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A CERTAIN INVESTIGATION OF OPERATORS AND FRAME THEORY

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ABSTRACT

In this paper, we have to define (P, k) – quasiposi n-perinormal operator on Complex Hilbert space H and using this definition, we have derive the theorem which satisfies inequality condition. if T is (P, k)– quasiposi normal operator, We prove, T is (P, k)– quasiposi n-perinormal operator and we have discussed characterization of (P, k)– quasiposin-perinormal operator in H. Finally, we proposed the relations between quasi normal operator and frames with shift invariant operator.

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Keywords: *Hilbert Space, (p, k) –quasiposiperinormal operator, invertible operator, Frames, orthonormal basis.*

1. INTRODUCTION

Kotaro Ttnahashi, Atsushi Uchiyama and Muneo Cho discussed [2] Isolated point of spectrum of (p, k)-quasihyponormal operators and they given some result on this operators in 2004. M. L. Lee and S. H. Lee [1] have been discussed theorem and some properties on (p, k) - quasiposinormal operators in 2005. D. Senthil kumar, P. Maheswari Naik and Kiruthika investigated [3, 4] and given some theorems on (p, k)- quasiposinormal operators in 2013. In this paper, In order to extend the operator and investigate on (p, k)–quasiposi n-perinormal operator T on complex Hilbert space H and we discussed characterization of these operators in the first section.

In the second section, Frames were formally defined in Hilbert spaces by R.J. Duffin and A.C. Schaffer [5] in 1952 to deal with non harmonic Fourier series. After a couple of years, frames were brought to life in 1986 by Daubechies, Grossmann and Meyer, in the context of Painless nonorthogonal expansions [6] and Peter G.Casazza and Ole Christensen discussed Perturbation of operators and Applications to Frame theory [7].

K.Raju pillai and S.Palaniammal are investigated a nice reconstruction theorem for Frames in Hilbert Space and application in Communication systems. Frames and they used shifting operators to transmitted information or signal one domain to another domain [8, 9]. In this paper, we discussed Frame theory with quasinormal operators, quasi unitary operator and shift invariant operators.

SOME RESULT ON (P, k) - QUASIPOSI N- PERINORMAL OPERATORE

2. DEFINITIONS AND PRELIMINARIES

Let H be a complex Hilbert space and $B(H)$ be a algebra of all bounded linear operator on H, kernel and angel of an operator T are ker(T) and Ran(T) respectively . An operator $T \in B(H)$ is called

- Hypo normal, if $T^*T \geq TT^*$.
- P-Hypo normal, if $(T^*T)^p \geq (TT^*)^p$ for a positive integer p.
- P-Posinormal, if $c^2(T^*T)^p \geq (TT^*)^p$, for some $c > 0$ and $0 < p \leq 1$.

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- n-perinormal, if $T^{*n}T^n \geq (T^*T)^n$ for all $n \geq 2$.
- An operator $T \in B(T)$ is said to be positive $(T \geq 0)$ if $\langle Tx, x \rangle \geq 0$ for all $x \in H$.
- An operator is called Paranormal if $\|Tx\|^2 \leq \|T^2x\|\|x\|\ for\ all\ x \in H$.
- An operator is said to be Hemi normal if T is hypo normal and T^*T commute with T^*T.
- An operator is said to be (p, k)- quasihyponormal, if $T^{*k}(T^*T)^p \geq (TT^*)^pT^k, 0 < p \leq 1$.
- (p, k) –quasiposinormal, if $T^{*k}c^2(T^*T)^p \geq (TT^*)^pT^k$ for some $c > 0$, $0 < p \leq 1$ and for integer $k > 0$.

