CENTRALIZING LEFT GENERALIZED DERIVATIONS ON SEMIPRIME RINGS

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ABSTRACT

In this paper we proved that if \( d \) is a nonzero derivation of a prime ring \( R \) and \( f \) be a left generalized derivation, then \( f \) is a strong commutativity preserving. Using this we proved that \( R \) is commutative.

Keywords: Prime ring, Derivation, Generalized derivation, Left generalized derivation, Homomorphism, Centralizing.

INTRODUCTION

Bell and Martindale [2] studied centralizing mappings of semi prime rings and proved that if \( d \) is a nonzero derivation of prime ring \( R \) such that \( d(x)x = 0 \) for all \( x \) in a nonzero left ideals of \( R \), then \( R \) is commutative. Bell and Daif [3] investigated commutativity in prime and semiprime rings admitting a derivation or an endomorphism which is strong commutativity preserving on a nonzero right ideal. Ali and Shah [1] extended some results of Bell and Martindale [2] or generalized derivations. Throughout this paper, \( R \) will denote a semiprime ring and \( Z \) its center. Recall that prime if \( aRb = (0) \) implies that \( a = 0 \) or \( b = 0 \) and semi prime if \( aRa = (0) \) implies that \( a = 0 \). As usual \( [x, y] \) will denote the commutator \( xy - yx \). An additive mapping: \( R \to R \) is called a derivation if \( d(xy) = d(x)y + xd(y) \), holds for all \( x, y \in R \). An additive mapping \( F: R \to R \) is called a generalized derivation if there exists a derivation \( d: R \to R \) such that \( F(xy) = F(x)y + xd(y) \) for all \( x, y \in R \). An additive mapping \( F: R \to R \) is called a left generalized derivation if there exists a derivation \( d: R \to R \) Such that \( F(xy) = d(xy) + xF(y) \) for all \( x, y \in R \). A mapping \( f \) is commuting on a right ideal \( Z \) of \( R \) if \( [f(x), x] = 0 \), for all \( x \in Z \) and \( f \) is centralizing if \( f(xy), x \in Z \), for all \( x \in U \). A mapping \( f: R \to R \) is called strong commutativity preserving if \( [f(x), f(y)] = [x, y] \), for all \( x, y \in R \).

Remark: For a nonzero elements \( a \in Z \), if \( ab \in Z \), then \( b \in Z \).

To prove main result we require the following lemmas:

Lemma 1: If \( f \) is an additive mapping from \( R \) to \( R \) such that \( f \) is centralizing on a right ideal \( U \) of \( R \), then \( f(x) \in Z \), for all \( x \in U \cap Z \).

Proof: Since \( f \) is centralizing on \( U \), we have \( f(x + y), x + y \in Z \)
\[ f(x) + f(y), x + y \in Z \]
\[ f(x), x + f(y), y + f(x), x + f(y), y \in Z \]
\[ f(x), y + f(y), x + Z \]

Now if \( x \in Z \), then from above equation we have
\( f(x), y \in Z \) we replaced \( y \) by \( f(x), y \), then
\( f(x)[f(x), y] \in Z \)

If \( f(x), y = 0 \), then \( f(x) \in c^{1}(U) \), the centralizer of \( U \) in \( R \) and by [1] belongs to \( Z \). But on the other hand, if \( f(x), y \neq 0 \), it again follows from the remark 1 that \( f(x) \in Z \)

Lemma 2: Let \( R \) be a semiprime ring and \( U \) a nonzero ideal of \( R \). If \( Z \) in \( R \) centralizes the set \( [U, U] \), then \( Z \) centralizes \( U \).
Now we prove the following result:

**Theorem 1:** Let \( d: R \to R \) be a non zero derivation of prime ring \( R \) and \( f \) be a left generalized derivation on a nonzero right ideal \( U \) of \( R \). If \( f \) acts as a homomorphism on \( U \), then \( f \) is strong commutativity preserving on \( U \).

**Proof:** We assume that \( f \) acts as homomorphism on \( U \) and \( f \) be a left generalized derivation on \( U \). Then
\[
f(xy) = f(x)f(y) = d(x)y + xf(y)
\]
for all \( x, y \in U \) (1)

We replace by \( zy, z \in U \), the second equality of (1) we have
\[
f(xzy) = f(x)f(zy) = d(x)zy + xf(zy) = d(x)zy + xf(z)f(y).
\]
(2)

Since \( f \) is a homomorphism. On the other hand we have
\[
f(xzy) = f(xz)f(y) = (d(x)z + xf(z))f(y)
\]
\[
f(xzy) = d(x)zf(y) + xf(z)f(y).
\]
(3)

From equation (2) \& (3), we get
\[
d(x)zy + xf(z)f(y) = d(x)zf(y) + xf(z)f(y)
\]
\[
d(x)z(f(y) - y) = 0.
\]
(4)

We replace \( y \) by \( [x, y] \) in equation (4), then
\[
d(x)z([x, y] - [x, y]) = 0
\]
By replacing \( z \) by \( zr, r \in R \) in the above equation then
\[
d(x)z R([x, y] - [x, y]) = 0
\]
By the prime ness of \( R \), we have either \( d(x)z = 0 \) or \( f[x, y] - [x, y] = 0 \)

Since \( d \neq 0 \), then \( f[x, y] - [x, y] = 0 \).
\[
f[x, y] = [x, y]
\]
\[
[f(x), f(y)] = [x, y].
\]

Hence \( f \) is strong commutativity preserving on \( U \)

**Theorem 2:** Let \( U \) be right ideal of a semiprime \( R \) such that \( U \cap Z \neq 0 \). Let \( d \) be a non zero derivation and \( f \) be a left generalized derivation on \( R \) such that \( f \) is centralizing on \( U \). Then \( R \) is commutative.

**Proof:** We assume that \( Z \neq 0 \) because \( f \) is commuting on \( U \) and there nothing to prove.

Since \( f \) is centralizing on \( U \), we have
\[
[f(x), x] \in Z \text{ for all } x, y \in U
\]

Linearizing the above equation we have
\[
[f(x + y), x + y] \in Z \text{ for all } x, y \in U
\]
\[
[f(x), x] + [f(x), y] + [f(y), x] + [f(y), y] \in Z
\]
\[
[f(x), y] + [f(y), x] \in Z \text{ for all } x, y \in U.
\]
(5)

We replaced \( x \) by \( yz \) in equation (5), we get
\[
[f(yz), y] + [f(y), yz] \in Z
\]
\[
[(d(y)z + y[f(z)], y] + [f(y), y]z + y[f(y), z] \in Z
\]
\[
[d(y), y]z + d(y)x, x + y[f(z), y] + [f(y), y]z + y[f(y), z] = Z, \text{ then}
\]
\[
[d(y), y]z + y[f(z), y] + [f(y), y]z \in Z
\]

Now by lemma 1, \( f(z) \in Z \) and there fore \( [d(y), y]z + y[f(y), y]z \in Z \)

But \( f \) is centralizing on \( U \). We have \([f(y), y]z \in Z \) and consequently \([d(y), y]z \in Z \).

Since \( z \) is non zero, it follows from remark1 that \([d(y), y] \in Z \). This implies that \( d \) is centralizing on \( U \) and hence we conclude that \( R \) is commutative.
REFERENCE


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