

PERSISTENCE AND OPTIMAL HARVESTING
OF MARINE POPULATION AFFECTED WITH PRIMARY AND SECONDARY TOXICANT

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ABSTRACT

This work investigates the affect of toxicant which is structured into two stages, primary and secondary on a single species fishery model. Boundedness and positivity of solution has been shown in order to ensure feasibility of biological model. The time lag required for transmission of primary toxicant to secondary toxicant is incorporated and resulting delayed model is analyzed for stability. Optimal harvesting policy has been discussed by using Pontryagin's Principal. Butler-Mc Gehee lemma is used to identify the condition which influences the persistence of the system. Finally, some numerical simulations are given to verify the mathematical conclusions.

Keywords: Stability, Time-delay, Optimal harvesting, Persistence.

1. INTRODUCTION

With the rapid development of modern technology, industry and agriculture, a large quantity of toxicant and contaminants enter into the ecosystem one after another. These toxicants seriously threaten the survival of the exposed population. In order to regulate toxic substances wisely, we must assess the risk of the population exposed to toxicant. Therefore, it is important to study the effects of toxicant on biological population and to find a theoretical threshold value, which determines permanence or extinction of biological population.

In recent years, some investigations have been conducted to study the effect of toxicant emitted into the environment from industrial and household resources on biological species by using mathematical models. In particular, Hallam *et.al.* [2–5] in series of their papers studied qualitative approach of toxicants on population. They assumed growth rate density of single species as decreasing function of toxicant concentration whereas carrying capacity is taken as constant.

Freedman and Shukla in their paper [1] studied the effect of toxicant on a single species and on a prey–predator community by taking into account the introduction of toxicant from an external source and both growth rate and carrying capacity are taken as decreasing function of toxicant. They assumed the same nature of toxicant and not taken into consideration different stages of toxicant. But in reality, we can structure toxicant according to its level of intensity or according to its chemical composition. Since in some cases, toxicant at low intensity level does not effect the growth of biological species but when its intensity is increased, it affects the biological population adversely. One example is the emission of carbon and sulphur dioxide through industries and vehicles. These pollutants do not affect fishery habitat in initial stage but in more toxic stage, in the form of acid rain these pollutant affect the growth of marine ecosystem seriously.

Shukla *et al.* [6] studied the effects of primary and secondary toxicants on resource biomass. They showed that the decrease in biomass density of resource is more than one, in corresponding case of a single toxicant due to large transformation, uptake rates and high toxicity of secondary toxicant.

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Keeping the above thing in mind, we propose a fishery model to discuss the effects of primary (low intensity) and secondary (more toxic) toxicant on the stability and harvesting of marine species. The organization of paper is as follows : Section 2 deals with the mathematical model and some basic results on positivity and boundedness of the system. In section 3 existence of equilibrium points and their stability behaviour is discussed in the absence of delay. The critical value of delay is calculated at which stability change can occur and occurrence of Hopf bifurcation. Optimal harvesting policy is discussed in section 4. In Section 5, we derive the sufficient conditions for persistence and numerical simulations are included to illustrate the applicability of the results obtained in Section 6, and lastly discussion is presented in Section 7.

2.THE MATHEMATICAL MODEL

We consider a single species fishery model with toxicant affect governed by following differential equations:

$$\begin{aligned}\frac{dx}{dt} &= r(T_2, \in T_1) x \left(1 - \frac{x}{K(T_2)} \right) - qxE, \\ \frac{dT_1}{dt} &= Q_0 - \alpha_1 T_1 - \beta T_1(t - \tau) - h(x, \in T_1), \\ \frac{dT_2}{dt} &= \theta \beta T_1(t - \tau) - \alpha_2 T_2 - \gamma g(x, T_2),\end{aligned}\tag{1}$$

with initial conditions

$$x(0) > 0, T_1(t) = \phi_1(t) > 0, T_2(t) = \phi_2(t) > 0 \text{ for } -\tau \leq t \leq 0.$$

Here $x(t)$ is the concentration of single fish species, $T_1(t)$ is the concentration of primary toxicant that is of low intensity and $T_2(t)$ is highly toxic secondary toxicant concentration at any time $t > 0$. In modelling the system (1), we made following assumption:

H₁: The fish population grows logistically with its secondary toxicant dependent carrying capacity $K(T_2)$ and toxicant dependent growth function $r(T_2, \in T_1)$. It is assumed that growth is slightly affected by primary toxicant i.e. $\in \ll 1$. These functions satisfy the following properties:

$$\begin{aligned}K(0) &= K_0 > 0, K'(T_2) < 0 \quad \forall T_2 > 0, \\ r(0) &= r_0 > 0, \frac{\partial r}{\partial T_2} < 0, \frac{\partial r}{\partial T_1} < 0, \quad \forall T_1, T_2 > 0,\end{aligned}$$

there exists some values of T_1 and T_2 such that

$$K(\bar{T}_2) = 0, r(\bar{T}_2, \in \bar{T}_1) = 0.$$

H₂: The fish population is harvested with constant harvesting effort E in direct proportion to its concentration with constant catchability coefficient $q > 0$.

H₃: The primary toxicant is emitted into the environment with a prescribed rate Q_0 by an external source and transmitted into highly toxic form after time τ with toxicant dependent conversion function as $\theta \beta T_1(t - \tau)$ where β is constant conversion rate which converted into secondary toxicant from primary toxicant.

H₄: In the absence of toxicant, when $r_0 - qE < 0$, $\frac{dx}{dt} < 0$, and fish population approaches to extinction. So throughout in this paper, we assume $r_0 - qE > 0$.

H₅: The uptake of primary and secondary toxicant by the population is presented by increasing functions $h(x, T_1)$ and $g(x, T_2)$ respectively :

$$\begin{aligned}h(x, T_1) &= h(x, 0) = 0, \quad \frac{\partial h}{\partial x} > 0, \quad \frac{\partial h}{\partial T_1} > 0, \\ g(0, T_2) &= g(x, 0) = 0, \quad \frac{\partial g}{\partial x} > 0, \quad \frac{\partial g}{\partial T_2} > 0 \quad \forall x, T_1, T_2 > 0.\end{aligned}$$

The constants α_1 and α_2 are depletion rate coefficients of primary and secondary toxicant respectively due to various factors in the environment. For simplicity, we assume simplest form of the functions as below:

$h(x, T_1) = \epsilon x T_1$, $0 < \epsilon < 1$ is the constant uptake coefficient of primary toxicant by the population.

$g(x, T_2) = \gamma x T_2$, $\gamma > 0$ is the proportionality constant for the uptake of secondary toxicant by the population.

Then the system (1) takes the form

$$\begin{aligned} \frac{dx}{dt} &= r(T_2, \epsilon T_1) x \left(1 - \frac{x}{K(T_2)} \right) - qx E, \\ \frac{dT_1}{dt} &= Q_0 - \alpha_1 T_1 - \beta T_1(t - \tau) - \epsilon x T_1, \\ \frac{dT_2}{dt} &= \theta \beta T_1(t - \tau) - \alpha_2 T_2 - \gamma x T_2, \end{aligned} \quad (2)$$

with initial condition

$$x(0) > 0, T_1(t) = \phi_1(t) > 0, T_2(t) = \phi_2(t) > 0 \quad \text{for } -\tau \leq t \leq 0.$$

Next, we have theorems regarding the positivity and boundedness of solutions of system (2).

Theorem 1: All solutions of the system (2) with initial conditions are non-negative.

Proof: Let $t = t_1$ be the first time when $T_1(t_1) = 0$. So $t_1 = \{\min t > 0, T_1(t) = 0\}$,

$$\text{then } \left. \frac{dT_1}{dt} \right|_{t=t_1} = Q_0 - \beta T_1(t_1 - \tau) > 0,$$

(since concentration of emitted toxicant should be sufficiently greater than transmitted toxicant). Hence for sufficiently small $\eta_1 > 0$, $\dot{T}_1(t_1 - \eta_1) > 0$. But by definition of t_1 , $\dot{T}_1(t_1 - \eta_1) \leq 0$, this contradiction proves that

$$T_1(t) > 0 \quad \forall t > 0. \text{ Similarly, let } t_2 = \{\min t > 0, T_2(t) = 0\}, \text{ then } \left. \frac{dT_2}{dt} \right|_{t=t_2} = \beta T_1(t_2 - \tau) > 0. \text{ Hence for}$$

sufficiently small $\eta_2 > 0$, $\dot{T}_2(t_2 - \eta_2) > 0$. But by definition of t_2 , $\dot{T}_2(t_2 - \eta_2) \leq 0$. again contradiction proves

$$\text{that } T_2(t) > 0 \quad \forall t > 0 \text{ and also } \left. \frac{dx}{dt} \right|_{x=0} = 0 \text{ proves the non-negativity of } x(t).$$

Hence, all solutions of system (2) are non-negative.