The quasi nilpotent part of T defined to be the set $H_0(T) = \{ x \in T : \lim_{n\to\infty}\|T^nx\|^{1/n} \} = 0$ and T is said to be Quasi nilpotent if its spectral radius $\gamma(T) = \inf_{n\in N}\|T^n\|^{1/n} = \lim_{n\to\infty}\|T^n\|^{1/n} = 0$, clearly $H_0(T)$ is a linear subspace of H. Let $\sigma_{jp}(T)$ be the joint spectrum of T , then $\lambda \in \sigma_{jp}(T)$ if and only if there exists a non zero vector x such that $Tx = \lambda x\ and\ T^*x = \lambda^*x$, this implies $\sigma_{jp}(T) \subset \sigma_p(T)$, equality condition hold , if T is normal. Let $\sigma_{ap}(T)$ be the approximate point spectrum of T and λ be Eigen value of T, then $\sigma_{ap}(T) = \{\lambda \in c : \exists\ a\ unit\ vector\ (x_n) \in H\}$, $such\ that$ $\lim_{n\to\infty}\|(T-\lambda)x_n\| = 0$ it is evident that $\sigma_p(T) \subset \sigma_{ap}(T)$, we have $\sigma_{jp}(T) \subset \sigma_p(T) \subset \sigma_{ap}(T)$ and the equality conditions hold for normal operator.

In this paper, we given a definition of (p, k)–quasiposi n-perinormal operator on Complex Hilbert space H and we derive theorems using this operator in Hilbert space.

Definition 2.1: Let $T \in B(H)$ be a (p, k)–quasiposi n-perinormal operator on Complex Hilbert space H, for all $n \geq 2$ such that $T^{*k}c^2(T^{*n}T^n)^p - (T^*T)^{np}T^k \geq 0$, for a positive integer k, $0 < p \leq 1$, and fixed constant $c > 0$.

Example 2.2: Let $l_2(C) = \{ x\,; x = (x_1, x_2, x_3, \dots\dots), \sum_{i=1}^{\infty}|x_i|^2 < \infty \}$, the unilateral shift operator on H is defined by $U(x_1, x_2, x_3, \dots\dots\dots) = (0, x_1, x_2, x_3, \dots\dots)$, then U is (p, k)–quasiposi n-perinormal operator, on Complex Hilbert space , the Bilateral shift operator is define by for the vector $x \in H$ such that $B(x_1, x_2, x_3, \dots\dots\dots) = (x_2, x_3, \dots\dots\dots)$. This statement is not trivial for the Bilateral shift Operator on Complex Hilbert space.

Example 2.3: In which follows we will show that T is not n-perinormal operator, T is n-perinormal if and only if $\||T|^nx\| \leq \|T^nx\|$, for all $n \geq 1$ and for a non-zero vector $\in H$,

$$|T|^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad and \quad T^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

For $e_1 = (0,1,0)$, the above relation does not hold, i.e. T is not n-perinormal operator for all $n \geq 1 \Rightarrow$ T is not (p, k)-quasiposi n-perinormal operator on complex Hilbert space.

Theorem 3.1: Let $T \in B(H)$ be an invertible, isometry operator on Hilbert space, If T is (p, k)-quasiposinormal operator on H for $0 < p \leq 1$ and for positive integer k, then T is (p, k) –quasiposi n-perinormal operator on H.

Proof: Let T be (p, k) -quasiposinormal operator for positive integer k,
i.e $T^{*k}c^2(T^*T)^p \geq (TT^*)^pT^k$,

For a positive integer k and fixed constant $c > 0$. T is invertible operator on H and then, we have

i.e $c^2(T^*T)^p \geq (TT^*)^p$, for all $0 < p \leq 1$.

$\Rightarrow N(T^*T) \geq TT^*$, for fixed constant $N = c^{2/p} > 0$ and $\frac{1}{p} > 1$

i.e $N(T^*T)(T^*T) \geq TT^*(T^*T)$

$$\begin{aligned} N(T^*T)(T^*T) &= N(T^*T^*)(TT) \\ &= NT^{*2}T^2 \end{aligned} \tag{1.1}$$

We know that

$$\begin{aligned} TT^*(T^*T) &= TT^*TT^* \\ &= (TT^*)^2 \end{aligned} \tag{1.2}$$

Therefore by (1.1) and (1.2) it follows that $NT^{*2}T^2 \geq (TT^*)^2$, for fixed constant $N > 0$,

