M^2M_2M_4ck_2$, be such that $0 \le \lambda \le 1$**.** Then the problem (3.1) is controllable on J **.**

Proof: By using definition (3.2) and equation (3.3) we get that

$$z_1 = z(b) = T(b)z_0 + \int_0^b T(b-s)(Bu)(s)ds + \int_0^b T(b-s)\, f(s,z(s))ds+\int_0^b T(b-s)Q(s,K(s,z(s)))ds.$$

Condition (H_4) leads to
$z_1 = T(b)z_0 + Wu+\int_0^b T(b-s)\, f(s,z(s))ds+ \int_0^b T(b-s)\, Q(s,K(s,z(s)))ds$**.**

Therefore,
$Wu = z_1 - T(b)z_0- \int_0^b T(b-s)f(s,z(s))ds- \int_0^b T(b-s)Q(s,K(s,z(s)))ds$**.**

From constraction of $\widetilde{W}^{-1}$ in (H_4), we have that $u(t) = \widetilde{W}^{-1}\,(Wu(t))$, then

$$u(t) = \widetilde{W}^{-1}(z_1-T(b)z_0- \int_0^b T(b-s)\, f(s,z(s))ds - \int_0^b T(b-s)\, Q(s,K(s,z(s)))ds)(t)) \tag{3.4}$$

we shall now show that, when using this control the operator defined by
$(\Phi z)(t) = T(t)z_0+\int_0^t T(t-s)(Bu)(s)ds+\int_0^t T(t-s)f(s,z(s))ds+\int_0^t T(t-s)Q(s,K(s,z(s)))ds$ has a fixed point. This fixed point is then a solution of equation (3.3).

Clearly, $(\Phi z)(b)= z_1$, which means that the control u steers the nonlinear control system from the initial z_0 to z_1 in time b, provided we can obtain a fixed point of the nonlinear operator Φ**.**

Let $S=L_p(J,X)$, for $1 < p < \infty$ with X a Banach space and $S_0=\{z : z\in C(J,X)\subset S,\ z(0)= z_0,\ \|z(t)\|_p\le r$, for $t \in J\ \}$, where r is a positive constant**.** Then S_0 is clearly a bounded, closed, convex subset of S [10]. Now we define a mapping, $\Phi : S \to S_0$ by,

$$(\Phi z)(t) = T(t)z_0 + \int_0^t T(t-s)\, f(s,z(s))ds +\int_0^t T(t-s)Q(s,K(s,z(s)))ds + \int_0^t T(t-\eta)B\widetilde{W}^{-1}[z_1 - T(b)z_0 - \int_0^b T(b-s)f(s,z(s))ds- \int_0^b T(b-s)Q(s,K(s,z(s)))ds](\eta)d\eta \tag{3.5}$$

Taking the norm of both sides of **(3.5)**

$$\|(\Phi z)(t)\|_p = \|T(t)z_0 + \int_0^t T(t-s)f(s,z(s))ds+ \int_0^t T(t-s)Q(s,K(s,z(s)))ds+ \int_0^t T(t-\eta)B\widetilde{W}^{-1}\,[z_1-T(b)z_0 - \int_0^b T(b-s)f(s,z(s))ds- \int_o^b T(b-s)\, Q(s,K(s,z(s)))\, ds]\,(\eta)d\eta\|_p$$

$$\|(\Phi z)(t)\|_p \le \|T(t)z_0\|_p+\|\int_0^t T(t-s)f(s,z(s))ds\|_p+\|\int_0^t T(t-s)Q(s,K(s,z(s)))ds\|_p+\|\int_0^t T(t-\eta)B\widetilde{W}^{-1}[z_1-T(b)z_0 - \int_0^b T(b-s)f(s,z(s))d- \int_0^b T(b-s)\, Q(s,K(s,z(s)))\, ds]\,(\eta)d\eta\|_p$$

$$\|(\Phi z)(t)\|_p \leq \|T(t)z_0\|_p + \|\int_0^t T(t-s)[f(s,z(s))-f(s,0)+f(s,0)]ds\|_p$$
$$+\|\int_0^t T(t-s)\ [Q(s,K(s,z(s)))-Q(s,K(s,0))+Q(s,K(s,0))]ds\|_p$$
$$+\|\int_0^t T(t-\eta)B\widetilde{W}^{1}[\|z_1\|_p+\|T(b)z_0\|_p+\|\int_o^b T(b-s)[f(s,z(s))$$
$$-f(s,0)+f(s,0)]ds\|_p+\|\int_0^b T(b-s)[Q(s,K(s,z(s)))-Q(s,K(s,0))+Q(s,K(s,0))]ds\ \|_p](\eta)d\eta\ \|_p$$

