A CLASSIFICATION OF NEARLY LORENTZIAN SASAKIAN MANIFOLD

L. K. PANDEY*

D. S. Institute of Technology & Management, Ghaziabad, (U.P.) – 201007, India.

(Received On: 30-09-15; Revised & Accepted On: 25-10-15)

ABSTRACT


Keywords: Generalized nearly Lorentzian Sasakian manifold, generalized nearly Lorentzian Special Sasakian manifold, generalized Lorentzian Co-symplectic manifolds and semi-symmetric metric F-connection.

1. INTRODUCTION

An n (=2m+1) dimensional differentiable manifold $M_n$, on which there are defined covariant vector fields $A_i$, where $i = 3,4,5,...,(n-1)$, the associated contravariant vector fields $T_i$, where $i = 3,4,5,...,(n-1)$, a tensor field $F$ of type $(1,1)$ and a Lorentzian metric $g$, satisfying

$(1.1)\ F^2 = -I_n - \sum_{i=3}^{n-1} A_i \otimes T_i, \quad FT_i = 0, \quad A_i(T_i) = -1, \quad A_i(FX) = 0, \quad \text{Rank } F = n - i$

$(1.2)\ g(FX,Y) = g(X,Y) + \sum_{i=3}^{n-1} A_i(X) A_i(Y), \quad \text{where } A_i(X) = g(X,T_i), i = 3,4,5,...,(n-1),$

Then $M_n$ is called a generalized Lorentzian contact manifold (a generalized L-contact manifold) and the structure $(F, T_i, A_i, g)$ is known as generalized L-contact structure [5]

Let D be a Riemannian connection on $M_n$, then we have

$(1.3)\ (a) \quad (D^X F)(Y,Z) - (D^Y F)(X,Z) + \sum_{i=3}^{n-1} A_i(Y)(D^X A_i)(Z) + \sum_{i=3}^{n-1} A_i(Z)(D^X A_i)(Y) = 0$

$(b) \quad (D^X F)(Y,F^2 Z) = (D^F X)(F^2 Y, F^2 Z)$

A generalized L-contact manifold is called a generalized Lorentzian Sasakian manifold, if

$(1.4)\ (a) \quad i(D_F X)(Y) - F^2 X \sum_{i=3}^{n-1} A_i(Y) = g(FX, F^2 Y) \sum_{i=3}^{n-1} T_i = 0 \iff$

$(b) \quad i(D_F X)(Y, Z) + g(FX, F^2 Z) \sum_{i=3}^{n-1} A_i(Y) - g(FX, F^2 Y) \sum_{i=3}^{n-1} A_i(Z) = 0 \iff$

$(c) \quad iD_F X = F^2 X + T_i - \sum_{i=3}^{n-1} T_i,$

From which, we get

$(1.5)\ (a) \quad i(D_F A_i)(Y) = g(FX, F^2 Y) \iff$

$(b) \quad i(D_F A_i)(Y) - A_i(Y) + \sum_{i=3}^{n-1} A_i(Y) = F^2 X, Y)$

A generalized L-contact manifold is called a generalized Lorentzian Special Sasakian manifold (a generalized LS-Sasakian manifold), if

$(1.6)\ (a) \quad i(D_F X)(Y) + F^2 X \sum_{i=3}^{n-1} A_i(Y) - F(X,Y) \sum_{i=3}^{n-1} T_i = 0 \iff$

$(b) \quad i(D_F X)(Y, Z) + F(X, Z) \sum_{i=3}^{n-1} A_i(Y) - F(X, Y) \sum_{i=3}^{n-1} A_i(Z) = 0 \iff$

$(c) \quad iD_F X = F^2 X + T_i - \sum_{i=3}^{n-1} T_i,$

Corresponding Author: L. K. Pandey*

D. S. Institute of Technology & Management, Ghaziabad, (U.P.) – 201007, India.
From which, we get
(1.7) (a) \( i(D_X A_i) (FY) = F(X, Y) \) \( \Leftrightarrow \)
(b) \( i(D_X A_i) (Y) = A_i (Y) + \sum_{i=3}^{n} A_i (Y) = -g (FX, FY) \)