By induction, it follows that

$$NT^{*n-1}T^{n-1} \geq (TT^*)^{n-1}$$

$$\Rightarrow\ NT^{*n}T^{*n} \geq (TT^*)^n,\ \text{for all}\ n \geq 1$$

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For positive number $p \neq 0$ such that $c^2(T^{*n}T^n)^p \geq (TT^*)^{np}$, for all $n \geq 1$ and fixed constant $N^p = c^2 > 0$. i.e $\langle c^2(T^{*n}T^n)^p - (TT^*)^{np})x, x\rangle \geq 0$, for non-zero vector $x \in H$ and T^k and T is positive operator in H and if $x = T^k y$, such that $\langle T^{*k}c^2(T^{*n}T^n)^p - (TT^*)^{np})T^k y, y\rangle \geq 0$, for non zero vector $y \in H$, Therefore, we have $T^{*k}c^2(T^{*n}T^n)^p - (TT^*)^{np})T^k \geq 0$, for all $n \geq 1$, $o < p \leq 1$ and positive integer k.

Hence, the proof of the theorem 3.1 is completed.

Theorem 3.2: Let $T \in B(H)$ be an operator on Hilbert space and T is (p,k) –quasiposi n-perinormal operator, if ϕ is homeomorphism for dimension, then $\phi(T)$ is (p,k) –quasiposi n-perinormal $\sigma_{jp}(T) \setminus \{0\} = \sigma_{ap}(T) \setminus \{0\}$.

Proof: Suppose T is (p,k) –quasiposi n-perinormal operator and let ϕ be the representation of Berberian, in first, we have to show that $\phi(T)$ is (p,k) –quasiposi n-perinormal,

$$\begin{aligned}\phi(T)^{*k}c^2(\phi(T^{*n})\phi(T^n))^p - (\phi(TT^*))^{np})\phi(T)^k &= \phi(T^{*k})c^2(\phi(T^{*n}T^n))^p - \left(\phi(TT^*)\right)^{np}\phi(T^k)\\ &= \phi(T^{*k})c^2(\phi(T^{*n}T^n))^p - \left(\phi(TT^*)\right)^{np}\phi(T^k)\\ &= \phi(T^{*k}c^2(T^{*n}T^n)^p) - \phi(TT^*)^{np})T^k)\\ &= \phi(T^{*k}c^2(T^{*n}T^n)^p - (TT^*)^{np})T^k)\end{aligned}$$

Since $T^{*k}c^2(T^{*n}T^n)^p - (TT^*)^{np})T^k \geq 0$ i.e T is positive operator

$\Rightarrow \phi(T^{*k}c^2(T^{*n}T^n)^p - (TT^*)^{np})T^k) \geq 0$

Thus $\phi(T)^{*k}c^2(\phi(T^{*n})\phi(T^n))^p - (\phi(TT^*))^{np})\phi(T)^k \geq 0$

Therefore $\phi(T)$ is (p,k) –quasiposi n-perinormal and by [3], hence we have the following result
$\sigma_a(T) \setminus \{0\} = \sigma_a(\phi(T)) \setminus \{0\} = \sigma_p(\phi(T)) \setminus \{0\} = \sigma_{jap}(T) \setminus \{0\}$.

Therefore the proof is completed.

Theorem 3.3: Let $T \in B(H)$ be an invertible quasi normal operator on complex Hilbert space, If T is (p,k) –quasiposi n-perinormal operator, then T is (p,k) –quasiposi $(n+1)$-perinormal operator ,Its converse is also true , if T has dense range in H.

Proof: Suppose T is (p,k) –quasiposi n-perinormal operator, we have to show that, T is (p,k) –quasiposi $(n+1)$-perinormal operator, i.e $(T^{*k}c^2\left(T^{*n+1}T^{n+1}\right)^p - (TT^*)^{(n+1)p})T^k) \geq 0$ for all $0 < p \leq 1,$ and fixed constant $c > 0$

$$\Rightarrow \langle(T^{*k}c^2(T^{*n}T^n)^p - (TT^*)^{np}T^k)x, x\rangle \geq 0, \ for\ non\ vector\ x \in H$$

there exists a non zero vector x , $y \in H\ such\ that\ a\ system\ T^p y = x$, then we have

$$\begin{aligned}\langle(T^{*k}c^2(T^{*n}T^n)^p - (TT^*)^{np}T^k)T^p y, T^p y\rangle &= \langle T^{*p}(T^{*k}c^2(T^{*n}T^n)^p - (TT^*)^{np}T^k)T^p y, y\rangle\\ &= \langle(T^{*k}c^2\left(T^{*n+1}T^{n+1}\right)^p - (TT^*)^{(n+1)p}T^k)y, y\rangle,\end{aligned}$$

