International Journal of Mathematical Archive-2(7), July - 2011, Page: 1040-1045
Available online through www.ijma.info ISSN 2229 – 5046

UNIQUE FIXED POINT THEOREMS ON HAUSDORFF SPACES

[1]Zaheer K. Ansari*, [2]Rajesh Shrivastava and [3]Arun Garg

[1]Department of Applied Mathematics, JSS Academy of Technical education,
C – 20/1, Sector 62, Noida – 201301, U.P. (India)
E-mail: zkansari@rediffmail.com

[2]Department of Mathematics, Govt. Science & Commerce College
Benazir, Bhopal, M.P. (India)
E-mail: rajeshraju0101@rediffmail.com

[3]Department of Mathematics, Institute of Technology & Management,
ITM Campus, N. H. – 75, Jhansi Road, Gwalior, M. P. – 474001 (India)
E-mail: arungargitm@yahoo.com

(Received on: 11-06-11; Accepted on: 30-06-11)

ABSTRACT

In present paper, some fixed point theorems in Hausdorff spaces have been proved, which generalize the theorem of YEH, C.C. [2].

$$d(Tx,Ty) < d(x,y);\ \forall\ x, y \in X.$$

Keywords and Phrases: *Result on Hausdorff spaces, fixed point theorem.*

MSC (2000): *54H25, 28A78.*

INTRODUCTION:

In 1962, M. Edelstein [1] proved the following result:

Theorem 1: Let (X,d) be a metric space, T be a contractive self-mapping of X. If for some $x \in X$, the sequence of iterates $\{T^n x\}$ has a convergent subsequence $\{T^{n_i} x\}$ converging to a point $x_0 \in X$, then $x_0 = \lim\{T^n x\}$ is unique fixed point.

Singh, S.P. and Zorzitto [3] generalized the result of M. Edelstein [1].

Here, we prove the following theorems, which generalized the result of YEH, C.C. [2]:

Theorem 2: Let T be a continuous mapping of a Housdorff space X into itself and let f be a continuous mapping of X×X into non-negative reals such that

$$f(x,x) = 0;\ \forall x \in X \text{ and } f(x,y) \neq 0,\ \forall x, y \in X,\ x \neq y \tag{2.1}$$

$$f(x,z) \leq f(x,y) + f(y,z), \text{ for all x, y, z in X} \tag{2.2}$$

$$\begin{aligned} f(Tx,Ty) \leq q \max\Bigg\{ & f(x,y), f(x,Tx), f(y,Ty), f(y,Tx), \frac{1}{2} f(x,Ty), \frac{f(x,Tx)f(y,Ty)}{f(x,y)}, \\ & \frac{f(x,Ty)f(y,Tx)}{f(x,y)}, \frac{f(x,y)f(x,Ty)}{f(x,y)+f(Tx,Ty)}, \frac{f(x,Ty)f(Tx,Ty)}{f(x,y)+f(Tx,Ty)}, \frac{f(x,Tx)f(x,Ty)}{f(x,y)+f(Tx,Ty)}, \frac{f(y,Ty)f(x,Ty)}{f(x,y)+f(Tx,Ty)}, \\ & \frac{[f(x,y)f(Tx,Ty) + f(x,Ty)f(y,Ty)]}{2f(x,y)+f(Tx,Ty)} \Bigg\} \end{aligned} \tag{2.3}$$

***Corresponding author:** [1]*Zaheer K. Ansari**, ***E-mail:** zkansari@rediffmail.com*

hold for all $x, y \in X$ where $q \in [0,1)$. Then T has a unique fixed point in X.

Proof: Let x_0 be an arbitrary point in X and a sequence $\{x_n\}$ defined by $Tx_n = x_{n+1}$, n a positive integer. If for some n, $Tx_n = x_n$ then x_n is a fixed point. If not $Tx_n \neq x_n$ then by (2.3), we have.

