International Journal of Mathematical Archive-6(10), 2015, 171-176
Available online through www.ijma.info ISSN 2229 – 5046

3-RD ORDER VOLTERRA-FREDHOLM INTEGRO-DIFFERENTIAL EQUATION USING BERNSTEIN POLYNOMIALS METHOD

MOHAMMED KHALID SHAHOOTH*[1] AND MISBAH. M. SHASAN

**Pure Mathematics, Department of Mathematics,
Faculty Science and Technology, Notional University of Malaysia.**

(Received On: 22-08-15; Revised & Accepted On: 15-09-15)

ABSTRACT

In this paper, the Bernstein Polynomial method is used to find an approximate solution to initials values problem for 3^{rd}-order linear Volterra-Fredholm integro differential equation of the second kind. Some different examples considered and the solution discussed numerically and display graphically. By enhancing the degree of Bernstein Polynomial, we can improve the accuracy results.

Keywords: *Bernstein polynomials method, Volterra-Fredholm integro-differential equations.*

1. INTRODUCTION

Mathematical modelling of real-life problems usually results in functional equations, such an ordinary or partial differential equations, integral and integro-differential equations and stochastic equations. Many mathematical formulation of physical phenomena contain integro-differential equations, these equations arises in many fields like fluid dynamics, biological models and chemical kinetics. In fact, integro-differential equations are usually difficult to solve analytically so it is required to obtain an efficient approximate or numerical solution [1], [2]. There are several numerical and analytical methods have been of great interest by several authors to study integro-differential equations, Therefore, in literature, there exist many numerical and semi-analytical-numerical techniques to solve Integro-differential equation. For Example, Integral Collocation Approximation Methods for the Numerical Solution of High-Orders Linear Fredholm-Volterra Integro-Differential Equations is found in [3].Wavelet-Galerkin method (WGM) to solve integro-differential equation can be found in [4]. Solution of forth-order integro-differential equation using variational iteration method in [5].In [6] Numerical Methods For Solving The First Order Linear Fredholm-Volterra Integro-Differential Equation. Numerical solution of nonlinear Volterra–Fredholm integro-differential equations via direct method using triangular functions was employed in [7]. In [8, 9] integro-differential equation is studied by using the differential transform method. In [10] Lagrange interpolation method is applied to solve integro-differential equation. The Tau method is applied to the integro-differential equation in [11]. Application of Adomian's decomposition method on Integro-differential equation are investigated in [12, 13, 14]. In [15] rationalized Haar functions method is applied on system of linear integro-differential equations. An Approximate solution of linear integro-differential equations by using modified Taylor expansion method is found in [16]. Numerical Solution of Volterra-Fredholm Integro-Differential Equation by Block Pulse Functions and Operational Matrices can be found in [17]. In [18] Collocation method is used to solve fractional integro-differential equation.

In recent years, many researchers have been successfully applying Bernstein polynomials method (BPM) to various linear and nonlinear integro-differential equation. For example, Bernstein polynomials method applied to find an approximate solution for Fredholm integro-Differential equation and integral equation of the second kind in [19]. The propose method is applied to find an approximate solution to initials values problem for high-order nonlinear Volterra-Fredholm integro differential equation of the second kind in [20]. Application of the Bernstein Polynomials for Solving the Nonlinear Fredholm Integro-Differential Equations is found in [21]. This method is used to find an approximate Solution of Fractional Integro-Differential Equations in [22].

Corresponding Author: Mohammed Khalid Shahooth*[1]
Pure Mathematics, Department of Mathematics,
Faculty Science and Technology, Notional University of Malaysia.

In this paper, we propose Bernstein polynomials method to solve second kind 3^{-rd} order Volterra-Fredholm integro-differential equations. We have introduced that the BPM is very powerful and efficient technique in finding analytical solutions for such as equations. The present paper is organized as follows: The Bernstein Polynomial method is described in the second section. In third section we present the Bernstein Polynomial approximation which method required for our function. In the fourth section the numerical finding to demonstrate the accuracy and applicability of the propose method by considering two examples are reported.

