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SOLUTION OF LINEAR HIGHER, NONLINEAR HIGHER ORDER BOUNDARY VALUE PROBLEMS WITH FRACTIONAL ORDER FUNCTION BY USING DIFFERENTIAL TRANSFORM METHOD

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ABSTRACT

In this paper we solve some numerical examples of linear and nonlinear higher order with fractional order function by using Differential Transform Method, it gives the convergent series form effectively and the result obtains by Differential Transform Method shows the approximate solution of the Differential equations.

Keywords: *Series solutions, Differential transform Method, boundary value problem, Numerical solution.*

INTRODUCTION

A variety of methods, exact, approximate and purely numerical are available for the solution of differential equations. Most of these methods are computationally intensive because they are trial-and error in nature, or need complicated symbolic computations. The differential transformation technique is one of the numerical methods for ordinary differential equations. The concept of differential transformation was first proposed by Zhou in 1986 [15] and Arikhoglu[1] and Ozkol, Ayaz [3], Chen [6-7] and Ho, 1996, 1999; Hassan[9] and Abdel-Halim[2] 2008, Duan [8], Khaled Batiha[10], it was applied to solve linear and nonlinear initial value problems in electric circuit analysis, Kho, Mantri [11-12], Bert [4].This method constructs a semi analytical, numerical technique that uses the Taylor series for the solution of differential equations in the form of a polynomial. It is different from the high-order Taylor series method which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally time-consuming, especially for higher order equations [5].The differential transform is an iterative procedure for obtaining analytic Taylor series solutions of differential equations. The Differential transformation method is very effective and powerful for solving various kinds of differential equation.

In this paper, we solve some numerical examples of linear, nonlinear higher order and fractional order boundary value problem Shahid [13], V. Ananathaswamy [14] by using Differential Transform Method the numerical results are compared with their exact solutions. The main advantage of the DTM is that, it can be applied directly to linear and nonlinear ordinary differential equations without linearization; it reduces the size of computational work and providing the series solution with convergence.

BASIC IDEAS OF DIFFERENTIAL TRANSFORM METHOD

The transformation of the k^{th} Derivative of a function with one variable follows:

$$U(k) = \frac{1}{k!}\left(\frac{d^k u(x)}{dx^k}\right)_{x=x_0} \tag{1}$$

Where $u(x)$ is the original function and $U(k)$is the transformed function and the differential inverse transformation $U(k)$ is defined by,

$$u(x) = \sum_{k=0}^{\infty} U(k)(x-x_0)^k \tag{2}$$

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When $x_0 = 0$, the function $u(x)$ Defined in (2) is expressed as

$$u(x) = \sum_{k-0}^{k=\infty} U(k)x^k \tag{3}$$

Equation (3) implies that the concept of one dimensional differential transform is almost is same as the one dimensional Taylors series expansion. We use following fundamental theorems on differential transform method

Theorem 1: If $u(x) = \alpha g(x) \pm \beta h(x) \;\; then \;\; U(k) = \alpha G(k) \pm \beta H(k)$

Theorem 2: If $u(x) = x^m \;\; then \;\; U(K) = \delta(k-m) \;\; where \;\; \delta(k-m) = \begin{cases} 1, & if\ k = m \\ 0, & if\ k \neq m \end{cases}$

Theorem 3: If $u(x) = e^x \;\; then \;\; U(k) = \frac{1}{k!}$

Theorem 4: If u (x) = g (x) h (x) then $U(k) = \sum_{l=0}^{k} G(l)H(k-l)$

Theorem 5: If $y(x) = y_1(x)\, y_2(x)\ then\ Y(k) = \sum_{k_1=0}^{k} Y_1(k_1)\, Y_2(k-k_1)$

Theorem 6: If $y(x) = \frac{d^n y_1(x)}{dx^n}$, then $Y(k) = \frac{(k+n)!}{k!} Y_1(k+n)$

Theorem 7: If $y(x) = e^{\lambda x}$ then $Y(k) = \frac{\lambda^k}{k!}$, λ is constant

Theorem 8: If y $(x) = \sin(wx+\alpha) \;\; then \;\; Y\ (k) = \frac{w^k}{k!}\sin(\frac{k\pi}{2} + \alpha)$, where α, w are constant.