$$f(x_1, x_2) = f(Tx_0, Tx_1)$$
$$\leq q \max\left\{ f(x_0,x_1), f(x_0,x_1), f(x_1,x_2), f(x_1,x_1)\frac{1}{2}f(x_0,x_2), \frac{f(x_0,x_1)f(x_1,x_2)}{f(x_0,x_1)}, \frac{f(x_0,x_2)f(x_1,x_1)}{f(x_0,x_1)}, \right.$$
$$\frac{f(x_0,x_1)f(x_0,x_2)}{f(x_0,x_1)+f(x_1,x_2)}, \frac{f(x_0,x_2)f(x_1,x_2)}{f(x_0,x_1)+f(x_1,x_2)}, \frac{f(x_0,x_1)f(x_0,x_2)}{f(x_0,x_1)+f(x_1,x_2)}, \frac{f(x_1,x_2)f(x_0,x_2)}{f(x_0,x_1)+f(x_1,x_2)},$$
$$\left. \frac{f(x_0,x_1)f(x_1,x_2)+f(x_0,x_2)f(x_1,x_2)}{2f(x_0,x_1)+f(x_1,x_2)} \right\}$$
$$\leq q \max\left\{ f(x_0,x_1), f(x_0,x_1)f(x_1,x_2), 0, \frac{1}{2}\{f(x_0,x_1)+f(x_1,x_2)\}, f(x_1,x_2), 0, f(x_0,x_1), \right.$$
$$\left. f(x_1,x_2), f(x_0,x_1), f(x_1,x_2), f(x_1,x_2) \right\}$$

i.e. $\leq q \max\left\{ f(x_0,x_1), f(x_1,x_2)\frac{1}{2}[f(x_0,x_1)+f(x_1,x_2)] \right\}$

Then we have either $f(x_1,x_2) < qf(x_1,x_2)$ which is a contradiction as q<1

or $f(x_1,x_2) < qf(x_0,x_1) < f(x_0,x_1)$ as q<1

and $f(x_1,x_2) \leq \frac{q}{2-q} f(x_0,x_1) \leq qf(x_0,x_1) < f(x_0,x_1)$ as q<1 Proceeding in the same manner we have

$$f(x_0,x_1) > f(x_1,x_2) > f(x_2,x_3) > \ldots$$

Thus we have a monotone sequence of positive reals which converges with all it subsequences to some $x \in X$, x being real.

Again $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ in X which converges to some $x \in X$. From the continuity of T,

we have

$$Tx = T\{\lim x_n\} = \lim Tx_n$$
$$= \lim (x_{n+1})$$

$$T^2(x) = T(Tx) = T\{\lim x_{n_k+1}\}$$
$$= \lim\{Tx_{n_k+1}\} = x_{n+2}$$

Now we are to prove that x is a fixed point for T.

$$f(x,Tx) = f(\lim x_n, \lim x_{n+1})$$
$$= \lim f(x_n, x_{n+1})$$
$$= \lim f(x_{n+1}, x_{n+2})$$
$$= \lim f(Tx, T^2x)$$

Again by (2.3) we have

$$f(Tx,T^2x) \le q\max\left\{ f(x,Tx), f(x,Tx), f(Tx,T^2x), f(Tx,Tx), \frac{1}{2} f(x,T^2x)\ , \frac{f(x,Tx)f(Tx,T^2x)}{f(x,Tx)}, \right.$$
$$\frac{f(x,T^2x)f(Tx,Tx)}{f(x,Tx)}, \frac{f(x,Tx)f(x,T^2x)}{f(x,Tx)+f(Tx,T^2x)}, \frac{f(x,T^2x)f(Tx,T^2x)}{f(x,Tx)+f(Tx,T^2x)},,$$
$$\left. \frac{f(Tx,T^2x)f(x,T^2x)}{f(x,Tx)+f(Tx,T^2x)}, \frac{f(x,Tx)f(Tx,T^2x)+f(x,T^2x)f(Tx,T^2x)}{2f(x,Tx)+f(Tx,T^2x)} \right\}$$

$$\le q\max\left\{ f(x,Tx), f(x,Tx), f(Tx,T^2x), 0, \frac{1}{2}\left\{f(x,Tx)+f(Tx,T^2x)\right\}, \right.$$
$$\left. f(Tx,T^2x), 0, f(x,Tx), f(Tx,T^2x), f(x,Tx), f(Tx,T^2x), f(Tx,T^2x) \right\}$$

$$\le q\max\left\{ f(x,Tx), f(Tx,T^2x), \frac{1}{2}[f(x,Tx)+f(Tx,T^2x)] \right\}$$

which gives

$$f(Tx,T^2x) \le q\, f(x,Tx) \le f(x,Tx) \qquad \text{as } q<1$$

So we arrive at a contradiction hence $x = Tx$ i.e. x is a fixed point for T.

