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# A NOTE ON RATE OF APPROXIMATION OF BOUNDED VARIATION FUNCTIONS BY THE BÉZIER VARIANT OF CHLODOWSKY OPERATORS

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***ABSTRACT***

*In this present paper, we introduce the concept of the point wise approximation of the Bézier variant of the Chlodowsky operators for bounded variation function. By means of the analysis techniques and some result of probability theory, we obtain an estimate formula on this type approximation. We use the result of Lian Bo-Yong [3].*

## 1. INTRODUCTION

For a function $f$ which is define on the interval$[0, a_m]$ the Chlodowsky operators $C_m(f,x)$ are define by

$$C_m(f,x) = \sum_{i=0}^{m} f\left(\frac{ia_m}{m}\right) p_{mi}\left(\frac{x}{a_m}\right), \tag{1}$$

where $p_{mi}\left(\frac{x}{a_m}\right) = \binom{m}{i}\left(\frac{x}{a_m}\right)^i\left(1-\frac{x}{a_m}\right)^{m-i}$ and $(a_m)$ is a sequence of increasing positive numbers, with the properties $\lim_{m\to\infty} a_m = \infty$ and $\lim_{m\to\infty} a_m/m = 0$. when $a_m \equiv 1$, the operators $C_m(f,x)$ become the well-known Bernstein operators

$$B_m(f,x) = \sum_{i=0}^{m} f\left(\frac{i}{m}\right) p_{mi}(x). \tag{2}$$

The Bézier variant of Chlodowsky operators $C_{m,\vartheta}$ introduced by H.Karsli and E. Ibikli [2], which is defined by

$$C_{m,\vartheta}(f,x) = \sum_{i=0}^{m} f\left(\frac{ia_m}{m}\right) Q_{mi}^{(\vartheta)}\left(\frac{x}{a_m}\right), \tag{3}$$

where $\vartheta > 0$ and

$$Q_{mi}^{(\vartheta)}\left(\frac{x}{a_m}\right) = J_{m,i}^{\vartheta}\left(\frac{x}{a_m}\right) - j_{m,i+1}^{\vartheta}\left(\frac{x}{a_m}\right),$$

$$J_{m,i}\left(\frac{x}{a_m}\right) = \sum_{j=i}^{m} p_{mj}\left(\frac{x}{a_m}\right)$$

Obviously for $\vartheta = 1$, the operators $C_{m,\vartheta}$ reduce to the operators $C_m$. Let

$$K_{m,\vartheta}\left(\frac{x}{a_m},\frac{y}{a_m}\right) = \begin{cases} \sum_{ka_m \le my} Q_{mi}^{(\vartheta)}\left(\frac{x}{a_m}\right), & 0 < y \le a_m; \\ 0, & y = 0. \end{cases} \tag{4}$$

By Lebesgue-Stieltjes integral representation, we have

$$C_{m,\vartheta}(f,x) = \int_0^{a_m} f(y)\, d_y K_{m,\vartheta}\left(\frac{x}{a_m},\frac{y}{a_m}\right), \tag{5}$$

In this paper, we discuss the pointwise approximation of $C_{m,\vartheta}$ to bounded variation functions for the case $\vartheta > 0$ which includes $\vartheta \ge 1$. We also mention some of the important result on this subject by Gupta [1] and Pych-Taberska[5].

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## 2. AUXILIARY RESULTS

In this section we give certain results, which are necessary to prove the main result.

**Lemma 2.1:** For every $x \in (0, a_m)$ and $0 \le i \le m$, we have

$$P_{mi}(x/a_m) \le \frac{a_m}{\sqrt{2emx(a_m - x)}}. \tag{6}$$

**Proof:** By [7 Theorem 1], we have $P_{mi}(y) < \frac{1}{\sqrt{2emt\,(1-y)}}\ for\ 0 < y < 1$.
Replacing $y\ for\ x/a_m$, we can get (6) easily.

The following lemma is the well-known Berry-Esseen bound for the central limit theorem of probability theory. Its proof can be found in Shiryayev [6].