$for\ all\ non\ vector\ y \in H.$
$i.e\ (T^{*k}c^2\left(T^{*n+1}T^{n+1}\right)^p - (TT^*)^{(n+1)p})T^k) \geq 0$, for all $0 < p \leq 1, n \geq 1$

Thus $T\ is\ (p,k)$ –quasiposi $(n+1)$-perinormal operator.

Conversely Suppose $T\ is\ (p,k)$ –quasiposi $(n+1)$-perinormal operator and T has dense range in H, i.e $H = \overline{R(T)}$, let $x \in H$, then if there exists $(x_n), (y_n) \in H, y_n = T^p x_n$, such that $T^p x_n \to y$

$$\langle(T^{*k}c^2(T^{*n}T^n)^p - (TT^*)^{np}T^k)y_n, y_n\rangle = \langle(T^{*k}c^2(T^{*n}T^n)^p - (TT^*)^{np}T^k)T^p x_n, T^p x_n\rangle$$

Since T has dense range in H, If $x_n \to x\ and\ y_n \to y\ then\ \langle x_n, y_n\rangle \to \langle x, y\rangle, y_n = T^p x_n$, $y_n \to y$ and $(T^{*k}c^2(T^{*n}T^n)^p - (TT^*)^{np}T^k)$ is continuous in H, we have

i.e $\langle(T^{*k}c^2(T^{*n}T^n)^p - (TT^*)^{np}T^k)y_n, y_n\rangle \to \langle(T^{*k}c^2(T^{*n}T^n)^p - (TT^*)^{np}T^k)y, y\rangle$

$\Rightarrow \langle(T^{*k}c^2(T^{*n}T^n)^p - (TT^*)^{np}T^k)y, y\rangle \geq 0$ i.e positive operator on H

$i.e\ T^{*k}c^2(T^{*n}T^n)^p - (TT^*)^{np}T^k \geq 0, for\ all\ n \geq 1\ and\ 0 < p \leq 1$

Thus T is (p,k) – quasiposi n-perinormal operator in H,
Therefore, the proof of theorem 3.3 is completed.

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Hölder-McCarthy Inequality 3.4 ([1])

Let A be positive operator on H, then the followings hold,

(i) $\langle A^r x, x\rangle \leq \|x\|^{2(1-r)}\langle Ax\, x\rangle^r \quad if \;\; 0 < r \leq 1$

(ii) $\langle A^r x, x\rangle \geq \|x\|^{2(1-r)}\langle Ax,, x\rangle$ if $r \geq 1$

Theorem 3.5: Let $T \in B(H)$ be an invertible operator on complex Hilbert space, If T is (p,k)–quasiposi n-perinormal, $for\ all \quad n \geq 1\ and\ 0 < p \leq 1$ such that $\|T^{k+1}x\|^2 \leq c^{1/p}\|T^{k+n+1}x\|\|T^{k-n+1}x\|$ for all unit vector $x \in H.$

Proof: Suppose T is (p,k) –quasiposi n-perinormal operator and T is invertible operator on H and we have