Uniqueness: Let y ($\neq x$) be another fixed point. Then by (2.3) we have

$$f(x,y) \le q\ \max\left\{ f(x,y), f(x,x), f(y,y), f(y,x), \frac{1}{2} f(x,y)\ \ \frac{f(x,x)f(y,y)}{f(x,y)}, \frac{f(x,y)f(y,x)}{f(x,y)}, \right.$$
$$\left. \frac{f(x,y)f(x,y)}{f(x,y)+f(x,y)}, \frac{f(x,y),f(x,y)}{f(x,y)f(x,y)}, \frac{f(x,y)f(x,y)}{f(x,y)+f(x,y)}, \frac{f(y,y)f(x,y)}{f(x,y)+f(x,y)}, \frac{f(x,y)f(x,y)+f(x,y)f(y,y)}{2f(x,y)+f(x,y)} \right\}$$

$f(x,y) \le q\, f(x,y) < f(x,y)$ which is a contradiction as q<1

Hence $x = y$ i.e. x is a unique fixed point of T.

This completes the proof of the theorem.

Theorem 3: Let T_1 and T_2 be two commuting continuous mapping of a Hausdorff space X into it self and let *f* be a symmetric continuous mapping of X×X into non-negative reals such that

$$f(x,x) = 0 \quad \forall\ x \in X \ \text{ and } \ f(x,y) \neq 0 \quad \forall\ x, y \in X,\ x \neq y \tag{3.1}$$

$$f(x,z) \le f(x,y) + f(y,z) \tag{3.2}$$

$$f(T_1x,T_2y) \le q\ \max\left\{ f(x,y), f(y,T_1x)\frac{1}{2} f(x,T_2y), f(y,T_2y), f(y,T_1x)\ , \frac{f(x,T_1x)f(y,T_2y)}{f(x,y)} \right.$$
$$\frac{f(y,T_2y)f(x,T_2y)}{f(x,y)}, \frac{f(x,y)f(x,T_2y)}{f(x,y)+f(T_1x,T_2y)}, \frac{f(T_1x,T_2y)f(x,T_2y)}{f(x,y)+f(T_1x,T_2y)}, \frac{f(x,T_1x)f(y,T_2y)}{f(x,y)+f(T_1x,T_2y)}, \frac{f(x,T_1x)f(y,T_2y)}{f(x,y)+f(T_1x,T_2y)},$$
$$\left. \frac{f(x,y)f(T_1x,T_2y)+f(x,T_2y)f(y,T_2y)}{2f(x,y)+f(T_1x,T_2y)} \right\} \tag{3.3}$$

hold for all $x, y \in X$ where $q \in [0,1)$. If for some $x_0 \in X$, the sequence $\{x_n\}$ where $T_1 x_{2n} = x_{2n+1}$ and $T_2 x_{2n+1} = x_{2n+2}$ for n = 0, 1, 2, 3, ... has a convergent. Subsequence of the type $\{x_{(2p+1)n}\}$ where $p \in N$, is fixed and $n \in N$, then T_1 and T_2 have a unique fixed point.

Proof: Using (3.3) we have

$$f(x_{1,} x_2) = f(T_1 x_0, T_2 x_1) \le q \max\Bigg\{ f(x_0,x_1), f(x_1,x_2), f(x_1,x_2), f(x_1,x_1), \frac{1}{2} f(x_0,x_2),$$
$$\frac{f(x_0,x_1) f(x_1,x_2)}{f(x_0,x_1)}, \frac{f(x_1,x_1) f(x_0,x_2)}{f(x_0,x_1)}, \frac{f(x_0,x_1) f(x_0,x_2)}{f(x_0,x_1) + f(x_1,x_2)},$$
$$\frac{f(x_1,x_2) f(x_0,x_2)}{f(x_0,x_1) + f(x_1,x_2)}, \frac{f(x_0,x_1) f(x_0,x_2)}{f(x_0,x_1) + f(x_1,x_2)}, \frac{f(x_1,x_2) f(x_0,x_2)}{f(x_0,x_1) + f(x_1,x_2)}, \frac{f(x_0,x_1) f(x_1,x_2) + f(x_0,x_2) f(x_1,x_2)}{2f(x_0,x_1) + f(x_1,x_2)}\Bigg\}$$
$$\le q \max\left\{ f(x_0,x_1), f(x_1,x_2), 0, \frac{1}{2}\{f(x_0,x_1) + f(x_1,x_2)\}\right\}$$

Then we have either $f(x_1,x_2) \le q\, f(x_1,x_2)$ which is a contradiction as q<1.

or $f(x_1,x_2) \le qf(x_0,x_1)$ as q<1. and $f(x_1,x_2) \le \frac{q}{2-q} f(x_0,x_1)$ as q<1.

