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Pairwise-Ψ-Open Sets in Bitopological Spaces

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ABSTRACT

The aim of this paper is to present a new class of sets namely pairwise Ψ- open sets and their properties in bitopological spaces. Further the notion of pairwise Ψ- operators are also studied to obtain their characterization.

Key words: *Ψ-open, (i, j)-Ψ-open, (i, j)-Ψ-closed, (i, j)-Ψ-D(A)*

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1. INTRODUCTION:

The study of bitopological spaces was first initiated by Kelly [8] in 1963. A large number of papers have been published to generalize the topological concepts to bitopological settings. The term 'preopen' was used for the first time by A.S. Mashhour, M. E. Abd El-Deeb[10] On pre-continuous and weak pre-continuous mappings. Many concepts of topology have been generalized by considering the concept of preopen sets by several authours [5],[11],[12]]. Recently Alias B. Khalaf and Haji M. Hasan et al [1] have defined a new type of set which is conditional ξ-open set in bitopological spaces.

In this paper we introduce the concept of pairwise Ψ-open set in abitopological spaces and obtain number of characterization and relationship of this class with other concepts of sets. Throughout this paper (X,τ_1,τ_2) is a bitopological space and if A⊆Y⊆ X then i-Int(A) and i-Cl (A) denotes respectively the interior and closure of A with respect to the induced topology on Y.

2. PRELIMINARIES:

Definition 2.1 A subset A of a space(X, τ) is called:

1. preopen [10], if A⊆ Int(Cl(A))
2. semi-open [9], if A⊆ Cl(Int(A))
3. α-open [12], if A⊆ Int Cl (Int (A))
4. regular open[6],if A= Int(Cl(A))
5. regular semi-open[15], if A=sInt(sCl(A))

Definition 2.2 A subset A of a space(X, τ) is called:

1. preclosed [10], if Cl(Int(A)) ⊆ A
2. semi-closed [9], if Int(Cl(A)) ⊆ A
3. α-closed [12], if Cl Int (Cl (A)) ⊆ A
4. regular closed[6],if A= Cl(Int(A))
5. regular semi-closed[15], if A=sCl(sInt(A))

Definition 2.3: The intersection of all preclosed sets of X containing A is called preclosure.

Definition 2.4: The intersection of all semi-closed sets of X containing A is called semi-closure.

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Definition 2.5: The intersection of all α-closed sets of X containing A is called α-closure.

Definition 2.6: The union of all preopen sets of X contained in A is called preinterior of A.

Definition 2.7: The union of all semi-open sets of X contained in A is called semi-interior of A.

Definition 2.8: The union of all α-open sets of X contained in A is called α-interior of A.

Definition 2.11: A subset A of a space X is called δ-open [16], if for each x ∈ A, there exists an open set G such that x ∈ G ⊆ Int (Cl (G)) ⊆ A. The complement of δ-open sets is δ-closed.

Definition 2.12: A subset A of a space (X, τ) is called η-open [15], if A is a union of δ-closed sets. The complement of η-open sets is called η-closed.

Definition 2.13: A subset A of a space X is called θ-semi-open [6](resp., semi-θ-open[3]), if for each x ∈ A, there exists an semi-open set G such that x ∈ G ⊆ Cl (G)⊆ A.(resp., x ∈ G ⊆ sCl (G)⊆ A).

Definition 2.14: A topological space X is called,

1. Externally disconnected [3], if Cl (U) ∈ τ for every U∈ τ.

2. Locally indiscrete [5], if every open subset of X is closed.

Definition 2.15: [10] A space X is semi-T_1 if and only if for any point x ∈ X the singleton set$\{x\}$ is semi-closed.

Definition 2.16: [11] For any space (X, τ) and (Y, τ) if A ⊆ X, B ⊆ Y $then$:

1. $pInt_{X\times Y}(A\times B) = pInt_X(A)\times pInt_Y(B)$
2. $sCl_{X\times Y}(A\times B) = sCl_X(A)\times sCl_Y(B)$

Definition 2.17[13]**:** A space X is said to be semi-regular if for any open set U of X and each point x ∈ U, there exists a regular open set V of X such that x ∈ V⊆ U.

3. BASIC PROPERTIES:

Definition 3.1: A subset A of a bitopological space (X, τ_1, τ_2) is said to be (i, j)-Ψ-open, if A is a j-open set and for all x in A, there exists an i-pre closed set F such that x ∈ F⊆ A.

A subset B of X is called (i, j)-Ψ-closed, if B^c is (i, j)-Ψ-open.