Theorem 2: The region

$$\Omega = \left\{ (x, T_1, T_2) \in R_+^3 : 0 \leq T_1(t) + T_2(t) \leq \frac{Q_0}{\alpha}, \alpha = \min(\alpha_1, \alpha_2) \right\}$$

is a region of attraction for all solutions initiating in the positive octant.

Proof: From system (2)

$$\frac{dx}{dt} \leq r_0 x \left(1 - \frac{x}{K_0} \right).$$

On integrating and taking limit $t \rightarrow \infty$

$$x(t) \leq K_0.$$

$$\text{Also } \frac{dT_1}{dt} + \frac{dT_2}{dt} \leq Q_0 - \alpha(T_1 + T_2),$$

where $\alpha = \min(\alpha_1, \alpha_2)$. According to comparison principle, it follows that $T_1 + T_2 \leq \frac{Q_0}{\alpha}$.

3. THE MATHEMATICAL ANALYSIS

Existence of Equilibriums: The system (2) has only two non-negative equilibrium points namely

$$\bar{P}(0, \bar{T}_1, \bar{T}_2) \text{ and } P^*(x^*, T_1^*, T_2^*).$$

$\bar{P}(0, \bar{T}_1, \bar{T}_2)$ is given by

$$\bar{T}_1 = \frac{Q_0}{\alpha_1 + \beta}, \bar{T}_2 = \frac{\theta\beta Q_0}{\alpha_2(\alpha_1 + \beta)}.$$

Interior equilibrium point $P^*(x^*, T_1^*, T_2^*)$ is given by the solution of following algebraic equations:

$$r(T_2, \in T_1) \left(1 - \frac{x}{K(T_2)} \right) - qE = 0, \quad (3a)$$

$$Q_0 - \alpha_1 T_1 - \beta T_1 - \in x T_1 = 0, \quad (3b)$$

$$\theta\beta T_1 - \alpha_2 T_2 - \gamma x T_2 = 0. \quad (3c)$$

From eqs. (3b) and (3c), we get

$$T_1^* = \frac{Q_0}{\alpha_1 + \beta + \in x} = f_1(x),$$

$$T_2^* = \frac{\theta\beta Q_0}{(\alpha_1 + \beta + \in x)(\alpha_2 + \gamma x)} = f_2(x).$$

Using the values of T_1^* and T_2^* in eq. (3a), we get

$$F(x) = r(f_2(x), \in f_1(x)) \left(1 - \frac{x}{K(f_2(x))} \right) - qE,$$

$$F(0) = r(f_2(0), \in f_1(0)) - qE,$$

$$F(K_0) = r(f_2(K_0), \in f_1(K_0)) \left(1 - \frac{K_0}{K(f_2(K_0))} \right) - qE < 0,$$

since $K_0 > K(f_2(K_0))$.

So, there exists a unique x^* such that $0 < x^* < K_0$ and $F(x^*) = 0$, when $F(0) > 0$, and $F'(x) < 0$.

Stability Analysis: In order to study the local behavior of system, we consider variational matrix corresponding to each equilibrium points:

$$M_1 = \begin{bmatrix} r(\bar{T}_2, \in \bar{T}_1) - qE & 0 & 0 \\ -\in \bar{T}_1 & -\alpha_1 - \beta e^{-\lambda \tau} & 0 \\ -\gamma \bar{T}_2 & \beta e^{-\lambda \tau} & -\alpha_2 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} -r(T_2^*, \in T_1^*) \frac{x^*}{K(T_2^*)} & \frac{\partial r}{\partial T_1} \Big|_{P^*} x^* \left(1 - \frac{x^*}{K(T_2^*)} \right) & \frac{\partial r}{\partial T_2} \Big|_{P^*} x^* \left(1 - \frac{x^*}{K(T_2^*)} \right) + \frac{r(T_2^*, \in T_1^*) x^{*2} K'(T_2^*)}{K^2(T_2^*)} \\ -\in T_1^* & -\alpha_1 - \beta e^{-\lambda \tau} - \in x^* & 0 \\ -\gamma T_2^* & \theta\beta e^{-\lambda \tau} & -\alpha_2 - \gamma x^* \end{bmatrix},$$