Repeating this argument we have

$$f(x_0,x_1) > f(x_1,x_2) > f(x_2,x_3) > .....$$

Thus, the sequence $\{f(x_n, x_{n+1})\}$ is converging to some z. Again sequence $\{x_n\}$ has a subsequence $\{x_{(2p+1)2n}\}$. From the continuity of T_1 and T_2, we have $T_1 x = T_1 \lim x_{(2p+1)2n} = \lim x_{(2p+1)2n+1}$

$T_2 T_1 x = T_2 \lim x_{(2p+1)2n+1} = \lim x_{(2p+1)2n+2}$

Now, we have to show that x is a fixed point for T_1 and T_2, we get

$$\begin{aligned} f(x, T_1 x) &= f\left(\lim x_{(2p+1)2n}, x_{(2p+1)2n+1}\right) \\ &= f\left(\lim x_{(2p+1)2n}, x_{(2p+1)2n+1}\right) \\ &= f\left(\lim x_{(2p+1)2n+1}, x_{(2p+1)2n+2}\right) \\ &= f(T_1 x, T_2 T_1 x) \end{aligned}$$

Suppose $x \ne T_1 x$ then by (3.1), we have

$$f(T_1x, T_2T_1x) \le q \max\Bigg\{ f(x,T_1x), f(x,T_1x), f(T_1x,T_2T_1x) f(T_1x,T_1x), \frac{1}{2} f(x,T_2T_1x),$$
$$\frac{f(x,T_1x) f(T_1x,T_2T_1x)}{f(x,T_1x)}, \frac{f(x,T_2T_1x) f(T_1x,T_1x)}{f(x,T_1x)}, \frac{f(T_1x,T_2T_1x) f(x,T_2T_1x)}{f(x,T_1x) + f(T_1x,T_2T_1x)},$$
$$\frac{f(x,T_1x) f(x,T_2T_1x)}{f(x,T_1x) + f(T_1x,T_2T_1x)}, \frac{f(x,T_1x) f(x,T_2T_1x)}{f(x,T_1x) + f(T_1x,T_2T_1x)}, \frac{f(x,T_2T_1x) f(T_1x,T_2T_1x)}{f(x,T_1x) + f(T_1x,T_2T_1x)},$$
$$\frac{f(x,T_1x) f(T_1x,T_2T_1x) + f(x,T_2T_1x) f(T_1x,T_2T_1x)}{2f(x,T_1x) + f(T_1x,T_2T_1x)}\Bigg\}$$

Implies

$f(T_1x, T_2T_1x) \le qf(x,T_1x) < f(x,T_1x)$ a contradiction as q<1.

Thus $x = T_1 x$, similarly if $\{x_{(2p+1)2n+1}\}$ be subsequence of $\{x_{(2p+1)n}\}$ then we can get easily $x = T_2 x$. Therefore x is a fixed point for T_1 and T_2.

Uniqueness: Let $y(\neq x)$ be another fixed point of T_1 and T_2. By applying (3.3) we can prove that $f(T_1 x, T_2 y) \le f(x, y)$, a contradiction as q<1.

Hence x is a unique fixed point of T_1 and T_2.

This completes the proof of the theorem.

Theorem 4: Let $T_1, T_2, T_3, ...T_n$ be continuous mappings of a Hausdorff space X into itself and let f be a continuous mapping of X x X into the non-negative reals such that

$$f(x,x) = 0 \quad \forall\, x \in X \text{ and } f(x,y) \neq 0 \quad \forall\, x, y \in X, x \neq y \tag{4.1}$$