The linear Volterra-Fredholm integro-differential equation of the second kind can be expressed in general form as follows [13]

$$u^{(n)}(x) = f(x) + \lambda_1 \int_a^x k_1(x,t)u(t)dt + \lambda_2 \int_a^b k_2(x,t)u(t)dt, \tag{1}$$

Where $u^{(n)}(x) = \dfrac{d^n x}{dx^n}$ and $f(x),\ k_1(x,t),\ k_2(x,t)$ are known functions, $a,\ b,\ \lambda_1,\ \lambda_2$ are constant Values, and $u(x)$ is the unknown function which must be determined. The linear Volterra-Fredholm integro-differential equation of the second kind can be expressed as follows [23]

$$\sum_{i=0}^{m} \mu_i(x)\left[u^{(i)}(x)\right] = f(x) + \lambda_1 \int_a^x k_1(x,t)u(t)dt + \lambda_2 \int_a^b k_2(x,t)u(t)dt, \tag{2}$$

with the initial condition $y^{(i)}(a) = y_i, i = 1, 2, \cdots, m-1.$ where $\mu_i(x), i = 1, \cdots, m$, $\mu_i(x) \neq 0$ are known functions that have derivatives on an interval $a \leq x \leq t \leq b.$

2. BERNSTEIN POLYNOMIALS METHOD

Polynomials are incredibly useful mathematical tools as they are simple to define, can be calculated quickly on computer systems and represent a tremendous variety of functions. The Bernstein polynomials of degree-*n* are defined by [24]

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i} \quad \text{for } i = 0,1,2,\ldots,n \tag{3}$$

where $\binom{n}{i} = \dfrac{n!}{i!(n-i)!}$, (*n*) is the degree of polynomials, (*i*) is the index of polynomials and (*t*) is the variable. The exponents on the (t) term increase by one as (i) increases, and the exponents on the $(1-t)$ term decrease by one as (i) increases. The Bernstein polynomials of degree 2 are:

$$B_0^2(t) = (1-t)^2$$

$$B_1^2(t) = 2t(1-t)$$

$$B_2^2(t) = t^2$$

In addition, derivatives of the n^{th}-degree Bernstein polynomials are polynomials of degree $(n-1)$ By using the definition of the Bernstein polynomial in Equation (3). The derivative can be written as a linear combination of Bernstein polynomials [24].

$$\frac{d}{dt}B_k^n(t) = \frac{d}{dt}\binom{n}{k} t^k (1-t)^{n-k} = n(B_{k-1}^{n-1}(t) - B_k^{n-1}),\ 0 \leq k \leq n. \tag{4}$$

3. BERNSTEIN POLYNOMIALS APPROXIMATION METHOD

In this section, we consider the Bernstein polynomial approximation solution. Any function $u(x) \in L^2([a,b])$ can be expanded into a Bernstein polynomial series of finite terms [24], [25].

$$u(x) = c_0 B_0^n(t) + c_1 B_1^n(t) + c_2 B_2^n(t) + \cdots + c_n B_n^n(t),\ -\infty \leq a \leq x \leq b \leq \infty \tag{5}$$

Where $B_0^n(x), B_1^n(x), B_2^n(x), \cdots, B_n^n(x)$ are Bernstein polynomials terms which defined in Equation (3). $c_0, c_1, c_2, \cdots, c_n$ are unknown Bernstein polynomial coefficients, then Equation (5) can be decomposed as

$$u(x) = \sum_{i=0}^{n} c_i B_i^n(x) \tag{6}$$

By utilizing Equation (6) and substituting into Equation (2), we obtain

$$\sum_{i=0}^{m} \mu_i(x) \left[\sum_{j=0}^{n} c_j B_j^n(x) \right]^i = f(x) + \lambda_1 \int_a^x k_1(x,t) \left[\sum_{j=0}^{n} c_j B_j^n(x) \right] dt + \lambda_2 \int_a^b k_2(x,t) \left[\sum_{j=0}^{n} c_j B_j^n(x) \right] dt \tag{7}$$

The Equation (7) can be rewritten in a simplify form as:

$$\begin{aligned} \sum_{i=0}^{m} \mu_i(x) \left[c_0 B_0^n(x) + c_1 B_1^n(x) + c_2 B_2^n(x) + \cdots + c_n B_n^n(x) \right]^i &= f(x) \\ &+ \lambda_1 \int_a^x k_1(x,t) \left[c_0 B_0^n(x) + c_1 B_1^n(x) + c_2 B_2^n(x) + \cdots + c_n B_n^n(x) \right] dt \\ &+ \lambda_2 \int_a^b k_2(x,t) \left[c_0 B_0^n(x) + c_1 B_1^n(x) + c_2 B_2^n(x) + \cdots + c_n B_n^n(x) \right] dt, \end{aligned} \tag{8}$$