From above analysis, we observe following points about the equilibrium points:

1. The three eigenvalues of variational matrix M_1 are $r(\bar{T}_2, \in \bar{T}_1) - qE$, $-\alpha_1 - \beta e^{-\lambda \tau}$ and $-\alpha_2$. The equilibrium point \bar{P} is stable in $T_1 - T_2$ plane and saddle point in $x - T_1 - T_2$ plane if $r(\bar{T}_2, \in \bar{T}_1) > qE$.
2. The characteristic equation corresponding to interior equilibrium $P^*(x^*, T_1^*, T_2^*)$ is given by

$$\lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 + e^{-\lambda \tau} (b_1 \lambda^2 + b_2 \lambda + b_3) = 0, \quad (4)$$

where

$$\begin{aligned} a_1 &= \frac{x^* r(T_2^*, \in T_1^*)}{K(T_2^*)} + \alpha_1 + \alpha_2 + (\epsilon + \gamma) x^*, \\ a_2 &= \frac{x^* r(T_2^*, \in T_1^*)}{K(T_2^*)} (\alpha_1 + \alpha_2 + (\epsilon + \gamma) x^*) + (\alpha_2 + \gamma x^*) (\alpha_1 + \epsilon x^*) + \\ &\quad \in \frac{\partial r}{\partial T_1} \bigg|_{P^*} x^* T_1^* \left(1 - \frac{x^*}{K(T_2^*)} \right) + \gamma T_2^* H^*, \\ a_3 &= \frac{x^* r(T_2^*, \in T_1^*)}{K(T_2^*)} (\alpha_2 + \gamma x^*) (\alpha_1 + \epsilon x^*) + \\ &\quad \in \frac{\partial r}{\partial T_1} \bigg|_{P^*} x^* T_1^* (\alpha_2 + \gamma x^*) + \gamma T_2^* H^* (\alpha_1 + \epsilon x^*), \\ b_1 &= \beta, b_2 = \beta \left(\frac{x^* r(T_2^*, \in T_1^*)}{K(T_2^*)} + \alpha_2 + \gamma x^* \right), \\ b_3 &= \beta \left(\frac{x^* r(T_2^*, \in T_1^*)}{K(T_2^*)} (\alpha_2 + \gamma x^*) + \epsilon H^* T_1^* \theta + \gamma H^* T_2^* \right), \\ \text{where } H^* &= \frac{\partial r}{\partial T_2} \bigg|_{P^*} x^* \left(1 - \frac{x^*}{K(T_2^*)} \right) + \frac{r(T_2^*, \in T_1^*) x^{*2} K'(T_2^*)}{K^2(T_2^*)}. \end{aligned}$$

When $\tau = 0$, i.e. in case of instantaneous transmission of primary toxicant into secondary toxicant, characteristic equation becomes

$$\lambda^3 + (a_1 + b_1) \lambda^2 + (a_2 + b_2) \lambda + a_3 + b_3 = 0. \quad (5)$$

Since $a_1 + b_1 > 0$, when $a_3 + b_3 > 0$, and $(a_1 + b_1)(a_2 + b_2) > (a_3 + b_3)$, then by Routh-Hurwitz criterion all roots of eq. (5) have negative real parts and P^* is locally asymptotically stable equilibrium point in the absence of delay.

When $\tau \neq 0$, stability of system can change only if there exists at least one root of eq. (4) such that $\text{Re}(\lambda) = 0$. Let $\lambda = iw$ be one such root. Substituting this in eq. (4) and on equating real and imaginary parts, we have

$$(b_1 w^2 - b_3) \cos w \tau - b_2 w \sin w \tau = -a_1 w^2 + a_3, \quad (6)$$

$$(b_1 w^2 - b_3) \sin w \tau + b_2 w \cos w \tau = w^3 - a_2 w. \quad (7)$$

On squaring and adding these two equations, we get

$$w^6 + (a_1^2 - 2a_2 - b_1^2) w^4 + (a_2^2 - 2a_1 a_3 - b_2^2 + 2b_1 b_3) w^2 + a_3^2 - b_3^2 = 0. \quad (8)$$

On substituting $w^2 = u$, in above equation

$$u^3 + (a_1^2 - 2a_2 - b_1^2) u^2 + (a_2^2 - 2a_1 a_3 - b_2^2 + 2b_1 b_3) u + a_3^2 - b_3^2 = 0. \quad (9)$$

When P^* is locally asymptotically stable equilibrium point in the absence of delay and following inequalities hold $a_1^2 - 2a_2 - b_1^2 > 0, a_3^2 - b_3^2 > 0$ and $(a_1^2 - 2a_2 - b_1^2)(a_2^2 - 2a_1a_3 + b_2^2 + 2b_1b_3) > a_3^2 - b_3^2$, (10) then all conditions of Routh Hurwitz criteria for the existence of negative roots are satisfied. Hence, the eq. (9) has only negative real roots and there exist no real solution of eq. (8), so P^* remains stable for all $\tau > 0$.