**Lemma 2.2:** Let $\{\xi_i\}_{i=1}^{+\infty}$ be a sequences of independent and identically distributed random variance such that the expectation $E(\xi_1) = b_1 \in R$, the variance $Var(\xi_1) = E(\xi_1 - b_1)^2 = a_1^2 > 0$ and $E|\xi_1 - E(\xi_1)|^3 < +\infty$. Then there exists a constant $C, 1/\sqrt{2\pi} \le C < 0.8$, such that for all $m$ and $y$,

$$\left| P\left(\frac{1}{a_1\sqrt{m}}\sum_{i=1}^{m}(\xi_1 - b_1) \le y\right) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-u^2/2}du \right| < C\frac{E|\xi_1 - E(\xi_1)|^3}{a_1^3\sqrt{m}} \tag{7}$$

**Lemma 2.3:** For $x \in (0, a_m)$, we have

$$\left| \sum_{mx/a_m < i \le m} P_{mi}(x/a_m) - \frac{1}{2} \right| < \frac{0.8a_m}{\sqrt{mx(a_m - x)}}. \tag{8}$$

**Proof:** Let $\xi_1$ be the random variable with two-point distribution $P(\xi_1 = k) = \left(\frac{x}{a_m}\right)^k \left(1 - \frac{x}{a_m}\right)^{1-k}$ $(k = 0,1,\ x \in (0, a_m)$ is a parameter). Hence $b_1 = E(\xi_1) = x/a_m$, $a_1^2 = E(\xi_1 - b_1)^2 = \frac{x}{a_m}\left(1 - \frac{x}{a_m}\right)$, and $E|\xi_1 - E(\xi_1)|^3 = \frac{x}{a_m}\left(1 - \frac{x}{a_m}\right)\left[\left(\frac{x}{a_m}\right)^2 + \left(1 - \frac{x}{a_m}\right)^2\right]$. Let $\{\xi_i\}_{i=1}^{+\infty}$ be a sequence of independent random variables identically distributed with $\xi_1$, $\eta_m = \sum_{j=1}^{m}\xi_1$. Then the probability distribution of the random variable $\eta_m$ is

$$P(\eta_m = i) = \binom{m}{i}\left(\frac{x}{a_m}\right)^i \left(1 - \frac{x}{a_m}\right)^{m-i} = P_{mi}\left(\frac{x}{a_m}\right).$$

So

$$\sum_{mx/a_m < i \le m} P_{mi}\left(\frac{x}{a_m}\right) = P\left(\frac{mx}{a_m} < \eta_m \le m\right) = 1 - P\left(\eta_m \le \frac{mx}{a_m}\right)$$

$$= 1 - P\left(\frac{\eta_m - mx/a_m}{\sqrt{m}\sqrt{\frac{x}{a_m}\left(1 - \frac{x}{a_m}\right)}}\right) \le 0.$$

By (7), we get

$$\left| \sum_{mx/a_m < i \le m} P_{mi}(x/a_m) - \frac{1}{2} \right| = \left| P\left(\frac{\eta_m - mx/a_m}{\sqrt{m}\sqrt{\frac{x}{a_m}\left(1 - \frac{x}{a_m}\right)}} \le 0\right) - \frac{1}{2} \right|$$

$$< \frac{E|\xi_1 - E(\xi_1)|^3}{a_1^3\sqrt{m}} < \frac{0.8\left[\left(\frac{x}{a_m}\right)^2 + \left(1 - \frac{x}{a_m}\right)^2\right]a_m}{\sqrt{mx(a_m - x)}} < \frac{0.8a_m}{\sqrt{mx(a_m - x)}}.$$

This completes the proof of (8).