Finally, extension the Bernstein polynomial terms into Equation (8) by using Equation (3), we obtained

$$\begin{aligned} \sum_{i=0}^{m} \mu_i(x) \left[c_0 (1-x)^n + c_1 x (1-x)^{n-1} + c_2 x^2 (1-x)^{n-2} + \cdots + c_n x^n \right]^i &= f(x) \\ &+ \lambda_1 \int_a^x k_1(x,t) \left[c_0 (1-t)^n + c_1 t (1-t)^{n-1} + c_2 t^2 (1-t)^{n-2} + \cdots + c_n t^n \right] dt \\ &+ \lambda_2 \int_a^x k_1(x,t) \left[c_0 (1-t)^n + c_1 t (1-t)^{n-1} + c_2 t^2 (1-t)^{n-2} + \cdots + c_n t^n \right] dt, \end{aligned} \tag{9}$$

Now by integrating the terms in the right hand side of Equation (9) after simplifying and derivatives the terms in the left hand side using Equation (4).

Once this equation is simplified and represent as a linear equation include x as a variable. Then substitute the collocation points in the interval $[a,b]$, which can be calculated as follows:

$$x_i = a + i\frac{b-a}{n}, \quad i = 0,1,2,...,n \tag{10}$$

The result obtained from Equation (9). At the end, we establish a system of linear equations involve $(n+1)$ the unknown coefficients $c_i, i = 0,1,2,..,n$ which can be determined by solving the linear system using the Gauss elimination method.

4. NUMERICAL EXAMPLES AND RESULTS

In this section, we consider the following examples of linear 3rd-order Volterra-Fredholm integro-differential equation. These examples are studied in [13] by using series solution method and variation iteration method respectively. The computations associated with the examples were performed using Matlab ver.2013a.

Example 1: Consider a 3-rd order linear Volterra-Fredholm integro-differential equation as follows:

$u'''(x) = -\frac{1}{2}x^2 + \int_0^x u(t)dt + \int_{-\pi}^{\pi} xu(t)dt,\ u(0) = u'(0) = -u''(0) = 1.$ and the exact solution is $u(x) = x + \cos(x).$

Table-1: Numerical results for Example 1 with exact solution by using BPM.

t	$yexact$	*Bernstein polynomial method*	
		$yapp, n=2$	$Error, n=2$
0	1.0000	1.0000	0.0000
0.1000	1.0950	1.0950	0.0000
0.2000	1.1801	1.1800	0.0000
0.3000	1.2553	1.2550	0.0000
0.4000	1.3211	1.3200	0.0000
0.5000	1.3776	1.3750	0.0000
0.6000	1.4253	1.4200	0.0000
0.7000	1.4648	1.4550	0.0001
0.8000	1.4967	1.4800	0.0003
0.9000	1.5216	1.4950	0.0007
1.0000	1.5403	1.5000	0.0016

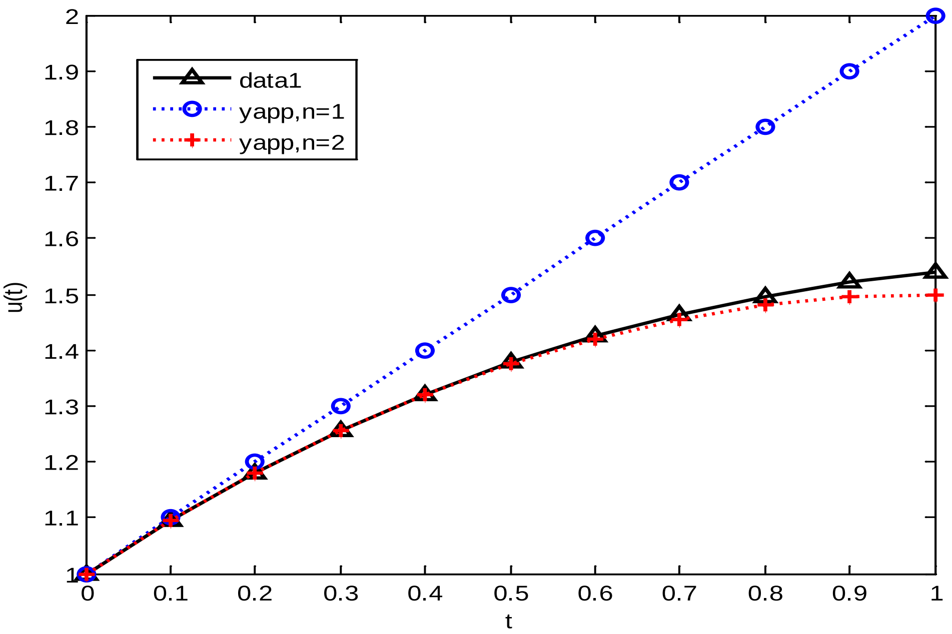

Figure-1: Approximation solutions and exact solution for Example 1 by using BPM.