The family of (i, j)-Ψ-open (resp., (i, j)-Ψ-closed) subset of X is denoted by (i, j)-ΨO(X) (resp., (i, j)-ΨC(X)). The following result shows that any union of $(i,j)\Psi O$(X) sets in bitopological space (X, τ_1 ,τ_2) is $(i,j)\Psi O$(X).

Proposition 3.2: Let $\{A_\lambda : \lambda \in \Delta\}$ be family of $(i,j)\Psi$- open sets in bitopological space (X, τ_1, τ_2), then $\cup\{A_\lambda : \lambda\in\Delta\}$ is an $(i,j)\Psi$- open set.

Proof: Let $\{A_\lambda : \lambda \in \Delta\}$ be family of $(i,j)\Psi$- open sets in bitopological space (X, τ_1, τ_2).Since A_λ is j-open for each $\lambda \in \Delta$, then $\cup\{A_\lambda : \lambda\in\Delta\}$ is j-open set in a space X. Suppose that x∈ $\cup A_\lambda$. This implies that there exists $\lambda_0 \in \Delta$ such that x ∈ A_{λ_0}.

Since $A_{\lambda_0}$is an $(i,j)\Psi$- open set, there exists i-pre closed set F in X such that that x ∈ $F \subseteq A_{\lambda_0} \subseteq \cup A_\lambda$ for all $\lambda \in \Delta$. Therefore $\cup\{A_\lambda : \lambda\in\Delta\}$ is an $(i,j)\Psi$- open set.

The following result shows that any intersection of $(i,j)\Psi O$(X) sets in bitopological space (X, τ_1 ,τ_2) is $(i,j)\Psi O$(X).

Proposition 3.3: Any finite intersection of $(i,j)\Psi$-open sets in bitopological space (X, τ_1 ,τ_2) is an $(i,j)\Psi$-open set.

Proof: Let A_i be $(i,j)\Psi$-open for I = 1, 2…n in bitopological space (X, τ_1 ,τ_2) . Let x ∈ $\cap A_i$ for I = 1, 2…n. But A_i is $(i,j)\Psi$-open. So there exists pre closed F_i for each i= 1, 2…n such that x ∈ $F_i \subseteq A_i$.$This\ implies\ that$ x ∈ $\cap F_i \subseteq \cap A_i$. Therefore ⊆ $\cap A_i$ is an $(i,j)\Psi$-open set.
Hence the family of $(i,j)\Psi$-open subset of (X, τ_1 ,τ_2) forms a bitopology on X.

Definition 3.4: A space X is pre-T_1 if and only if for any point x ∈ X the singleton set{x} is pre-closed.

Remark 3.5: A subset A of a bitopological space X is (i, j)-Ψ-open, if A is j-open set and it is a union of i-pre closed sets. This means that A= $\cup F_\propto$, where A is a j-open and $F_\propto$, is an i-pre closed set for each α.

Result 3.6: Every (i, j)-Ψ-open set is j-open, but the converse is not true in general as shown in the following example.

Example 3.7: Let X={a, b ,c}, $\tau_1 = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$, $\tau_2 = \{X, \phi, \{a\}, \{b, c\}\}$, then (i, j)-ΨO(X) = $\{X, \phi, \{b, c\}\}$ .It is clear that {a} is j-open, but not (i, j)-ΨO(X).

Proposition 3.8: Let (X, τ_1, τ_2) be a bitopological space, if (X, τ_1) is a pre-T_1 space, then (i, j)-ΨO(X) =τ_j

Proof: Let A be any subset of a space X and A is j-open set. If A = ϕ then A ∈ (i, j)ΨO(X).

If A ≠ ϕ, now let x ∈ A. Since (X, τ_1) is pre-T_1 space, then every singleton set is i-pre closed set. Hence x ∈ $\{x\}$ ⊆ A.

Therefore A ∈ (i, j)ΨO(X). Hence $\tau_j(X)$ ⊆ (i, j)ΨO(X).

But (i, j)ΨO(X) ⊆ $\tau_j(X)$ generally. Thus (i, j)ΨO(X) = $\tau_j(X)$.

Proposition 3.9: Let (X, τ_1, τ_2) be a bitopological space and Let A be a subset of a space X. If A ∈ j-δO(X) and A is an i-closed set then A ∈ (i, j)ΨO(X).

Proof: Let A be a subset of a space X. If A = ϕ then A ∈ (i, j)ΨO(X). If A ≠ ϕ,

Now let x ∈ A. Since A ∈ j-δO(X) and j-δO(X) ⊆ $\tau_j(X)$ in general, so A∈ $\tau_j(X)$. Since A is i-closed so A is i-pre closed and x ∈ A⊆A. Hence A ⊆ (i, j) ΨO(X).