**Lemma 2.4:** For $\vartheta \ge 1$ and $x \in (0, a_m), i' = mx/a_m$, we have

$$(i)\quad \left| \left( \sum_{mx/a_m < i \le m} P_{mi}(x/a_m) \right)^{\vartheta} - \frac{1}{2^{\vartheta}} \right| \le \frac{0.8\vartheta a_m}{\sqrt{mx(a_m - x)}}, \tag{9}$$

$$(ii)\ Q_{mi'}^{(\vartheta)}(x/a_m) < \frac{\vartheta a_m}{\sqrt{2emx(a_m - x)}}. \tag{10}$$

**Proof:**
(i) From the fact that $|x^\vartheta - z^\vartheta| \le \vartheta|x-z|$ with $0 \le x, z \le 1$ and $\vartheta \ge 1$, we get (9) from (8) easily.
(ii) Using the same method of (i), we obtain $Q_{mi'}^{(\vartheta)}(x/a_m) \le \vartheta P_{mi'}(x/a_m)$.

The condition (10) now follows from (6) immidetialely.

**Lemma 2.5:** For $0 < \vartheta \le 1$ and $x \in (0, a_m)$, as $m > \frac{256a_m^2}{25x(a_m-x)}$ and $i' = mx/a_m$, we have

$$(i) \quad \left|\left(\sum_{mx/a_m < i} P_{mi}(x/a_m)\right)^\vartheta - \frac{1}{2^\vartheta}\right| < \frac{a_m}{\sqrt{mx(a_m-x)}}, \tag{11}$$

$$(ii) \quad Q_{mi'}^{(\vartheta)}(x/a_m) < \frac{a_m}{\sqrt{mx(a_m-x)}}. \tag{12}$$

**Proof:** (i) By mean value theorem, we have

$$\left|\left(\sum_{mx/a_m < i \le m} P_{mi}(x/a_m)\right)^\vartheta - \frac{1}{2^\vartheta}\right| = \vartheta\big(\xi_{mi}(x/a_m)\big)^{\vartheta-1}\left|\left(\sum_{mx/a_m < i \le m} P_{mi}(x/a_m)\right) - \frac{1}{2}\right| \tag{13}$$

where $\xi_{mi}(x/a_m)$ lies between ½ and $\sum_{mx/a_m < i \le m} P_{mi}(x/a_m)$.

In view of (8) and all $m > \frac{256a_m^2}{25x(a_m-x)}$, we have

$$\sum_{mx/a_m < i \le m} P_{mi}(x/a_m) > \frac{1}{4}. \tag{14}$$

Hence $\xi_{mi}(x/a_m) > \frac{1}{4}$ hold for $m > \frac{256a_m^2}{25x(a_m-x)}$.

From (13), (8) and noting $3.2\vartheta < 4^\vartheta$, we get (11) immediately.

(ii) Using the mean value theorem, we get

$$\begin{aligned} Q_{mi'}^{(\vartheta)}(x/a_m) &= \vartheta\big(\eta_{mi'}(x/a_m)\big)^{\vartheta-1}\big[J_{m,i'}(x/a_m) - \eta_{m,i'+1}(x/a_m)\big] \\ &= \vartheta\left(\frac{1}{\eta_{mi'}(x/a_m)}\right)^{1-\vartheta} P_{m,i'}(x/a_m), \end{aligned} \tag{15}$$

where $J_{m,i'+1}(x/a_m) < \eta_{mi'}(x/a_m) < j_{m,i'}(x/a_m)$.

But in view of (14), we know

$$\eta_{mi'}(x/a_m) > J_{m,i'+1}(x/a_m) = \sum_{J > mx/a_m} P_{mj}(x/a_m) > \frac{1}{4}.$$

From (15), (6) and noting $2\vartheta < 4^\vartheta$, we deduce that

$$Q_{mi'}^{(\vartheta)}(x/a_m) < \frac{\vartheta 4^{1-\vartheta} a_m}{\sqrt{2emx(a_m-x)}} < \frac{a_m}{\sqrt{mx(a_m-x)}}.$$

**Lemma 2.6:** (i) For $\vartheta \ge 1$ and $0 \le y < x$, there holds

$$K_{m,\vartheta}\left(\frac{x}{a_m}, \frac{y}{a_m}\right) \le \frac{\vartheta x(a_m-x)}{m(x-y)^2}. \tag{16}$$