Example 2: Consider a 3-rd order linear Volterra-Fredholm integro-differential equation as follows:

$$u'''(x) = 2\sin(x) - x - 3\int_0^x (x-t)u(t)dt + \int_0^{\pi/2} u(t)dt,$$

$u(0)=u'(0)=1,\ u''(0)=-1.$ and the exact solution is $u(x)=\sin(x)+\cos(x).$

Table-2: Numerical results for Example 2 with exact solution by using BPM.

t	$yexact$	*Bernstein polynomial method*	
		$yapp, n=2$	$Error, n=2$
0	1.0000	1.0000	0.0000
0.1000	1.0948	1.0950	0.0000
0.2000	1.1787	1.1800	0.0000
0.3000	1.2509	1.2550	0.0000
0.4000	1.3105	1.3200	0.0001
0.5000	1.3570	1.3750	0.0003
0.6000	1.3900	1.4200	0.0009
0.7000	1.4091	1.4550	0.0021
0.8000	1.4141	1.4800	0.0043
0.9000	1.4049	1.4950	0.0081
1.0000	1.3818	1.5000	0.0140

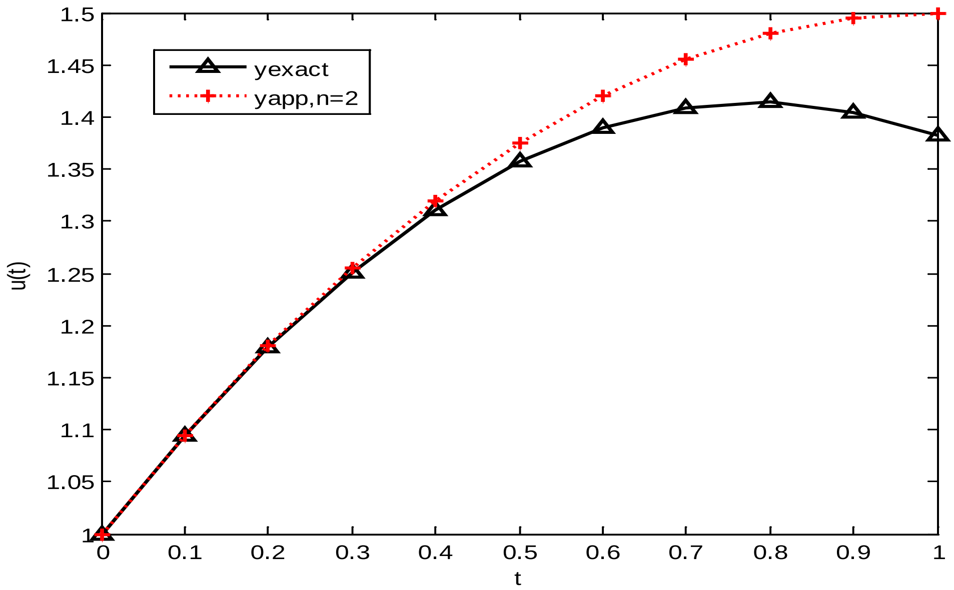

Figure-2: Approximation solutions and exact solution for Example 2 by using BPM.

5. CONCLUSION

Most integro differential equations are difficult to solve analytically, in many cases it requires to obtain the approximate solution. In this work, the Bernstein polynomial method for the solution of 3rd-order linear Volterra-Fredholm integro-differential equations is successfully implemented. The proposed approach is simple and has been tested on two examples to illustrate the efficiency of the present method. It is clear that from the results, more accurate results can be obtained by increasing the n^{th}-degree of the Bernstein polynomial.