A generalized L-contact manifold is called a generalized Lorentzian Co-symplectic manifold, if
(1.8) (a) \( (D_X F)(Y) - \sum_{i=3}^{n} A_i (Y) F D_X T_i - \sum_{i=3}^{n} (D_X A_i) (FY) T_i = 0 \) \( \Leftrightarrow \)
(b) \( (D_X F)(Y, Z) + \sum_{i=3}^{n} A_i (Y) (D_X A_i) (FZ) - \sum_{i=3}^{n} A_i (Z) (D_X A_i) (FY) = 0 \)

Therefore, a generalized Lorentzian Co-symplectic manifold will be a generalized Lorentzian Sasakian manifold, if
(1.9) (a) \( i(D_X A_i) (FY) = g (FX, FY) \) \( \Leftrightarrow \)
(b) \( i(D_X A_i) (Y) = A_i (Y) + \sum_{i=3}^{n} A_i (Y) = F (X, Y) \) \( \Leftrightarrow \)
(c) \( i(D_X T_i) = F^2 + T_i - \sum_{i=3}^{n} T_i \)

And a generalized Lorentzian Co-symplectic manifold will be a generalized LS-Sasakian manifold, if
(1.10) (a) \( i(D_X A_i) (FY) = F(Y, X) \) \( \Leftrightarrow \)
(b) \( i(D_X A_i) (Y) = A_i (Y) + \sum_{i=3}^{n} A_i (Y) = -g (FX, FY) \) \( \Leftrightarrow \)
(c) \( i(D_X T_i) = F^2 + T_i - \sum_{i=3}^{n} T_i \)

Nijenhuis tensor in a generalized almost contact metric manifold is given by
(1.11) \( \nabla (X, Y, Z) = (D_X F)(Y, Z) - (D_Y F)(X, Z) + (D_Y F)(Y, FZ) - (D_Y F)(X, FZ) \)

Where \( \nabla (X, Y, Z) \equiv g (N (X, Y, Z) \)

2. GENERALIZED NEARLY LORENTZIAN SASAKIAN MANIFOLD

A generalized L-contact manifold is called a generalized nearly Lorentzian Sasakian manifold, if
(2.1) \( i(D_X F)(Y, Z) + \sum_{i=3}^{n} A_i (Y) g (FZ, FX) - \sum_{i=3}^{n} A_i (Z) g (FX, FY) \)
\[ = i(D_Y F)(Z, X) + \sum_{i=3}^{n} A_i (Z) g (FX, FY) - \sum_{i=3}^{n} A_i (X) g (FY, FZ) \]
\[ = i(D_Z F)(X, Y) + \sum_{i=3}^{n} A_i (X) g (FY, FZ) - \sum_{i=3}^{n} A_i (Y) g (FX, FZ) \]

From which, we get
(2.2) (a) \( i(D_X F)(Y) + i(D_Y F)(X) - \sum_{i=3}^{n} A_i (Y) F^2 X - \sum_{i=3}^{n} A_i (X) F^2 Y - 2 \sum_{i=3}^{n} T_i g (FX, FY) = 0 \) \( \Leftrightarrow \)
(b) \( i(D_X F)(Y, Z) + i(D_Y F)(X, Z) + \sum_{i=3}^{n} A_i (Y) g (FX, FZ) + \sum_{i=3}^{n} A_i (X) g (FY, FZ) - \sum_{i=3}^{n} A_i (Z) g (FX, FZ) = 0 \)

From which, we get
(2.3) (a) \( i(D_X F)(Y) + i(D_Y F)(X) + \sum_{i=3}^{n} A_i (X) F(Y) + 2 \sum_{i=3}^{n} T_i F(Y, X) = 0 \) \( \Leftrightarrow \)
(b) \( i(D_X F)(Y) + i(D_Y F)(X, Z) + \sum_{i=3}^{n} A_i (X) g (FX, FZ) + 2 \sum_{i=3}^{n} A_i (Z) g (FX, FY) = 0 \)