(ii) For $\vartheta \ge 1$ and $0 \le x < y$ there holds

$$1 - K_{m,\vartheta}\left(\frac{x}{a_m}, \frac{y}{a_m}\right) \le \frac{\vartheta x(a_m-x)}{m(x-y)^2}. \tag{17}$$

**Proof:** (i) By a simple calculation, we get

$$C_m(1,x) = 1,$$
$$C_m(y,x) = x,$$
$$C_m(y^2,x) = x^2 + \frac{x(a_m-x)}{m}.$$

 

Thus

$$C_m((y-x)^2,x)=\frac{x(a_m-x)}{m}. \tag{18}$$

Now from the fact that $|x^\vartheta - z^\vartheta| \le \vartheta|x-z|$ with $0\le x,\ z\le 1$ and $\vartheta \ge 1$, we get

$$\begin{aligned} K_{m,\vartheta}\left(\frac{x}{a_m},\frac{y}{a_m}\right) &= \sum_{ia_m\le my} Q_{mi}^{(\vartheta)}(x/a_m)\\ &\le \vartheta \sum_{ia_m\le my} P_{mi}(x/a_m)\\ &\le \vartheta \sum_{ia_m\le my} \frac{(ia_m/m-x)^2}{(y-x)^2} P_{mi}(x/a_m)\\ &\le \vartheta \frac{C_m((y-x)^2,x)}{(y-x)^2}. \end{aligned}$$

(16) Now follows from (18).

(ii) Using a similar method we can get (17) easily.

**Lemma 2.7:** (i) For $0<\vartheta\le 1$ and $0\le y<x$, there holds

$$K_{m,\vartheta}\left(\frac{x}{a_m},\frac{y}{a_m}\right)\le K_{m,1}\left(\frac{x}{a_m},\frac{y}{a_m}\right)\le \frac{x(a_m-x)}{m(y-x)^2}. \tag{19}$$

(ii) For $0<\vartheta\le 1$ and $0\le x<y$, there holds

$$1-K_{m,\vartheta}\left(\frac{x}{a_m},\frac{y}{a_m}\right)\le \frac{A_\vartheta x(a_m-x)}{m(y-x)^2}, \tag{20}$$

where $A_\vartheta$ is a positive constant depending only on $\vartheta$.

**Proof:** (i) Along the some lines of the proof of [9, Lemma 4] and the inequality of (16), we can get (19) easily.

(ii) Since $0\le x<y$, so $\left|\frac{ia_m}{m}-x\right|/|y-x|\ge 1$ for all $i\ge my/a_m$. Thus we have

$$\begin{aligned} 1-K_{m,\vartheta}\left(\frac{x}{a_m},\frac{y}{a_m}\right) &= 1-\sum_{i\le my/a_m} Q_{mi}^{(\vartheta)}(x/a_m)\le \sum_{i\ge my/a_m} Q_{mi}^{(\vartheta)}(x/a_m)\\ &= \sum_{i\ge my/a_m}\left(Q_{mi}^{(\vartheta)}(x/a_m)-Q_{m,i+1}^{(\vartheta)}(x/a_m)\right)=\left(\sum_{i\ge my/a_m} P_{mi}(x/a_m)\right)^{\vartheta}\\ &\le \left(\sum_{i\ge my/a_m}\frac{\left|\frac{ia_m}{m}-x\right|^{2/\vartheta}}{|y-x|^{2/\vartheta}} P_{mi}(x/a_m)\right)^{\vartheta} \end{aligned}$$

Then, by Hölder's inequality with $p,q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\begin{aligned} \left(\sum_{i=0}^{m}\left|\frac{i}{m}-\frac{x}{a_m}\right|^{2/\vartheta} P_{mi}(x/a_m)\right)^{\vartheta} &= \left(\sum_{i=0}^{m}\left|\frac{i}{m}-\frac{x}{a_m}\right|^{2/\vartheta}\left(P_{mi}(x/a_m)\right)^{1/p}\left(P_{mi}(x/a_m)\right)^{1/q}\right)^{\vartheta}\\ &\le \left(\sum_{i=0}^{m}\left|\frac{i}{m}-\frac{x}{a_m}\right|^{2p/\vartheta} P_{mi}(x/a_m)\right)^{\vartheta/p}. \end{aligned}$$