Theorem 9: If $(x) = \cos(wx+\alpha) \;\; then \;\; Y\ (k) = \frac{w^k}{k!}\cos(\frac{k\pi}{2} + \alpha)$, where α, w are constant.

ANALYSIS OF METHOD

Consider an nth order boundary value problem of the form

$$y^n(x) = f(x, y, y', y'', ..., y^{n-1}), \qquad 0 \le x \le 1 \tag{4}$$

with the initial conditions

$$B\left(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, ... \frac{d^{n-1}y}{dx^{n-1}} \right) \tag{5}$$

The differential transform of (4) is

$$Y(k+n) = \frac{F(k)}{(k+n)!} \tag{6}$$

where $F(k)$ is the differential transform of $f(x, y, y', y'', ..., y^{n-1})$.

The transformed boundary condition can be written as

$$Y(k) = A, Y(m) = \sum_{k=0}^{N} \prod_{i=1}^{m-1} (k-i)Y(k) = B_m, (m < n), \tag{7}$$

where m is the order of the derivative in the boundary conditions and A, B_m are real constants.

Using (6) and (7), the value of $Y(i), i = 1,2,3,...$ can be determined and by using inverse differential transformation, the following approximate solution up $O(x^{N+1})$ can be determined

$$y(x) = \sum_{k-0}^{N} Y(k)x^k \tag{8}$$

ANALYSIS OF METHOD FOR NONLINEAR HIGHER FRACTIONAL ORDER BOUNDARY VALUE PROBLEMS

Consider an nth order boundary value problem of the form

$$y^n(x) = e^{-x} y^{-\frac{1}{m}}(x)$$

is given by

$$Y(k+n) = \frac{k!}{(k+n)!}\left[\left(\frac{1}{\left(\frac{1}{m}-1\right)^n}\right)\left(\frac{(-1)^K}{K!}\right)K!\,Y(k)\right], k = 0,2,4\ldots$$

$$Y(k+n) = \frac{k!}{(k+n)!}\left[\left(\frac{1}{\left(1-\frac{1}{m}\right)^n}\right)\left(\frac{(-1)^K}{K!}\right)K!\,Y(k)\right], k = 1,3,5,\ldots \tag{9}$$

Example 1: Consider a linear sixth order boundary value problem

$$y^6(x) = y(x) - 6e^x, 0 < x < 1, \tag{10}$$

with the boundary conditions

$$y(0) = 1, y(1) = 0, y'(0) = 0, y'(1) = -e, y''(0) = -1, y''(1) = -2e \tag{11}$$

By applying theorems (6), (7) on the equation (10) we get,

$$Y(k+6) = \frac{1}{\prod_{n=1}^{6}(k+n)}\left[Y(k) - \frac{6}{k!}\right] \tag{12}$$

Differential Transformation of the boundary condition equation (11) is

$$Y(0) = 1, Y(1) = 0, Y(2) = -\frac{1}{2}, \sum_{k=0}^{n} Y(k) = 0$$

$$\sum_{k=0}^{n} kY(k) = -e, \sum_{k=0}^{n} k(k-1)Y(k) = -2e \tag{13}$$

Put $n = 3, k = 0$ in (6) we get $Y(3) = -\frac{1}{3}$, put $n = 4, k = 0$ in (6) we get $Y(4) = -\frac{1}{8}$

Put $n = 5, k = 0$ in (6) we get $Y(5) = -\frac{1}{30}$, put $n = 6, k = 0$ in (6) we get $Y(6) = -\frac{1}{144}$