(2.4) (a) \( i(D_X F)(Y) + i(D_Y F)(X) + \sum_{i=3}^{n} A_i (X) F^2 Y + 2 \sum_{i=3}^{n} T_i g (FX, FY) = 0 \) \( \Leftrightarrow \)
(b) \( i(D_X F)(Y, Z) + i(D_Y F)(X, Z) + \sum_{i=3}^{n} A_i (X) g (FY, FZ) + 2 \sum_{i=3}^{n} A_i (Z) g (FX, FY) = 0 \)

(2.5) (a) \( i(D_X F)(Y) + i(D_Y F)(X) - \sum_{i=3}^{n} A_i (Y) (F D_X T_i - (D_Y F)(X)) - \sum_{i=3}^{n} A_i (X) (F D_Y T_i - (D_Y F)(Y)) \)
\[ - \sum_{i=3}^{n} T_i (D_i A_i)(FX) + (D_Y A_i)(FY) = 0 \) \( \Leftrightarrow \)
(b) \( i(D_X F)(Y, Z) + i(D_Y F)(X, Z) + \sum_{i=3}^{n} A_i (Y) (D_X A_i)(FZ) - (D_Y A_i)(FY) + \sum_{i=3}^{n} A_i (X) (D_Y A_i)(FY) + (D_Y A_i)(FX) = 0 \)

Pre-multiplying \( X, Y, Z \) by \( F \) in (1.11) and using equations (2.1), (1.3) (b), we see that a generalized nearly Lorentzian Sasakian manifold will be completely integrable, if
(2.6) \( D_X F)(\bar{X}, \bar{Z}) = (D_Y F)(\bar{X}, \bar{Z}) \)

3. GENERALIZED NEARLY LORENTZIAN SPECIAL SASAKIAN MANIFOLD

A generalized L-contact manifold is called a generalized nearly Lorentzian Special Sasakian manifold (a generalized nearly LS-Sasakian manifold), if
(3.1) \( i(D_X F)(Y, Z) - \sum_{i=3}^{n} A_i (Y) F(Z, X) - \sum_{i=3}^{n} A_i (Z) F(X, Y) \)
\[ = i(D_Y F)(Z, X) - \sum_{i=3}^{n} A_i (Z) F(X, Y) - \sum_{i=3}^{n} A_i (X) F(Y, Z) \]
\[ = i(D_Z F)(X, Y) - \sum_{i=3}^{n} A_i (X) F(Y, Z) - \sum_{i=3}^{n} A_i (Y) F(Z, X) \]

From which, we obtain
(3.2) (a) \( i(D_X F)(Y) + i(D_Y F)(X) + \sum_{i=3}^{n} A_i (Y) F(X) + \sum_{i=3}^{n} A_i (X) F(Y) = 0 \) \( \Leftrightarrow \)
(b) \( i(D_X F)(Y, Z) + i(D_Y F)(X, Z) - \sum_{i=3}^{n} A_i (Y) F(Z, X) + \sum_{i=3}^{n} A_i (X) F(Y, Z) = 0 \)
Where

In consequence of (5.2), we have

Let

\( (b) \)

This gives

(5.4)

\( (a) \)

Denoting by \( g_{\alpha\beta}(F,F) \) \( (F^2Y,Z) - \sum_{i=3}^{n-1} A_i(X)(F^2Y,FZ) = 0 \)

And a generalized nearly LS-Sasakian manifold will be a generalized nearly Lorentzian Co-symplectic manifold, if

(5.5)

\( (a) \)

\( (b) \)

\( \) (3.4) (a) \( \)

\( (b) \)

Pre-multiplying \( X, Y, Z \) by \( F \) in (1.11) and using equations (3.1), (1.3) (b), we see that a generalized nearly LS-Sasakian manifold will be completely integrable, if