$$c^2|T^n|^{2p} \geq |T|^{2np}$$

Now,

$$\begin{aligned}
\|T^{n+k+1}x\|^2 &= \langle\, T^{n+k+1}x\,,\, T^{n+k+1}x\,\rangle \\
&= \langle\, T^{*n}T^nT^{k+1}x\,,\, T^{k+1}x\,\rangle \\
&= \langle\, |T^n|^2\, T^{k+1}x\,,\, T^{k+1}x\,\rangle \\
&= \langle\, |T^n|^{2p\frac{1}{p}}\, T^{k+1}x\,,\, T^{k+1}x\,\rangle \\
&\geq \langle\, |T^n|^{2p}\, T^{k+1}x\,,\, T^{k+1}x\,\rangle^{\frac{1}{p}}\, \|T^{k+1}x\|^{2(1-\frac{1}{p})} \text{ by Hölder-McCarthy Inequality} \\
&\geq c^{-\frac{2}{p}}\langle\, |T|^{2np}\, T^{k+1}x\,,\, T^{k+1}x\,\rangle^{\frac{1}{p}}\, \|T^{k+1}x\|^{2(1-\frac{1}{p})} \text{ by proposition 2.5} \\
&\geq c^{-\frac{2}{p}}\langle\, |T|^{2np}\,|T|^{2n}\, T^{k-n+1}x\,,\, T^{k-n+1}x\,\rangle^{\frac{1}{p}}\, \|T^{k+1}x\|^{2(1-\frac{1}{p})} \\
&\geq c^{-\frac{2}{p}}\langle\, |T|^{2n(p+1)}\, T^{k-n+1}x\,,\, T^{k-n+1}x\,\rangle^{\frac{1}{p}}\, \|T^{k+1}x\|^{2(1-\frac{1}{p})} \\
&\geq c^{-\frac{2}{p}}\langle\, |T|^{2n}\, T^{k-n+1}x\,,\, T^{k-n+1}x\,\rangle^{\frac{p+1}{p}}\|T^{k-n+1}x\|^{-2} \;\; \|T^{k+1}x\|^{2(1-\frac{1}{p})} \\
&\geq c^{-\frac{2}{p}}\|T^{k+1}x\|^4\|T^{k-n+1}x\|^{-2}
\end{aligned}$$

$i.e\ \|T^{k+1}x\|^2 \leq c\|T^{k+n+1}x\|\|T^{k-n+1}x\|$, for all orthonormal vector $x \in H$

Where $c > 0$ is fixed constant and for $n \geq 1$.

Hence, proof is completed.

FRAME THEORY WITH OPERATOR

Frames are generalizations of orthonormal basis. The linear independence property for a basis which allows each element in the space to be written as a linear combination and this is very restrictive for practical problems. A frames allows each element in the space to be written as a linear combination of the elements in the frames, here linear independence between the frames element is not required. This fact plays important role in signal processing, image processing, coding theory and sampling theory.

4. PRELIMINARIES AND NOTATIONS

Let $\mathcal{H}$ be hilbert space and $\mathrm{l}(\mathcal{H})$ be a set of all linear bounded operators on $\mathcal{H}$. we can define the following operators

$$\mathrm{T} : \mathrm{l}^2 \rightarrow \mathcal{H}, \quad \mathrm{Ta} = \textstyle\sum_{n=1}^{\infty} a_n f_n\,, \text{ for } a = \{a_n\} \in l^2$$

is called synthesis operator or pre frame operator and the ad joint operator is given that

$$\mathrm{T}^* : \mathcal{H} \rightarrow l^2, \quad \mathrm{T}^*f = \{\langle f, f_n\rangle\}_{n=1}^{\infty}$$

is called the analysis operator. The composition operator T with its adjoint T^* is denoted by $S = T^*T$,

$$\text{i.e} \quad S : H \rightarrow H, \quad Sf = H$$

is called the frame operator.

Before going to definition of Stable and unstable, let us define bounded and unbounded signal or frames. If the signal is bounded, then its magnitude is always be finite. i.e $|f_n| \leq m_n$, otherwise unbounded. A system is said to be unstable if the output of the system is unbounded for bounded input. A system is called Stable if the output of system is bounded for every bounded input or BIBO stable.

We begin with frame definitions. Let $\mathcal{H}$ be separable Hilbert space with the inner product $\langle\cdot\,,\,\cdot\rangle$ linear in the first entry and all index sets are assumed to be countable.

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Definition 4.1: Let $\mathcal{H}$ be separable Hilbert space and a sequence $\{f_n\}_{n=1}^{\infty} \subset H$ is called an ordinary frames. If there exist constants A, B> 0, such that $A\,\|f\|^2 \leq \sum_{n=1}^{\infty}|\langle f, f_n\rangle|^2 \leq B\|f\|^2$, for all $f \in H$, where A and B are lower and upper frame bound.