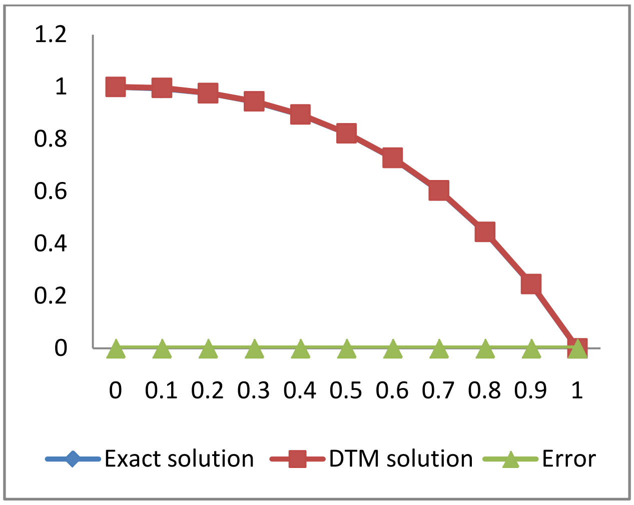
Put $n = 7, k = 0$ in (6) we get $Y(7) = -\frac{1}{840}$, put $n = 8, k = 0$ in (6) we get $Y(k) = -\frac{1}{5760}$

By using inverse differential transformation equation (3) and using boundary conditions we obtain the following series solution up to $O(x^8)$,

$$y(x) = \sum_{k-0}^{k=8} Y(k)x^k = 1 - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{8} - \frac{x^5}{30} - \frac{x^6}{144} - \frac{x^7}{840} - \frac{x^8}{5860}$$

Exact solution for equation (10) is $y(x) = (1-x)e^x$, In the following table we compare the exact solution and DTM solution,

x	Exact solution	DTM solution	Error
0	1	1	0
0.1	0.994350367	0.996320492	$1X10^{-3}$
0.2	0.976526077	0.976309706	$2 X 10^{-4}$
0.3	0.944036589	0.944901165	$8X10^{-5}$
0.4	0.894002981	0.895094824	$1X10^{-3}$
0.5	0.823103881	0.822436068	$1X10^{-3}$
0.6	0.727514353	0.728847758	$1X10^{-3}$
0.7	0.602836803	0.604126777	$1X10^{-3}$
0.8	0.444022296	0.445111434	$1X10^{-3}$
0.9	0.245255774	0.245969802	$1X10^{-4}$
1	0	0.000002480	$1X10^{-5}$



Example 2: Consider an eighth order boundary value problem having fractional order with nonlinear function

$$y^{8}(x) = e^{-x}\sqrt{y(x)} \tag{14}$$

with the boundary conditions

$$y(0) = 1, y(1) = -2, y'(0) = 1, y'(1) = e, y''(0) = 1,$$
$$y''(1) = 2, y'''(0) = 1, y'''(1) = -\frac{1}{6} \tag{15}$$

By applying theorems (6), (7) on the equation (14) and using (9) and the transformation of (15) becomes,

$if\;\; n = 1, m = \frac{1}{2}$

$$Y(k+1) = \frac{k!}{(k+1)!}\left[\left(\frac{1}{\left(\frac{1}{m}-1\right)}\right)\left(\frac{(-1)^{K}}{K!}\right)K!\,Y(k)\right], k = 0,2,4 \ldots$$

$$Y(k+1) = \frac{k!}{(k+1)!}\left[\left(\frac{1}{\left(1-\frac{1}{m}\right)}\right)\left(\frac{(-1)^{K}}{K!}\right)K!\,Y(k)\right], k = 1,3,5, \ldots$$

$Y(0) = 1$

$$Y(k+8) = \frac{k!}{(k+8)!}\left[\left(\frac{1}{\left(\frac{1}{m}-1\right)^{8}}\right)\left(\frac{(-1)^{K}}{K!}\right)K!\,Y(k)\right], k = 0,2,4 \ldots$$

$$Y(k+8) = \frac{k!}{(k+1)!}\left[\left(\frac{1}{\left(1-\frac{1}{m}\right)^{8}}\right)\left(\frac{(-1)^{K}}{K!}\right)K!\,Y(k)\right], k = 1,3,5, \ldots$$

$$if \quad k=0, Y(1)=-2, \quad if, k=1, Y(2)=2, \; f \quad k=2, Y(3)=-1.333333333,$$
$$if \quad k=3, Y(4)=0.111111111, if \quad k=4, Y(5)=-1.851851852X10^{-3},$$
$$f \quad k=5, Y(6)=5.144032922X10^{-6}, if \quad k=6, Y(7)=-2.041282905X10^{-9},$$
$$if \quad k=7, Y(8)=1.012541124X10^{-13}, if \quad k=8, Y(9)=-5.580583795X10^{-19},$$
$$if \quad k=9, Y(10)=3.075718582X10^{-25}, if \quad k=10, Y(11)=-1.695171176X10^{-31}$$