Choosing $p=\vartheta[1/\vartheta+1]$, then $2p/\vartheta=2[1/\vartheta+1]$ is an evenpositive integer. From [4, Theorem 1.5.1], we have

$$\left(\sum_{i=0}^{m}\left|\frac{i}{m}-\frac{x}{a_m}\right|^{2/\vartheta} P_{mi}(x/a_m)\right)^{\vartheta}\le A_\vartheta\frac{x}{a_m}\left(1-\frac{x}{a_m}\right)m^{-1},$$

where $A_\vartheta$ is a positive constant depending only on $\vartheta$. This completes the proof of (20).

**Lemma 2.8:** (i) For $\vartheta \geq 1,\ f \in BV[0,\infty)$ and $x \in (0, a_m)$, we have

$$\left|C_{m,\vartheta}(g_x, x)\right| \leq \frac{3\vartheta a_m^2}{mx(a_m - x)} \sum_{i=1}^{m} \bigvee_{x-x/\sqrt{i}}^{x+(a_m-x)/\sqrt{i}} (g_x). \tag{21}$$

(ii) For $0 < \vartheta \leq 1,\ f \in BV[0,\infty)$ and $x \in (0, a_m)$, when $m > \frac{256 a_m^2}{25(a_m - x)}$, we have

$$\left|C_{m,\vartheta}(g_x, x)\right| \leq \frac{A_\vartheta a_m^2}{mx(a_m - x)} \sum_{i=1}^{m} \bigvee_{x-x/\sqrt{i}}^{x+(a_m-x)/\sqrt{i}} (g_x). \tag{22}$$

**Proof:** (i) We recall the Lebesgue-Stieltjes integral representations

$$C_{m,\vartheta}(g_x, x) = \int_0^{a_m} g_x(y) d_y K_{m,\vartheta}\left(\frac{x}{a_m}, \frac{y}{a_m}\right). \tag{23}$$

Decompose the integral of (23) into three parts as follows

$$C_{m,\vartheta}(g_x, x) = \int_0^{a_m} g_x(y) d_y K_{m,\vartheta}\left(\frac{x}{a_m}, \frac{y}{a_m}\right) = I_1 + I_2 + I_3, \tag{24}$$

where

$$I_1 = \int_0^{x-\frac{x}{\sqrt{m}}} g_x(y) d_y K_{m,\vartheta}\left(\frac{x}{a_m}, \frac{y}{a_m}\right),$$
$$I_2 = \int_{x-\frac{x}{\sqrt{m}}}^{x+\frac{a_m-x}{\sqrt{m}}} g_x(y) d_y K_{m,\vartheta}\left(\frac{x}{a_m}, \frac{y}{a_m}\right),$$
$$I_3 = \int_{x+\frac{a_m-x}{\sqrt{m}}}^{a_m} g_x(y) d_y K_{m,\vartheta}\left(\frac{x}{a_m}, \frac{y}{a_m}\right).$$

Observing that $g_x(x) = 0$, we first have

$$\begin{aligned} I_2 &= \int_{x-\frac{x}{\sqrt{m}}}^{x+\frac{a_m-x}{\sqrt{m}}} |g_x(y) - g_x(x)| d_y K_{m,\vartheta}\left(\frac{x}{a_m}, \frac{y}{a_m}\right), \\ &\leq \bigvee_{x-x/\sqrt{m}}^{x+(a_m-x)/\sqrt{m}} (g_x) \leq \frac{1}{m-1} \sum_{i=2}^{m} \bigvee_{x-x/\sqrt{m}}^{x+(a_m-x)/\sqrt{m}} (g_x). \end{aligned} \tag{25}$$