Definition 4.2: Let $\mathcal{H}$ be separable Hilbert space. A sequence $\{f_n\}_{n=1}^{\infty} \subset H$ is called a Bessel Sequence. If here exists constant $B > 0$, such that $\sum_{n=1}^{\infty}|\langle f, f_n\rangle|^2 \leq B\|f\|^2$, for $f \in H$.

Definition 4.3: Let $\mathcal{H}$ be Hilbert space, then

(i) A sequence $\{g_n\}_{n=1}^{\infty}$ is called a dual frame for $\{f_n\}_{n=1}^{\infty}$ if $f = \sum_{n=1}^{\infty}\langle f, f_n\rangle g_n$, for $f \in H$

(ii) A sequence $\{g_n\}_{n=1}^{\infty}$ is called a canonical dual frame for $\{S^{-1}f_n\}_{n=1}^{\infty}$ if $f = \sum_{n=1}^{\infty}\langle f, f_n\rangle g_n$, for $f \in H$

Theorem 5.1: Let H be separable Hilbert space and $T\epsilon H$ is reductive quasi similar to quasi normal operator and S is frame operator of the frame $\{f_n\}_{n=1}^{\infty} \subseteq H$. Then T is quasi normal operator.

Proof: For each $g\epsilon H$, $g = \lim_{m\to\infty}(\sum_{n=1}^{\infty} g_{mn})$ and $S_n = T|g_n$ which is reductive quasi similar to quasi normal operator,

Now,

$$\begin{aligned}\langle T(TT^*)g, g\rangle &= \langle \lim_{m\to\infty}(\textstyle\sum_{n=1}^{\infty} T(TT^*)g_{mn}), g\rangle \\ &= \langle \lim_{m\to\infty}(\textstyle\sum_{n=1}^{\infty} S_n(S_n{S_n}^*)g_{mn}), g\rangle \\ &= \langle \lim_{m\to\infty}(\textstyle\sum_{n=1}^{\infty} S^*{}_n S_n S_n g_{mn}), g\rangle \\ &= \langle \lim_{m\to\infty}(\textstyle\sum_{n=1}^{\infty} T^*TT g_{mn}), g\rangle \\ &= \langle T^*TTg, g\rangle\end{aligned}$$

i.e $\langle (T(TT^*) - T^*TT)g, g\rangle = 0$

Therefore $TTT^* = T^*TT$

Therefore T is quasi normal operator.

Now,

$$\begin{aligned}\langle TSg, g\rangle &= \langle T(\textstyle\sum_{n=1}^{\infty}\langle g, g_n\rangle g_n), g\rangle \\ &= \langle (\textstyle\sum_{n=1}^{\infty}\langle g, g_n\rangle Tg_n), g\rangle \\ &= \textstyle\sum_{n=1}^{\infty}\langle g, g_n\rangle\langle (Tg_n), g\rangle \\ &= \textstyle\sum_{n=1}^{\infty}\langle g, g_n\rangle\langle \sum_{n=1}^{\infty} c_n g_n, g\rangle\end{aligned}$$

We have, $\langle TSg, g\rangle = K\sum_{n=1}^{\infty}|\langle g, g_n\rangle|^2$

Since $\{g_n\}_{n=1}^{\infty}$ is frame in H,

$\langle TSg, g\rangle = K\sum_{n=1}^{\infty}|\langle g, g_n\rangle|^2 \leq KB\|g\|^2$ and $\langle TSg, g\rangle = K\sum_{n=1}^{\infty}|\langle g, g_n\rangle|^2 \geq KA\|g\|^2$

There exist constants $\bar{B} = KB < \infty$ and $\bar{A} = KA > 0$, We have $\langle TSg, g\rangle \leq \bar{B}\|g\|^2$ and $\langle TSg, g\rangle \geq \bar{A}\|g\|^2$

Therefore, we get the inequality $\bar{A}\|g\|^2 \leq \langle TSg, g\rangle \leq \bar{B}\|g\|^2$

Where lower frame bound $\bar{A}$ and upper frame bound $\bar{B}$ with frame operator $S^q = TS$.