(3.6) (D"_X F) \( (\bar{\bar{F}}, \bar{Z}) + (D'_Y F) \( (\bar{\bar{F}}, \bar{X}) \)

4. GENERALIZED NEARLY LORENTZIAN CO-SYMPLECTIC MANIFOLD

A generalized L-contact manifold is called a generalized nearly Lorentzian Co-symplectic manifold, if

(4.1) \( (D'_X F)(Y,Z) + \sum_{i=3}^{n-1} A_i(X)(D'_Y A_i)(FZ) - \sum_{i=3}^{n-1} A_i(Z)(D'_Y A_i)(FY) \)

\( = (D'_Y F)(X,Z) + \sum_{i=3}^{n-1} A_i(Y)(D'_X A_i)(FX) - \sum_{i=3}^{n-1} A_i(X)(D'_X A_i)(FY) \)

\( = (D'_X F)(X,Y) + \sum_{i=3}^{n-1} A_i(X)(D'_Y A_i)(FY) - \sum_{i=3}^{n-1} A_i(Y)(D'_X A_i)(FX) \)

Therefore, a generalized nearly Lorentzian Sasakian manifold will be a generalized nearly Lorentzian Co-symplectic manifold, if

(4.2) (a) \( i(D'_X A_i)(F) = g(FX,FY) \)

(b) \( i(D'_X A_i)(Y) - A_i(Y) + \sum_{i=3}^{n-1} A_i(Y) = \gamma(F(X,Y) \)

(c) \( iD_X T_i = FX + T_i - \sum_{i=3}^{n-1} T_i \)

And a generalized nearly LS-Sasakian manifold will be a generalized nearly Lorentzian Co-symplectic manifold, if

(4.3) (a) \( i(D'_X A_i)(F) = \gamma(F(X,Y) \)

(b) \( i(D'_X A_i)(Y) - A_i(Y) + \sum_{i=3}^{n-1} A_i(Y) = -g(FX,FY) \)

(c) \( iD_X T_i = F^2X + T_i - \sum_{i=3}^{n-1} T_i \)

5. SEMI-SYMMETRIC METRIC F-CONNECTION IN A GENERALIZED LS-SASAKIAN MANIFOLD

Let \( M_{2m-1} \) be submanifold of \( M_{2m+1} \) and let \( c : M_{2m-1} \rightarrow M_{2m+1} \) be the inclusion map such that

\( d \in M_{2m-1} \rightarrow cd \in M_{2m+1} \),

Where \( c \) induces a Jacobian map (linear transformation) \( J : T'_{2m-1} \rightarrow T_{2m+1} \).

\( T'_{2m-1} \) is tangent space to \( M_{2m-1} \) at point \( d \) and \( T_{2m+1} \) is tangent space to \( M_{2m+1} \) at point \( cd \) such that

\( X \) in \( M_{2m-1} \) at \( d \) \( \rightarrow \bar{J^\alpha}X \) in \( M_{2m+1} \) at \( cd \)

Let \( \bar{g} \) be the induced Lorentzian metric in \( M_{2m-1} \), then

(5.1) \( \bar{g}(\bar{X}, \bar{Y}) = ((g(J\bar{X}, J\bar{Y}))b) \)

Semi-symmetric metric F-connection \( B \) in generalized Lorentzian special Sasakian manifold \( M_n \) is given by

(5.2) \( iB_{\alpha\beta}(Y) = iD_{\alpha\beta}Y = \sum_{i=3}^{n-1} A_i(Y)X + \sum_{i=3}^{n-1} g(X,Y)T_i - 2 \sum_{i=3}^{n-1} A_i(Y)Y \)

Where \( X \) and \( Y \) are arbitrary vector fields of \( M_{2m+1} \). Let

(5.3) \( T_i = \rho_t X + \sigma_i M + \sigma_i N, \)

Where \( i = 3, 4, 5, \ldots (n-1) \).