Again on solving eqs. (6) and (7), we get a critical value of delay that is given as follows :

$$\tau_c = \frac{1}{w} \sin^{-1} \left[\frac{(b_1 w^2 - b_3)(w^3 - a_2 w) + b_2 w(a_1 w^2 - a_3)}{(b_1 w^2 - b_3)^2 + b_2^2 w^2} \right], \text{ this is the least positive value of delay for which stability change occur.}$$

4. OPTIMAL HARVESTING POLICY

In the present section, we discuss the optimal policy that should be adopted by regulatory agency in order to maximize net revenue. That is given by

$$\pi = (pqx - c)E. \quad (11)$$

Our objective is to maximize following integral

$$J = \int_0^\infty e^{-\delta t} (pqx(t) - c)E(t)dt,$$

subject to state equations of system (2) and to control constraint $0 \leq E \leq E_{\max} = \frac{r_0}{q}$. Here δ is the instantaneous annual rate of discount, p is the price per unit biomass of landed fish and c is fishing cost per unit effort.

The associated Hamiltonian is given by

$$H(x, T_1, T_2, t) = e^{-\delta t} (pqx - c)E + \lambda_1(t) \left[r(T_2, \in T_1) x \left(1 - \frac{x}{K(T_2)} \right) - qx E \right] + \lambda_2(t) [Q_0 - \alpha_1 T_1 - \beta T_1 - \in x T_1] + \lambda_3(t) [\theta \beta T_1 - \alpha_2 T_2 - \gamma x T_2], \quad (12)$$

where λ_1, λ_2 and λ_3 are adjoint variables.

The necessary condition for E^* to be optimal over the control set $0 \leq E^* \leq E_{\max}$ is $\frac{\partial H}{\partial E} = 0$, which implies

$$\lambda_1 qx = e^{-\delta t} (pqx - c) = e^{-\delta t} \frac{\partial \pi}{\partial E}. \quad (13)$$

So, the user's cost of harvest per unit of effort equals the discounted value of the future marginal profit of the effort at the steady state level.

According to the Pontryagin's Maximum principle, the adjoint variables λ_1, λ_2 and λ_3 must satisfy

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x}, \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial T_1}, \frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial T_2}.$$

On considering interior equilibrium $P^*(x^*, T_1^*, T_2^*)$ and solving above equations we get

$$\lambda_1(t) = \frac{C_2}{C_1 + \delta} e^{-\delta t}, \lambda_2(t) = \frac{B_2}{B_1 + \delta} e^{-\delta t}, \lambda_3(t) = \frac{A_2}{A_1 + \delta} e^{-\delta t},$$

where

$$A_1 = \alpha_2 + \gamma x^*, A_2 = \left(p - \frac{c}{qx^*} \right) \left\{ \frac{\partial r}{\partial T_2} x^* \left(1 - \frac{x^*}{K(T_2^*)} \right) + \frac{r(T_2^*, \in T_1^*) x^{*2} K'(T_2^*)}{K^2(T_2^*)} \right\},$$

$$B_1 = \alpha_1 + \beta + \epsilon x^*, B_2 = \left(p - \frac{c}{qx^*} \right) \frac{\partial r}{\partial T_1} \bigg|_{p^*} x^* \left(1 - \frac{x^*}{K(T_2^*)} \right) + \frac{A_2 \theta \beta}{A_1 + \delta}, C_1 = \frac{r(T_2^*, \epsilon T_1^*)}{K(T_2^*)},$$

$$C_2 = pqE - \frac{B_2 \epsilon T_1^*}{B_1 + \delta} - \frac{A_2 \gamma T_2^*}{A_1 + \delta}.$$