$$\|(\Phi z)(t)\|_p \leq \|T(t)\|_p\|z_0\|_p + \int_0^t \|T(t-s)\|_p[\|f(s,z(s))-f(s,0)\|_p+\|f(s,0)\|_p]ds$$
$$+\int_0^t \|T(t-s)\|_p[\|Q(s,K(s,z(s)))-Q(s,K(s,0))\|_p+\|Q(s,K(s,0))\|_p]ds+\int_o^t \|T(t-\eta)\|_p\|B\|_p\|\widetilde{W}^{-1}\|_p[\|z_1\|_p$$
$$+\|T(b)\|_p\|z_0\|_p+\int_0^b \|T(b-s)\|_p[\|f(s,z(s))-f(s,0)\|_p+\|f(s,0)\|_p]ds$$
$$+\int_0^b \|T(b-s)\|_p[\|Q(s,K(s,z(s)))-Q(s,K(s,0))\|_p+\|Q(s,K(s,0))\|_p ds]](\eta)d\eta$$

By conditions from $(H_1)-(H_4)$, and since $\|B\|_p \leq c$ then we get that
$$\|(\Phi z)(t)\|_p \leq M\|z_o\|_p+\int_0^t M[M_1\|z(s)\|_p+M_3]ds+\int_0^t M[M_2M_4\|z(s)\|_p+M_5]ds+\int_0^t Mck[\|z_1\|_p$$
$$+M\|z_0\|_p+bM[M_1\|z(s)\|_p+M_3]+bM[M_2M_4\|z(s)\|_p+M_5]](\eta)d\eta$$

Since $z \in S$, then $\|z(s)\|_p \leq k_1$, and then :
$$\|(\Phi z)(t)\|_p \leq M\|z_0\|_p+bMM_1k_1+bMM_3+bMM_2M_4k_1+bMM_5+bMck_2$$
$$[\|z_1\|_p+M\|z_o\|_p+bMM_1k_1+bMM_3+bMM_2M_4k_1+bMM_5]$$

By condition (H_5), we have
$$\|(\Phi z)(t)\|_p \leq M\|z_0\|_p+h_1+h_2+h_3+h_4+bMck_2[\|z_1\|_p+M\|z_0\|_p+h_1+h_2+h_3+h_4] \leq r$$

Since f,K and Q are continuous and $\|(\Phi z)(t)\|_p \leq r$, it follows that Φ is also continuous and maps S_0 into itself **.**

Seconed, we have to show that Φ is *nonexpansive mapping* from S_0 into S_0. For $z_1(t)$, $z_2(t) \in S_0$ and from the definition of $(\Phi z)(t)$ in equation (3.5), we have
$$\|\Phi z_1(t)-\Phi z_2(t)\|_p = \|T(t)z_0+\int_0^t T(t-s)f(s,z_1(s))ds+\int_0^t T(t-s)Q(s,K(s,z_1(s)))ds+\int_0^t T(t-\eta)B\widetilde{W}^{-}b)z_0-\int_0^b T(b-s)f(s,z_1(s))ds$$
$$-\int_0^b T(b-s)Q(s,K(s,z_1(s)))ds](\eta)d\eta - T(t)z_0-\int_0^t T(t-s)f(s,z_2(s))ds-\int_0^t T(t-s)\ Q(s,K(s,z_2(s)))ds$$
$$-\int_0^t T(t-\eta)B\widetilde{W}^{-1}[z_1-T(b)z_0-\int_0^b T(b-s)f(s,z_2(s))ds-\int_0^b T(b-s)Q(s,K(s,z_2(s)))ds](\eta)d\eta\|_p$$

$$\|\Phi z_1(t)-\Phi z_2(t)\|_p = \|\int_0^t T(t-s)[f(s,z_1(s)) - f(s,z_2(s))]ds+\int_0^t T(t-s)[Q(s,K(s,z_1(s))) - Q(s,K(s,z_2(s)))]ds$$
$$+\int_0^t T(t-\eta)B\widetilde{W}^{-1}[\int_0^b T(b-s)\ [f(s,z_1(s))-f(s,z_2(s))]ds$$
$$+\int_0^b T(b-s)[Q(s,K(s,z_1(s))) - Q(s,K(s,z_2(s))ds)]](\eta)d\eta\|_p$$