Where \( \rho_t, \sigma_i \) are \( C^\infty \) vector fields in \( M_{2m-1} \). \( M, N \) are unit normal vectors to \( M_{2m-1} \).

Denoting by \( \bar{D} \) the connection induced on the submanifold from \( D \). Gauss equation is

(5.4) \( iB_{\alpha\beta}(Y) = iD_{\alpha\beta}Y = \sum_{i=3}^{n-1} A_i(Y)X + \sum_{i=3}^{n-1} g(J\bar{X}, J\bar{Y})T_i - 2 \sum_{i=3}^{n-1} A_i(J\bar{X})J\bar{Y} \)

Where \( p \) and \( q \) are symmetric bilinear functions in \( M_{2m-1} \). Also

(5.5) \( iB_{\alpha\beta}(Y) = iD_{\alpha\beta}Y = \sum_{i=3}^{n-1} A_i(J\bar{Y})\bar{X} + \sum_{i=3}^{n-1} g(J\bar{X}, J\bar{Y})T_i - 2 \sum_{i=3}^{n-1} A_i(J\bar{X})J\bar{Y} \)

In consequence of (5.2), we have

(5.6) \( iB_{\alpha\beta}(Y) = iD_{\alpha\beta}Y = \sum_{i=3}^{n-1} A_i(J\bar{Y})\bar{X} + \sum_{i=3}^{n-1} g(J\bar{X}, J\bar{Y})T_i - 2 \sum_{i=3}^{n-1} A_i(J\bar{X})J\bar{Y} \)

© 2015, IJMA. All Rights Reserved
Using (5.4), (5.5) and (5.6), we have
\begin{equation}
(i\tilde{\beta}_x\tilde{Y}) + \ i h(\tilde{x},\tilde{Y})M + i k(\tilde{x},\tilde{Y})N = \ i(\tilde{D}_x\tilde{Y}) + \ i p(\tilde{x},\tilde{Y})M + i q(\tilde{x},\tilde{Y})N - \sum_{i=3}^{n-1} a_i(\tilde{Y})\tilde{X} + \ \sum_{i=3}^{n-1} g(\tilde{X},\tilde{Y})\tilde{t}_i - 2 \sum_{i=3}^{n-1} a_i(\tilde{X})\tilde{Y}
\end{equation}

Using (5.3), we get
\begin{equation}
(i\tilde{\beta}_x\tilde{Y}) + \ i h(\tilde{x},\tilde{Y})M + i k(\tilde{x},\tilde{Y})N = \ i(\tilde{D}_x\tilde{Y}) + \ i p(\tilde{x},\tilde{Y})M + i q(\tilde{x},\tilde{Y})N - \sum_{i=3}^{n-1} a_i(\tilde{Y})\tilde{X} + \ \sum_{i=3}^{n-1} (\tilde{t}_i + \rho_i M + \sigma_i N) g(\tilde{x},\tilde{Y}) - 2 \sum_{i=3}^{n-1} a_i(\tilde{X})\tilde{Y}
\end{equation}

Where $\bar{g}(\tilde{Y},t_i) \equiv a_i(\tilde{Y})$

This implies
\begin{equation}
i(\tilde{D}_x\tilde{Y}) = \ i p(\tilde{x},\tilde{Y}) + \sum_{i=3}^{n-1} \rho_i \bar{g}(\tilde{x},\tilde{Y})
\end{equation}

If
\begin{equation}
i k(\tilde{x},\tilde{Y}) = \ i q(\tilde{x},\tilde{Y}) + \sum_{i=3}^{n-1} \sigma_i \bar{g}(\tilde{x},\tilde{Y})
\end{equation}

Thus, we have

**Theorem 5.1:** The connection induced on a submanifold of a generalized Lorentzian special Sasakian manifold with a Semi-symmetric metric F-connection with respect to unit normal vectors $M$ and $N$ is also Semi-symmetric metric F-connection iff (5.10) holds.

**REFERENCES**


Source of support: Nil, Conflict of interest: None Declared