On equating two values of $\lambda_1(t)$ we get

$$\frac{C_2}{C_1 + \delta} = p - \frac{c}{qx^*}, \quad (14)$$

this expression gives the optimal equilibrium level of species i.e. x_δ , then the optimal equilibrium levels of both toxicants and harvesting effort are given by

$$T_{1\delta} = \frac{Q_0}{\alpha_1 + \beta + \epsilon x_\delta}, T_{2\delta} = \frac{\theta \beta Q_0}{(\alpha_1 + \beta + \epsilon x_\delta)(\alpha_2 + \gamma x_\delta)}, E_\delta = \frac{r(T_{2\delta}, \epsilon T_{1\delta})}{q} \left(1 - \frac{x_\delta}{K(T_{2\delta})} \right).$$

It has been noted from above analysis that $\lambda_i(t)e^{\delta t}$ ($i = 1, 2, 3$) is independent of time and remains bounded as t tends to infinity, so they satisfy transversality condition.

From eq. (14), we observe that

$pqx^* - c = \frac{C_2 qx^*}{C_1 + \delta} \rightarrow 0$ as $\delta \rightarrow \infty$, this shows that the economic rent is completely dissipated when discount rate is infinite.

5. PERSISTENCE

Theorem 5.1: Let $r(\bar{T}_2, \epsilon \bar{T}_1) > qE$ holds, then the system (2) persists (does not persist) if P^* exist (does not exist).

Proof: To prove this theorem, we have to show that there are no omega limit points on the axes of orbits initiating in the interior of positive octant. Suppose u is a point in the positive octant and $\theta(u)$ is the orbit through u and ω is the omega limit set of the orbit through u . Note that $\omega(u)$ is bounded.

We claim that \bar{P} does not belong to $\omega(u)$. If $\bar{P} \in \omega(u)$, the condition $r(\bar{T}_2, \epsilon \bar{T}_1) > qE$ implies that \bar{P} is a saddle point, by Butler McGehee lemma there exists a point v in $\omega(u) \cap M^s(\bar{P})$ where $M^s(\bar{P})$ denote the stable manifold of \bar{P} . Now $M^s(\bar{P})$ is the $T_1 - T_2$ plane implies that an unbounded orbit $\theta(v)$ lies in $\omega(u)$, which is a contrary to the boundedness of the system.

Thus, $\omega(u)$ lies in the positive octant and system (2) is persist. Finally, since only the closed orbits and the equilibria form the omega limit set of the solutions on the boundary of R_+^3 and system (2) is dissipative, by main theorem in Butler et al. (1986) this implies that system (2) is uniformly persistent.

6. NUMERICAL EXAMPLE

In this section, we present numerical simulation to explain the applicability of the results obtained. We choose the following values of the parameters and functions in model (2) as below

$$(a) r(T_2, \epsilon T_1) = r_0 - r_{11}T_2 - \epsilon T_1, \quad K(T_2) = K_0 - K_{11}T_2,$$

$$r_0 = 3, \theta = 0.01, K_0 = 100, \alpha_1 = 0.05, r_{11} = 0.02, K_{11} = 0.01, Q_0 = 10, E = 1, \alpha_2 = 0.05, \gamma = 0.01,$$

$$q = 0.5, \beta = 0.1, \epsilon = 0.01.$$

The interior equilibrium point of system (2) with data (a) is

$$x^* = 82.7434, T_1^* = 10.2309, T_2^* = 0.01166.$$

Figures have been plotted between dependent variables and time for different parameter values to shows changes occurring in population with time under different conditions. The results of numerical simulation are displayed graphically. In figure (1) the x, T_1 and T_2 are plotted against time. From figure it is noted for given initial values the populations tend to their corresponding value of equilibrium point P^* and hence exists in the form of steady state assuring local stability of P^* . Figure shows that increasing the value of harvesting effort the population of species decreases and tends to zero if $E \geq 4.74$. Figure 3 and Figure 4 show that affect of primary and secondary toxicant on the population of species. From figures we can see that increasing the emission rates of primary toxicant and secondary toxicant the population of species decreases. It can also be checked that all inequalities given by eq. (10) are satisfied for above equilibrium points and chosen parameters, so our system is stable for all values of delay, if it is stable in the absence of delay refer Figure 5.