$$\|\Phi z_1(t)-\Phi z_2(t)\|_p \leq \int_0^t \|T(t-s)\|_p\|f(s,z_1(s))-f(s,z_2(s))\|_p ds+\int_o^t \|T(t-s)\|_p$$
$$\|Q(s,K(s,z_1(s)))-Q(s,K(s,z_2(s)))\|_p ds+\|\int_0^t T(t-\eta)B\widetilde{W}^{-1}[\int_0^b T(b-s)[f(s,z_1(s))-f(s,z_2(s))]ds$$
$$+\int_o^b T(b-s)[Q(s,K(s,z_1(s))) - Q(s,K(s,z_2(s))ds)]]\ (\eta)d\eta\|_p$$

$$\|\Phi z_1(t)-\Phi z_2(t)\|_p \leq \int_0^t \|T(t-s)\|_p\|f(s,z_1(s))-f(s,z_2(s))\|_p ds+\int_0^t \|T(t-s)\|_p$$
$$\|Q(s,K(s,z_1(s)))-Q(s,K(s,z_2(s)))\|_p ds+\int_0^t \|T(t-\eta)\|_p\|B\|_p\|\widetilde{W}^{-1}\|_p$$
$$[\int_0^b \|T(b-s)\|_p\|f(s,z_1(s))-f(s,z_2(s))\|_p ds+\int_0^b \|T(b-s)\|_p$$
$$\|Q(s,K(s,z_1(s)))-Q(s,K(s,z_2(s))ds)\|_p](\eta)d\eta$$

By using conditions $(H_1)-(H_4)$, we have that
$$\|\Phi z_1(t)-\Phi z_2(t)\|_p \leq \int_o^t MM_1\|z_1(s)-z_2(s)\|_p ds+\int_0^t MM_2M_4\|z_1(s)-z_2(s)\|_p ds$$
$$+\int_0^t Mck_2[bMM_1\|z_1(s)-z_2(s)\|_p+bMM_2M_4\|z_1(s)-z_2(s)\|_p]\ (\eta)d\eta$$

$$\|\Phi z_1(t)-\Phi z_2(t)\|_p \leq bMM_1\|z_1(t)-z_2(t)\|_p+bMM_2M_4\|z_1(t)-z_2(t)\|_p+bMck_2$$
$$[bMM_1\|z_1(t)-z_2(t)\|_p+bMM_2M_4\|z_1(t)-z_2(t)\|_p]$$

$$\|\Phi z_1(t)-\Phi z_2(t)\|_p \leq [bMM_1+bMM_2M_4]\|z_1(t)-z_2(t)\|_p+[b^2M^2M_1ck_2+b^2M^2\ M_2M_4ck_2]\|z_1(t)-z_2(t)\|_p$$

$$\|\Phi z_1(t)-\Phi z_2(t)\|_p \leq [bMM_1+bMM_2M_4+b^2M^2M_1ck_2+b^2M^2M_2M_4ck_2]\|z_1(t)-z_2(t)\|_p$$

By condition **(H_6)**, we get that
$\|\Phi z_1(t)-\Phi z_2(t)\|_p \le \lambda\|z_1(t)-z_2(t)\|_p \le \|z_1(t)-z_2(t)\|_p$

Therefore Φ is nonexpansive mapping, and hence by Theorem 2.3, there exists a fixed point $z \in S_0$, such that $\Phi z(t) = z(t)$, thus any fixed point of Φ is a mild solution of system (3.1) on J, which satisfies $z(b) = z_1$, and hence the system is controllable on J.

3.2 Problem Formulation (II)

Let the uniformly convex Banach space S and a Banach space U are defined as in section 3.1, and consider the optimal control problem in infinite dimensional state space :

$$\frac{d}{dt}(z(t)+G(t,z(t))) = Az(t) + Bu(t) + f(t,z(t)) + Q(t,K(t,z(t))), \text{ a.e. in } J = [0,b],\ z(0) = z_0, \tag{3.6}$$

where the linear operators A, B are defined as in problem (3.1), and the control $u(.) \in L^1(J,U)$ a Banach space of admissible control function. We assume with hypothesis (H_1) − (H_3) the following condition

(P_1) The nonlinear operator G: $J\times S \longrightarrow S$ is continuous and satisfies Lipschitz condition on the second argument, thus for all $z_1, z_2 \in S_0$, $L_1, L_2 > 0$, we have**:**
$\|G(t, z_1) - G(t, z_2)\|_p \le L_1\|z_1 - z_2\|_p$, and $L_2 = \max_{t\in J}\|G(t,0))\|_p$**.**

Now by using the same technique in subsection 3.1.1, it is easy to define the mild solution of problem (3.6) as follows:

Definition 3.3: A continuous function z: $[0,b] \to S$ defined by

$$z(t) = T(t)z_0 + T(t)G(0,z_0)-G(t,z(t)) +\int_0^t T(t-s)\,(Bu)(s)ds+\int_0^t T(t-s)\, f(s,z(s))ds$$
$$+\int_0^t T(t-s)\, Q(s,K(s,z(s)))ds - \int_0^t G(s,z(s)AT(t-s)ds \tag{3.7}$$

will be called a mild solution to the problem (3.6).