To estimate $I_1$, let $z = x - x/\sqrt{m}$. Using Legesgue-Stieltjes integration by parts and (16), we have

$$\begin{aligned} I_1 &= \int_0^z g_x(y) d_y K_{m,\vartheta}\left(\frac{x}{a_m}, \frac{y}{a_m}\right), \\ &= \left| g_x(z+) K_{m,\vartheta}\left(\frac{x}{a_m}, \frac{z}{a_m}\right) - \int_0^z g_x(y) d_y K_{m,\vartheta}\left(\frac{x}{a_m}, \frac{y}{a_m}\right) \right| \\ &\leq \bigvee_{z+}^{x} (g_x) K_{m,\vartheta}\left(\frac{x}{a_m}, \frac{z}{a_m}\right) + \int_0^z K_{m,\vartheta}\left(\frac{x}{a_m}, \frac{y}{a_m}\right) dy \left(-\bigvee_{y}^{x} (g_x)\right) \\ &\leq \bigvee_{z+}^{x} (g_x) \frac{\vartheta x(a_m - x)}{m(x-z)^2} + \frac{\vartheta x(a_m - x)}{m} \int_0^z \frac{1}{(x-y)^2} dy \left(-\bigvee_{y}^{x} (g_x)\right). \end{aligned}$$

Since

$$\int_0^z \frac{1}{(x-y)^2} dy \left(-\bigvee_{y}^{x} (g_x)\right) = \frac{\bigvee_y^x (g_x)}{(x-y)^2}\bigg|_0^{z+} + \int_0^z \frac{\bigvee_y^x (g_x)}{(x-y)^2} dy,$$

We have

$$|I_1| \leq \frac{\vartheta x(a_m - x)}{mx^2} \bigvee_{0}^{x} (g_x) + \frac{\vartheta x(a_m - x)}{m} \int_0^z \frac{2 \bigvee_y^x (g_x)}{(x-y)^2} dy.$$

Putting $y = x - x/\sqrt{u}$ for the last integral, we get

$$|I_1| \le \frac{\vartheta x(a_m - x)}{mx^2} \bigvee_0^x (g_x) + \frac{\vartheta x(a_m - x)}{mx^2} \int_1^m \bigvee_{x-x/\sqrt{u}}^x (g_x)\, du.$$

$$|I_1| \le \frac{\vartheta x(a_m - x)}{mx^2} \left[ \bigvee_0^x (g_x) + \sum_{i=1}^m \bigvee_{x-x/\sqrt{i}}^x (g_x) \right]. \tag{26}$$

Using the similar method and (17) to estimate $|I_3|$, we obtain

$$|I_3| \le \frac{\vartheta x(a_m - x)}{m(a_m - x)^2} \left[ \bigvee_0^{a_m} (g_x) + \sum_{i=1}^m \bigvee_x^{x+(a_m - x)/\sqrt{i}} (g_x) \right]. \tag{27}$$

Combining the estimates of (24), (25), (26), and (27), also noting the properties of $\bigvee_a^b(f)$ and $1/(m-1) \le \vartheta a_m^2/[mx(a_m - x)]$ for $x \in (0, a_m)$, we get

$$\begin{aligned} \left|C_{m,\vartheta}(g_x, x)\right| &\le \frac{\vartheta[(a_m - x)^2 + x^2]}{mx(a_m - x)} \left[ \bigvee_0^{a_m} (g_x) + \sum_{i=1}^m \bigvee_{x-x/\sqrt{i}}^{x+(a_m - x)/\sqrt{i}} (g_x) \right] + \frac{1}{m-1} \sum_{i=2}^m \bigvee_{x-x/\sqrt{i}}^{x+(a_m - x)/\sqrt{i}} (g_x) \\ &\le \frac{2\vartheta a_m^2}{mx(a_m - x)} \sum_{i=2}^m \bigvee_{x-x/\sqrt{i}}^{x+(a_m - x)/\sqrt{i}} (g_x) + + \frac{1}{m-1} \sum_{i=2}^m \bigvee_{x-x/\sqrt{i}}^{x+(a_m - x)/\sqrt{i}} (g_x), \\ &\le \frac{3\vartheta a_m^2}{mx(a_m - x)} \sum_{i=2}^m \bigvee_{x-x/\sqrt{i}}^{x+(a_m - x)/\sqrt{i}} (g_x) \end{aligned}$$

This completes the proof of (21).