REFERENCES

1. Batiha, B., Noorani, M. S. M., Hashim, I, Numerical solutions of the nonlinear integro-differential equations. *Int. J. Open Problems Compt. Math*, 1(1), (2008), 34-42.
2. Altawallbeh, Z., AL-Smadi, M. H, Abu-Gdairi, R, Approximate solution of second-order integro-differential equation of Volterra type in RKHS method. *Int. Journal of Math. Analysis*, *7*, (2013), 2145-2160.
3. Abubakar, A., Taiwo, O, A. Integral Collocation Approximation Methods for the Numerical Solution of High-Orders Linear Fredholm-Volterra Integro-Differential Equations. *American Journal of Computational and Applied Mathematics*, *4*(4), (2014), 111-117.
4. A. Avudainayagam, C. Vani, Wavelet-Galerkin method for integro-differential equations, Appl. Numer. Math., 32 (2000), 247-254.
5. N. Sweilam, Fourth order integro-differential equations using variational iteration method, Compu. Math. Appl., 54 (2007), 1086-1091.
6. Omran, H. H, Numerical Methods For Solving The First Order Linear Fredholm-Volterra Integro-Differential Equations. Journal Of Al-Nahrain University, Vol.12 (3), (2009), PP.139-143.
7. Babolian, E., Masouri, Z, Hatamzadeh-Varmazyar, S, Numerical solution of nonlinear Volterra–Fredholm integro-differential equations via direct method using triangular functions. Computers & Mathematics with Applications, 58(2), (2009), 239-247.
8. A. Arikoglu, I. Ozkol, Solution of boundary value problems for integro-differential equations by using differential transform method, Appl. Math. Compu., 168 (2005), 1145-1158.
9. P. Darnaia, A. Ebadian, A method for the numerical solution of the integro-differential equations, Appl. Math. Compu., 188 (2007), 657-668.
10. M. Rashed, Lagrange interpolation to compute the numerical solutions of differential, integral and integro-differential equations, Appl. Math. Compu., 151 (2004), 869-878.
11. S. Hosseini, S. Shahmorad, Numerical solution of a class of Integro-Differential equations by the Tau Method with an error estimation, Appl. Math. Compu., 136 (2003), 559-570.
12. I. Hashim, Adomian decomposition method for solving BVPs for fourth-order integro- differential equations, Jour. Compu. Appl. Math., 193 (2006), 658-664.
13. A. Wazwaz, Linear and Nonlinear Integral Equations: Methods and Applications, Higher education press, Springer, (2011).
14. A. Wazwaz, A reliable algorithm for solving boundary value problems for higher-order integro-differential equations, Appl. Math. Compu., 118 (2001), 327-342. 18.

15. K. Maleknejad, F. Mirzaee, S. Abbasbandy, Solving linear integro-differential equations system by using rationalized Haar function method, Appl. Math. Compu., 155 (2005), 317- 328.
16. Rashidinia, J Tahmasebi, A, Approximate solution of linear integro-differential equations by using modified Taylor expansion method. World Journal of Modelling and Simulation, Vol. 9, No. 4, (2013), pp. 289-301.
17. Rahmani, L., Rahimi, B, Mordad, M, Numerical Solution of Volterra-Fredholm Integro-Differential Equation by Block Pulse Functions and Operational Matrices. Gen, 4(2), (2011), 37-48.
18. E. Rawashdeh, Numerical solution of fractional integro-differential equation by collocation method, Appl. Math. Compu., 176 (2006), 1-6.
19. AL-Juburee, A. K., Approximate Solution for linear Fredholm Integro-Differential Equation and Integral Equation by Using Bernstein Polynomials method. Journal of the Faculty of Education .Al- Mustansiriyah University. Iraq. (2010).
20. Intisar Swedain Ali, Using Bernstein Polynomials Method for Solving High-order Nonlinear Volterra-Fredholm Integro Differential Equation, College of Education and Basic Iraq,v21, N:89,(2015).
21. Yadollah Ordokhani, Sara Davaei far, Application of the Bernstein Polynomials for Solving the Nonlinear Fredholm Integro-Differential Equations, Journal of Applied Mathematics & Bioinformatics, vol.1, no.2,(2011),13-31.
22. Osama H. Mohammed, Sarmad. A. Altaie, Approximate Solution of Fractional Integro-Differential Equations by Using Bernstein Polynomials, Eng. & Tech. Journal, Vol.30, No.8, (2012).
23. Salah H. Behiry, Saied I. Mohamed, Solving high-order nonlinear Volterra-Fredholm integro-differential equations by differential transform method, Natural Science, Vol.4, No.8, (2012), 581-587.
24. K. Joy, Bernstein polynomials, On-line Geometric Modeling Notes, (2000).
25. Henryk Gzyl, Jos´e Luis Palacios, On the Approximation Properties of Bernstein Polynomials via Probabilistic Tools, *Boletın de la Asociación Matemática Venezolana*, Vol. X, No. 1, (2003).

Source of support: Nil, Conflict of interest: None Declared