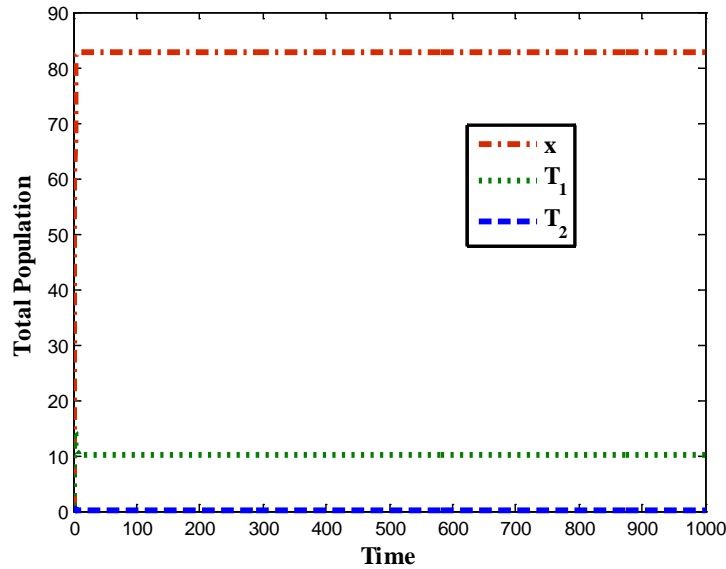


Figure-1: Stable behavior of x, T_1 and T_2 with time and other parameter values are same as (a).

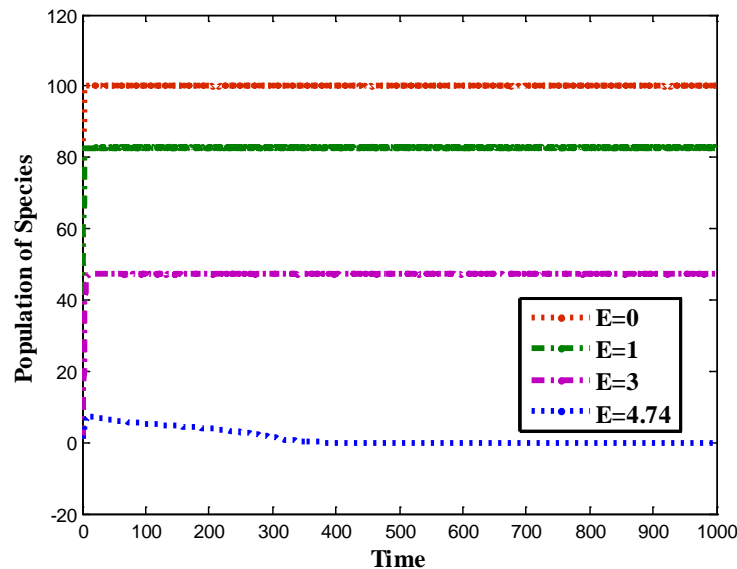


Figure-2: Variation of the population of species with time for different values of E and other parameter values are same as (a).

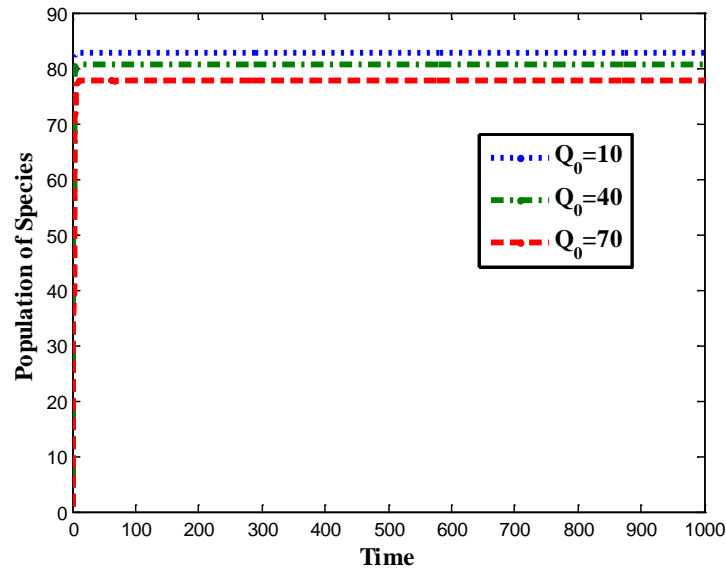


Figure-3. Variation of the population of species with time for different values of Q_0 and other parameter values are same as (a).