(ii) Using the same method and (19), (20), we can also get (22) easily.

## 3. MAIN RESULT

In this section we prove the following main theorems:

**Theorem 3.1:** Let $\vartheta \ge 1$, $f$ be a function of bounded variation on every finite subinterval of $[0, \infty)$ and $\lim_{x\to\infty} f(x)$ exists, i.e. $f \in BV[0, \infty)$. Then for every $x \in (0, a_m)$, we have

$$\begin{aligned} \left| C_{m,\vartheta}(f, x) - \frac{1}{2^\vartheta} f(x+) - \left(1 - \frac{1}{2^\vartheta}\right) f(x-) \right| &\le \frac{3\vartheta a_m^2}{mx(a_m - x)} \sum_{i=1}^m \bigvee_{x-x/\sqrt{i}}^{x+(a_m - x)/\sqrt{i}} (g_x) \\ &+ \frac{\vartheta a_m}{\sqrt{mx(a_m - x)}} \left(|f(x+) - f(x-)| + \varepsilon_n(x/a_m)|f(x) - f(x-)|\right). \end{aligned} \tag{28}$$

**Theorem 3.2:** Let $0 < \vartheta \le 1$, $f$ be a function of bounded variation on every finite subinterval of $[0,\infty)$ and $\lim_{x\to\infty} f(x)$ exists, i.e. $f \in BV[0, \infty)$. Then for every $x \in (0, a_m)$, and $m > \frac{256 a_m^2}{25x(a_m - x)}$, we have

$$\begin{aligned} \left| C_{m,\vartheta}(f, x) - \frac{1}{2^\vartheta} f(x+) - \left(1 - \frac{1}{2^\vartheta}\right) f(x-) \right| &\le \frac{A_\vartheta a_m^2}{mx(a_m - x)} \sum_{i=1}^m \bigvee_{x-x/\sqrt{i}}^{x+(a_m - x)/\sqrt{i}} (g_x) \\ &+ \frac{a_m}{\sqrt{mx(a_m - x)}} \left(|f(x+) - f(x-)| + \varepsilon_n(x/a_m)|f(x) - f(x-)|\right). \end{aligned} \tag{29}$$

where is a positive constant depending only on $\vartheta$,

$$g_x(y) = \begin{cases} f(y) - f(x+), & x < y \le a_m; \\ 0, & y = x; \\ f(y) - f(x-), & 0 \le y < x. \end{cases} \tag{30}$$

$$\varepsilon_m(x/a_m) = \begin{cases} 1, & if\ x = \dfrac{i' a_m}{m}, \quad for\ some\ i' \in N; \\ 0, & if\ x \ne \dfrac{i a_m}{m}, \quad for\ all\ i \in N. \end{cases} \tag{31}$$

when $a_m \equiv 1$, the operators $C_{m,\vartheta}(f,x)$ are just the Bernstein- Bézeir operators $B_{m,\vartheta}(f,x) = \sum_{i=0}^{m} f\left(\frac{i}{m}\right) Q_{mi}^{(\vartheta)}(x)$, which were studied by Zeng[8,9] . Therefore, our theorems extend the result of Zeng. Moreover, in the case $0 < \vartheta \leq 1$, Zeng [9] gave a rate of convergence of $B_{m,\vartheta}$ for bounded variation functions as follows:

Let $0 < \vartheta \leq 1$, $f$ be a function of bounded variation on [0, 1] ($f \in BV[0,1]$). Then for every $x \in (0, 1)$ and $m > \frac{256}{25}(x(1-x))^{-1}$ we have

$$\left|B_{m,\vartheta}(f,x) - \frac{1}{2^\vartheta} f(x+) - \left(1 - \frac{1}{2^\vartheta}\right) f(x-)\right| \leq \frac{A_\vartheta}{m(x(1-x))^{2-\vartheta}} \sum_{i=1}^{m} \bigvee_{x-x/\sqrt{i}}^{x+(1-x)/\sqrt{i}} (g_x)$$
$$+ \frac{a_m}{\sqrt{mx(1-x)}} (|f(x+) - f(x-)| + \varepsilon_n(x)|f(x) - f(x-)|). \qquad (32)$$

Obviously, for $a_m \equiv 1$, our Theorem 3.2 extends and improves the result of (32).