By using inverse differential transformation equation and using boundary conditions we obtain the following series solution up to $O(x^{11})$,

$$y(x)=\sum_{k-0}^{k=9} Y(k)x^k$$
$$=1-2x+2x^2-1.333333333x^3+0.111111111x^4-1.851851852X10^{-3}x^5$$
$$+5.144032922X10^{-6}x^6-2.041282905X10^{-9}x^7+1.012541124X10^{-1}\ x^{38}$$
$$-5.580583795X10^{-19}x^9+3.075718582X10^{-25}x^{10}-1.695171176X10^{-31}x^{11}$$

CONCLUSION

In this paper, the differential transformation method has been applied to obtain the numerical solution of linear and nonlinear higher order boundary value problems. The present method has been applied in a direct way without using linearization, discretization, or perturbation. By increasing the order of approximation more accuracy can be obtained. Comparison of the numerical results with the existing technique shows that the present method is more accurate.

REFERENCES

1. Arikoglu A, Ozkol I, *Solution of difference equations by using differential transform method,* Applied Mathematics and Computation, 174 (2006) 1216- 122.
2. Abdel-Halim Hassan I.H., *differential transform technique for solving higher order initial value problems,* Applied Mathematics and Computation, 154 (2004) 299-311 .
3. Ayaz F, "*Solutions of the system of differential equations by differential transform method",* Applied Mathematics and Computation, 147 (2004) 547-567.
4. Bert.W., H. Zeng, *Analysis of axial vibration of compound bars by differential transform method,* Journal of Sound and Vibration, 275 (2004) 641-647.
5. Che H.Hussain, Adem Kilicman, "*On The Solution Of Fractional Order Nonlinear Boundary Value Problems By Using Differential Transform Method", European* Journal Of Pure And Applied Mathematics, Vol.4No.2, 2011, 174-185, ISSN 1307-5543.
6. Chen C.K., S. S. Chen, *Application of the differential transform method to a non-linear conservative system,* Applied Mathematics and Computation 154 (2004) 431-441.
7. Chen C.L., Y.C. Liu, *Differential transformation technique for steady nonlinear heat conduction problems,* Appl. Math. Computer. 95 (1998) 155164.
8. Duan Y, R. Liu, Y. Jiang, *Lattice Boltzmann model for the modified Burger's equation,* Appl.Math. Computer. 202 (2008) 489-497.
9. HassanI.H. *Abdel-Halim Comparison differential transformation technique with Adomain decomposition method for linear and nonlinear initial value problems*, Choas Solutions Fractals, 36(1): 53-(2008)65.
10. Khaled Batiha and Belal Batiha, *"A New Algorithm for Solving Linear Ordinary Differential Equations",* World Applied Sciences Journal 15 (12):1774-1779, 2011, ISSN 1818-4952, IDOSI Publications, 2011.
11. Kuo B.L. *Applications of the differential transform method to the solutions of the free convection Problem,* Applied Mathematics and Computation 165 (2005) 63-79.
12. Montri Thongmoon, Sasitorn Pusjuso, "*The numerical solutions of differential transform method and the Laplace transforms method for a system of differential equations"* journal homepage: www.elsevier.com/ locate/nahs, Nonlinear Analysis: Hybrid Systems 4 (2010) 425-431.
13. Shahid S. Siddiqi, Ghazala Akram, "*Solution of Seventh Order Boundary Value Problem by Differential Transformation Method"*, World Applied Sciences Journal 16 (11): 1521- 1526, 2012ISSN 18-4952.
14. V. Ananthswamy, "*Solution of the linear and non-linear differential equations by using Homotopy perturbation method",* Indian Journal of Applied Research, Volume3, 2014, ISSN-2249-555X.
15. Zhou X, "*Differential Transformation and its applications for Electrical Circuit*", Huazhong University Press Wuhan, China, 1996.

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