Definition 3.4 : The system (3.6) is said to be controllable on the interval J if, for every $z_0, z_1 \in S$, there exists a control $u \in L^2(J,U)$ such that the mild solution $z(t)$ of (3.7) satisfying $z(b) = z_1$.

Theorem 3.2: Consider the optimal control problem (3.6) with hypothesis (H_1)−(H_4) and (P_1). Assume further

(P_2) There exists a positive function $v_0 \in L^1(0,b)$, such that: $\|AT(t)\|_p \le v_0(t)$
a.e., $t \in (0,b)$. And There exist a constant $k_3 > 0$, such that: $\int_0^b v_0(t)\, dt \le k_3$

(P_3) $M\|z_0\|_p+ Mh_1 + h_2 + h_3 + h_4 + h_2k_3+ bMck_2[\|z_1\|_p+M\|z_0\|_p+ Mh_1 + (L_1\|z_1\|_p+L_2) + h_3 + h_4 + h_2k_3] \le r$

$h_1= (L_1\|z_0\|_p+ L_2)$, $h_2 = (L_1k_1+L_2)$, $h_3 = bM(M_1k_1+ M_3)$, $h_4 = bM(M_2M_4k_1 + M_5)$

(P_4) $q = L_1 + bMM_1 + bMM_2M_4 + L_1k_3 + b^2M^2M_1ck_2 + b^2M^2M_2M_4ck_2 + bMck_2L_1k_3$, be such that $0 \le q \le 1$**.** Then the problem (3.6) is controllable on J .

***Proof*:** By using definition 3.4, and equation (3.7) we get that

$$z_1 = z(b) = T(b)z_0 +T(b)G(0,z_0) - G(b,z(b)) + \int_0^b T(b-s)\,(Bu)(s)ds + \int_0^b T(b-s)f(s,z(s))ds$$
$$+\int_0^b T(b-s)\, Q(s,K(s,z(s)))ds - \int_0^b G(s,z(s)AT(b-s)ds$$

Condition (H_4) leads to

$$z_1 = T(b)z_0 +T(b)G(0,z_0) - G(b,z(b)) + Wu + \int_0^b T(b-s)\, f(s,z(s))ds + \int_0^b T(b-s)\, Q(s,K(s,z(s)))ds - \int_0^b G(s,z(s)AT(b-s)ds.$$

Therefore,

$$Wu = z_1 - T(b)z_0 -T(b)G(0,z_0) + G(b,z(b)) - \int_0^b T(b-s)\, f(s,z(s))ds$$
$$- \int_0^b T(b-s)\, Q(s,K(s,z(s)))ds + \int_0^b G(s,z(s)AT(b-s)ds$$

From constraction of $\widetilde{W}^{-1}$ in **(H_4)**, we have that $u(t) = \widetilde{W}^{-1}(Wu(t))$, then

$$u(t) = \widetilde{W}^{-1}(z_1- T(b)z_0 -T(b)G(0,z_0) + G(b,z(b)) - \int_0^b T(b-s)\, f(s,z(s))ds$$
$$- \int_0^b T(b-s)\, Q(s,K(s,z(s)))ds + \int_0^b G(s,z(s)AT(b-s)ds(t)) \tag{3.8}$$

Now, by using this control we will define the following operator

$$(\Phi z)(t) = T(t)z_0 + T(t)G(0,z_0) - G(t,z(t)) + \int_0^t T(t-s)\,(Bu)(s)ds + \int_0^t T(t-s)\,f(s,z(s))ds$$
$$+\int_0^t T(t-s)\,Q(s,K(s,z(s)))ds - \int_0^t G(s,z(s)AT(t-s)ds \qquad (3.9)$$

By using the same manner used in the proof of Theorem 3.1, it is not difficiult to see that the operator Φ is *nonexpansive mapping* from S_0 into S_0, and has a fixed point $z(t)$ which is a solution of system (3.6) and satisfies $z(b) = z_1$. Hence the system (3.6) is controllable on J.

Remark 3.1: For study the controllability of the nonlinear control problem **(3.1)** and **(3.6)** in any Banach space $\mathbb{S} = C(J,X)$, the space of continuous functions $f(t)$ in the interval J=[0,b] with $\|f\| = \max_{o\le t\le b} |f(t)|$.