**Proof of Theorem 3.1 and Theorem 3.2:** Let $f$ satisfy the condition of Theorem 3.1 and Theorem 3.2. We can decompose *f(y)* into four parts as

$$f(y) = \frac{1}{2^\vartheta} f(x+) + \left(1 - \frac{1}{2^\vartheta}\right) f(x-) + g_x(y) + \frac{f(x+) - f(x-)}{2^\vartheta} \widehat{\text{sign}}(y-x)$$
$$+ \delta_x(y)\left[f(x) - \frac{1}{2^\vartheta} f(x+) - \left(1 - \frac{1}{2^\vartheta}\right) f(x-)\right],$$

where

$$\widehat{sign}(y-x) = \begin{cases} 2^\vartheta - 1, & y > x; \\ 0, & y = x; \\ -1, & y < x. \end{cases}$$

$$\delta_x(y) = \begin{cases} 1, & y = x; \\ 0, & y \neq x. \end{cases}$$

$g_x(y)$ is defined in (30). Therefore,

$$\left|C_{m,\vartheta}(f,x) - \frac{1}{2^\vartheta} f(x+) - \left(1 - \frac{1}{2^\vartheta}\right) f(x-)\right| \leq |C_{m,\vartheta}(g_x,x)| + \left|\frac{f(x+) - f(x-)}{2^\vartheta} C_{m,\vartheta}(\widehat{\text{sign}}(y-x),x)\right.$$
$$\left. + \left[f(x) - \frac{1}{2^\vartheta} f(x+) - \left(1 - \frac{1}{2^\vartheta}\right) f(x-)\right] C_{m,\vartheta}(\delta_x,x)\right|. \qquad (33)$$

By direct calculation, we get

$$C_{m,\vartheta}(\delta_x,x) = \varepsilon_m(x/a_m) Q_{mi'}^{\vartheta}(x/a_m)$$

and

$$C_{m,\vartheta}(\widehat{\text{sign}}(y-x),x) = \sum_{i > mx/a_m} (2^\vartheta - 1) Q_{mi}^{\vartheta}(x/a_m) + \sum_{i < mx/a_m} (-1) Q_{mi}^{\vartheta}(x/a_m)$$
$$= 2^\vartheta \sum_{i > mx/a_m} Q_{mi}^{\vartheta}(x/a_m) - 1 + \varepsilon_m(x/a_m) Q_{mi'}^{\vartheta}(x/a_m)$$
$$= 2^\vartheta \left(\sum_{i > mx/a_m} P_{mi}(x/a_m)\right)^\vartheta - 1 + \varepsilon_m(x/a_m) Q_{mi'}^{\vartheta}(x/a_m),$$

where $\varepsilon_m(x/a_m)$ is defined in (31).

Therefore, we have

$$\left|\frac{f(x+) - f(x-)}{2^\vartheta} C_{m,\vartheta}(\widehat{sign}(y-x),x) + \left[f(x) - \frac{1}{2^\vartheta} f(x+) - \left(1 - \frac{1}{2^\vartheta}\right) f(x-)\right] C_{m,\vartheta}(\delta_x,x)\right|$$
$$= \left|[f(x+) - f(x-)]\left[\left(\sum_{i > mx/a_m} P_{mi}\left(\frac{x}{a_m}\right)\right)^\vartheta - \frac{1}{2^\vartheta}\right] + [f(x+) - f(x-)]\varepsilon_m\left(\frac{x}{a_m}\right) Q_{mi'}^{\vartheta}(x/a_m)\right| \qquad (34)$$

By combining the estimates given by (33), (21), (34), (9) and (10), we obtain Theorem 3.1 and by combining the estimates given by (33), (22), (34), (11) and (12), we obtain Theorem 3.2.

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