$$f(x,z) \le f(x,y) + f(y,z) \tag{4.2}$$

$$\begin{aligned} f(T_r x, T_{r+1} y) \le q \max\Bigg\{ & f(x,y), f(x,T_r x), f(x,T_{r+1} y), f(y,T_r x), \frac{1}{2} f(x,T_{r+1} y), \\ & \frac{f(x,T_r x) f(y,T_{r+1} y)}{f(x,y)}, \frac{f(y,T_r x) f(x,T_{r+1} y)}{f(x,y)}, \frac{f(x,y) f(x,T_{r+1} y)}{f(x,y) + f(T_r x, T_{r+1} x)} \\ & \frac{f(x,T_r x) f(x,T_{r+1} y)}{f(x,y) + f(T_r x, T_{r+1} y)}, \frac{f(T_r x, T_{r+1} y) f(x,T_{r+1} y)}{f(x,y) + f(T_r x, T_{r+1} y)}, \frac{f(x,T_{r+1} y) f(y,T_{r+1} y)}{f(x,y) + f(T_r, T_{r+1} y)}, \\ & \frac{f(x,y) f(T_r x, T_{r+1} y) + f(x,T_{r+1} y) f(y,T_{r+1} y)}{2 f(x,y) + f(T_r x, T_{r+1} y)} \Bigg\} \end{aligned} \tag{4.3}$$

hold for all distinct $x, y \in X$ where $q \in [0,1)$ and $T_k x = x_{k+1}$.

If for some $x_0 \in X$, the sequence $\{x_n\}$ where

$$x_1 = T_1 x_0, \quad x_2 = T_2 x_1, \quad x_3 = T_3 x_2, ... x_k = T_k x_{r-1}$$

....

....

$$x_{k+1} = T_1 x_k, \; x_{k+2} = T_2 x_{k+1}, ... \; x_{2k} = T_k x_{2k-1}$$

$$x_{n_k+1} = T_1 x_{n_k}, \quad x_{n_k+2} = T_2 x_{n_k+1} .... \; x_{(n_k+1)_k} = T_k x_{(n_k+1)k-1}$$

for n = 0, 1, 2, 3, ... has a convergent subsequence of the type $\{x_{mk+1}\}$ where $m \in N$ is fixed and $n \in N$. Then $T_1, T_2, T_3, ...T_k$ has a unique fixed point.

Proof: Using (4.3) we have

$$\begin{aligned} f(x_1,x_2) = f(T_1 x_0, T_2 x_1) \le q \max\Bigg\{ & f(x_0,x_1), f(x_0,x_1), f(x_1,x_2), 0, \frac{1}{2}[f(x_0,x_1) + f(x_1,x_2)] \\ & f(x_1,x_2), 0, f(x_0,x_1), f(x_1,x_2), f(x_0,x_1), f(x_1,x_2) f(x_1,x_2) \Bigg\} \end{aligned}$$

$$\le q \max\left\{ f(x_0, x_1, f(x_1,x_2), \frac{1}{2}[f(x_0,x_1) + f(x_1,x_2)] \right\}$$

which implies

$$f(x_1,x_2) \le q f(x_0,x_1) < f(x_0,x_1) \text{ as q<1}$$

Analogously

$f(x_2, x_3) < f(x_1, x_2)$ .... $f(x_{k-1}, x_k) < f(x_{k-2}, x_{k-1})$

By (4.3) we have

$$f(x_k, x_{k+1}) = f(T_k x_{k-1}, T_k x_k) \leq q \max\left\{ f(x_{k-1}, x_k) f(x_k, x_{k+1}) \frac{1}{2}\left[f(x_{k-1}, x_k) + f(x_k, x_{k+1}) \right] \right.$$

implies

$$f(x_k, x_{k+1}) \leq qf(x_{k-1}, x_k) < f(x_{k-1}, x_k) \text{ as } q<1$$

Hence

$f(x_n, x_{n+1}) < f(x_{n-1}, x_n)$, n = 0, 1, 2, 3,…

The uniqueness of $T_1, T_2, T_3, \ldots T_k$ is similar as of theorem-3.

This completes the proof of theorem.

REFERENCE:

[1] Edelstein, M : *On fixed and periodic points under contractive mappings*. J. London Math Soc. 37, 74 – 79 (1962).

[2] Yeh, C.C. : *Some fixed point theorems*. Boll Greek Math Soc. 21 (1980), 47–58.

[3] Singh, S.P. and Zorzitto : *On fixed point theorem in metric spaces*. Ext. Des Annales De Lu Soc. Sci. Bruzelles, 85 (1971), 117 – 123.

****