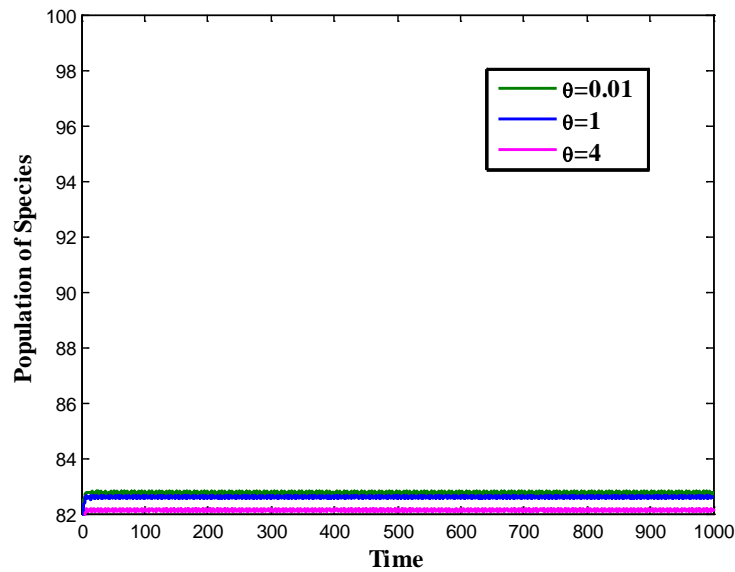


Figure-4: Variation of the population of species with time for different values of θ and other parameter values are same as (a).

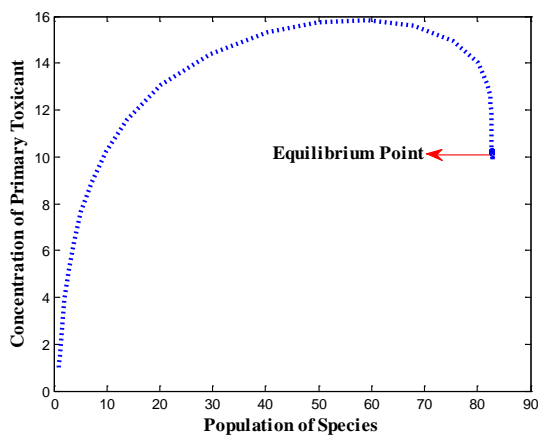


Figure-5a.

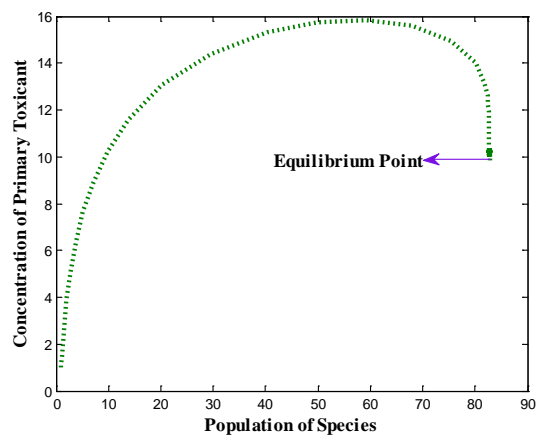


Figure-5b.

Figure 5: Graph of population of species verses concentration of primary toxicant for different values of delay. $\tau = 20$ in Figure 5a and $\tau = 30$ in Figure 5b and other parameter values are same as (a).

7. DISCUSSION

In the present paper, we have discussed non-selective harvesting of a fishery resource that is affected by toxicant. We have structured toxicant into two levels according to its intensity – Primary toxicant that is of low intensity and secondary toxicant is highly toxic. It is assumed that growth rate coefficient and carrying capacity of logistically growing fish species are adversely affected by presence of toxicant and decreases with increase in toxicant level. The primary toxicant gets transformed into secondary toxicant after a constant time lag τ .

We have proved positivity and boundedness of solution of system. Using stability theory of differential equation, we have also proved the existence of interior equilibrium and discussed stability of the system under certain conditions and found the condition for persistence of the system. It has been observed that increase in toxicant above certain level leads to extinction of species. We obtained the conditions under which system is stable for $\tau = 0$ and also obtained criteria for no stability change when $\tau \neq 0$. A least critical value of delay is also obtained at which stability change occur.

An optimal policy to harvest fish population is discussed by using Pontryagin's Maximum principle, optimal levels of harvesting effort, fish population and toxicants are obtained. It has been noted that increase in toxicant concentration decreases the optimal levels of harvesting effort and fish population. It has also been observed that sufficiently large value of discount rate decreases the net economic revenue to the society.

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