Theorem 5.2: Let $\mathcal{H}$ be separable Hilbert space and the frame operator S of the dual frame for $\{f_n\}_{n=1}^{\infty}$ is positive, self adjoint, invertible and commutes with synthesis operator, then

(i) T is quasi normal operator in Hilbert space H

(ii) If $\{\bar{f}_n\}_{n=1}^{\infty}$ is dual frame for $\{f_n\}_{n=1}^{\infty}$, then T is quasi unitary operators.

Proof: we know that (ref [8])

$T: l^2 \to \mathcal{H}, Ta = \sum_{i=1}^{\infty} a_n f_n$, for $a = \{a_n\} \in l^2$

is called synthesis operator or pre frame operator and the ad joint operator is given that

$T^*: \mathcal{H} \to l^2$, $T^*f = \{\langle f, f_n\rangle\}_{n=1}^{\infty}$

is called the analysis operator . The composition operator T with its adjoint T^* it denoted by

$S = T^*T$, i.e $S: \mathcal{H} \to \mathcal{H}$, $Sf = \sum_{n=1}^{\infty}\langle f, f_n\rangle f_n$ for $f \in \mathcal{H}$ is called the frame operator.

Since T is bounded linear operator,

$$\begin{aligned}TSf &= T(\textstyle\sum_{n=1}^{\infty}\langle f, f_n\rangle f_n) \\ &= (\textstyle\sum_{n=1}^{\infty}\langle f, f_n\rangle Tf_n) \\ &= \textstyle\sum_{n=1}^{\infty}\langle f, f_n\rangle \bar{f}_n \\ &= \textstyle\sum_{n=1}^{\infty}\langle f, \bar{f}_n\rangle f_n \qquad (1.1)\end{aligned}$$

Now

$$\begin{aligned} STf &= \left(\sum_{n=1}^{\infty} \langle Tf, f_n \rangle f_n\right) \\ &= \sum_{n=1}^{\infty} \langle f, \bar{f}_n \rangle f_n \end{aligned} \tag{1.2}$$

From (1.1) and (1.2), we have $TS = ST$

i.e $TTT^* = T^*TT$

Therefore T is quasi normal operator.

Since $\{\bar{f}_n\}_{n=1}^{\infty}$ is dual frame for $\{f_n\}_{n=1}^{\infty}$,

From (1.1) $TSf = \sum_{n=1}^{\infty} \langle f, f_n \rangle \bar{f}_n \implies TS = I$

From (1.2) $STf = \sum_{n=1}^{\infty} \langle f, f_n \rangle \bar{f}_n \implies ST = I$

So, we have $TS = ST = I$

Therefore T is quasi unitary operator in Hilbert space H Therefore T is quasi unitary operator in Hilbert space H.

Theorem 5.3 ([9]): Let H be Hilbert space and T_1 *and* T_2 be self ad joint operators in H, then

(i) $S = T_2 T_1 \geq 0$ is self adjoin operator.

(ii) If T_1 *and* T_2 are shift invariant operators in H then S is shift invariant operators in H.

(iii) If the sequence $(a_n) \to a \in H$, then the frame operator S is stable.

Proof: (i) is trivial for self adjoin operator

Since T_1 *and* T_2 are shift invariant operators, composition of any two shift invariant operators is also shift invariant operators in H.

For every $(f_{n-k}) \in l^2$, $g^{\circ}_{n-k} = T(f_{n-k})$ and $g_{n-k} = T^*(g^{\circ}_{n-k})$

$$\begin{aligned} g_{n-k} &= T^*\big(T(f_{n-k})\big) \\ &= T^*T(g^{\circ}_{n-k}) \\ &= S(g^{\circ}_{n-k}) \end{aligned}$$

Therefore S is shift invariant. Since the sequence $(a_n) \to a$ in the Hilbert space which is bounded (BIBO). Therefore the system is stable.

CONCLUSION

We conclude that the main problem of communication systems is noise, which is eliminated by frame theory operator in the modes of linear, Shift invariant and orthogonal. The shift invariant operators are used data or information transmitted one domain to another domain. The orthonormal Frames in Hilbert space used for reduce noise to received original data. Frames with operators play an important role not only the theoretic but also many applications in Engineering and Technology.

POTENTIAL CONFLICTS OF INTEREST

The authors declare no conflict of interest.

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