(T$_1$) Since the set S_0, which is defined in section 3.1, is closed subset of a Banach space $\mathbb{S}$, then S_0 is a Banach space. Thus, if we assuming that $0 \le \lambda < 1$ in condition (H$_6$), then we can prove that the operator Φ defined from S_0 into S_0 by equation (3.5) is a contraction mapping from a Banach space into a Banach space. Hence by Theorem 2.1, Φ has a unique fixed point $z(t)$ which is a mild solution to the problem (3.1) on J and satisfies $z(b) = z_1$. Therefore the system (3.1) is controllable on J .

By using similar way, when take $0 \le q < 1$ in condition (P_4) we can prove that the system (3.6) is controllable on J by using Banach Theorem 2.1.

(T$_2$) If we assume that in section 3.1, the semigroup $T(t)$, $t > 0$ is compact on a Banach space $\mathbb{S}$ (see condition (H$_1$)), and the nonlinear operators f, K and Q are all uniformly bounded continuous operators, then the operator Φ which is defined from S_0 into S_0 by equation (3.5) satisfies the Schauder Theorem 2.2, and hence Φ has a fixed point $z(t)$ which is a solution to the system (3.1) and satisfies $z(b) = z_1$. Thus the system (3.1) is controllable on J. For more details see [8] .

If in (T$_2$) we also assume that the nonlinear operator G in (P_1) is uniformly bounded continuous and using the similary way above, we can prove that the system (3.6) is controllable on J by using Schauder Theorem 2.2.

(T$_3$) For initial value problem (3.1), if we assume that the operators f, K and Q are also satisfy Lipchitz condition on the first argument, and since $S=L_P(J,X)$, $1 < p < \infty$ is reflexive Banach space [see example 2.1], then for every $z_0 \in D(A)$ the problem (3.1) has a unique strong solution $z(.)$ on [0,b] given by (3.3) (for more details see [11, Ch.6, Theorem 1.6]) .

4. APPLICATION

Consider the partial integrodifferential equation of the form

$$y_t(t,x)=y_{xx}(t,x)+(Bu)(t)+\sigma_1(t,y_{xx}(t,x))+\int_o^t \sigma_3(t,s,y_{xx}(s,x),\int_0^t \sigma_2(s,\tau,y_{xx}(\tau,x))d\tau)ds,$$
$$x \in I = (0,1),\ t \in J = [0,b], \qquad (3.10)$$

And given initial and boundary conditions

$$y(0,1) = y(1,t) = 0, \qquad (3.11a)$$

$$y(x,0) = y_0(x),\ x \in I, \qquad (3.11b)$$

where B:$U \to X$, with $U \subset J$ and $X=L^2[I,R]$, is a linear operator such that there exists an invertibled operator W^{-1} on $L^2[J,U]/\ker W$, where W is defined by,

$Wu = \int_0^b T(b-s)Bu(s)ds$.

$T(t)$ is a C_0 - semigroup, and

$\sigma_1 : J\times X \to X$

$\sigma_2 : J\times J\times X \to X$

$\sigma_3 : J\times J\times X\times X \to X$, are all continuously differentiable function by positive constants,

The problem (3.10)-(3.11) can brought to the form (3.1), as given in [14], by making suitable choices of A,B,f, K, Q as follows.

Let $X= L^2[I, R]$, $Az = z_{xx}$, B: $U \to X$, and $D(A) = \{z \in X,\ z_{xx} \in X;\ z(0) = z(1) = 0\}$ be such that the condition in hypothesis (H$_4$) is satisfied, and let $f(t,z)(x)= \sigma_1(t,z_{xx}(x))$, $(t,z) \in J\times X$,

$K(t,z)(x)= \int_0^t \sigma_2(s,\tau,z_{xx}(x))d\tau$, $Q(t,z,\mu)(x)= \int_o^t \sigma_3(t,s,z_{xx}(x),\mu(x))dt$, $x\in I$.

Then the system (3.10) - (3.11) becomes an abstract formulation of (3.1). Also by [14, theorem 3] the solutions are all bounded. Further, all the conditions stated in the above theorem 3.1 is satisfied. Hence the system (3.10)-(3.11) is controllable on J.

5. CONCLUSIONS

1. Generalize nonlinear control problem by taking *f*, *K*, *Q* and *G* in systems (3.1) and (3.6) any nonlinear operators which are satisfy Lipschitz condition on the seconed argument, and study the controllability of these systems by using C_0 – semigroup and Kirk fixed point theorem.
2. The idea of studing the controllability of problems (3.1) and (3.6) by using Banach fixed point theorem and Schauder fixed point theorem are introduced.

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