Corollary 3.10: Let (X, τ_1, τ_2) be a bitopological space, if a subset of a space X is i-regular closed and j-open then A ∈ (i, j)ΨO(X).

Theorem 3.11: In a bitopological space (X, τ_1, τ_2) , if a space (X, τ_i) is locally indiscrete then (i, j)-ΨO(X) ⊆ τ_i .

Proof: Let V∈ (i, j)ΨO(X) then V∈ $\tau_j(X)$. For each x ∈ V, there exists i-pre closed F in X such that x ∈ F⊆V. Then F is i-open. This implies that V∈ τ_i . Hence (i, j)-ΨO(X) ⊆ τ_i .

The converse of the theorem is not true in general as shown in the following example:

Example 3.12: Let X={a, b ,c}, $\tau_1 = \{X, \phi, \{b\}, \{a, c\}\}$, $\tau_2 = \{X, \phi, \{a, c\}\}$, then (i, j)-ΨO(X) = $\{X, \phi, \{a, c\}\}$ .It is clear that (X , τ_1) is locally indiscrete but τ_1 is not a subset of (i, j)-ΨO(X).

Theorem 3.13: LetX_1 , X_2 be bitopological space and $X_1 \times X_2$ be the bitopological product, let A_1 ∈ (i, j)ΨO(X_1) and A_2 ∈ (i, j)ΨO(X_2) then $A_1 \times A_2$ ∈ (i, j)ΨO($X_1 \times X_2$).

Proof: Let (x_1 ,x_2) ∈ $A_1 \times A_2$. Then $x_1 \in A_1$ *and* $x_2 \in A_2$.

Since A_1 ∈ (i, j)ΨO(X_1) and A_2 ∈ (i, j)ΨO(X_2), then A_1 ∈ j − ΨO(X_1) and A_2 ∈ j − ΨO(X_2).

Therefore there exists F_1 ∈ i-PC (X_1) and F_2 ∈ i-PC (X_2) such that $x_1 \in F_1 \subseteq A_1$ and $x_2 \in F_2 \subseteq A_2$. Therefore ($x_1$,x_2) ∈ $F_1 \times F_2 \subseteq A_1 \times A_2$.Since A_1 ∈ j − ΨO(X_1) and A_2 ∈ j − ΨO(X_2) then $A_1 \times A_2$ = j −Ψ $Int_{x_1}(A_1)$ × j −Ψ $Int_{x_2}(A_2)$ = j −Ψ $Int_{x_1 \times x_2}(A_1 \times A_2)$.

Hence $A_1 \times A_2$ ∈ j −Ψ O ($X_1 \times X_2$). Since F_1 ∈ i-PC (X_1) and F_2 ∈ i-PC (X_2) then we get $F_1 \times F_2$ = i-$pCl_{x_1}(F_1)$ × i-$pCl_{x_2}(F_2)$ = i-$pCl_{x_1 \times x_2}(F_1 \times F_2)$.

Hence $F_1 \times F_2 \in$ i-PC(X). Therefore $A_1 \times A_2 \in (\mathrm{i},\mathrm{j})\Psi O(X)$.

Definition 3.14: A subset A of A space X is Called θ- preopen, if for each x∈A,there exists a preopen set G such that x ∈ G ⊆ cl(G) ⊆ A.

Definition 3.15: A subset A of A space X is Called pre-θ-open, if for each x∈A,there exists a preopen set G such that x ∈ G ⊆ pCl(G) ⊂ A.

Theorem 3.16: For any bitopological space (X, τ_1, τ_2) , if A∈ $\tau_j(X)$ and either A ∈ j-ηO(X) or A ∈ i-$P\theta$O(X), then A ∈ $(\mathrm{i},\mathrm{j})\Psi \mathrm{O}$(X).

Proof: Let A ∈ j-ηO(X) and A∈ $\tau_j(X)$. If A = ϕ then A ∈ $(\mathrm{i},\mathrm{j})\Psi\mathrm{O}$(X). If A ≠ ϕ, A ∈ j-$\eta$O(X) then A= ∪ $F_\propto$, where $F_\propto$ ∈ i-δC(X) for each α. Then $F_\propto$ ∈ i-PC(X) for each α and A∈ $\tau_j(X)$. Then A ∈ $(\mathrm{i},\mathrm{j})\Psi\mathrm{O}$(X).

On the other hand, suppose that A ∈ i-$P\theta$O(X) and A∈ $\tau_j(X)$. If A = ϕ then A ∈ $(\mathrm{i},\mathrm{j})\Psi\mathrm{O}$(X). If A ≠ ϕ, A ∈ i-$P\theta$O(X) then for x ∈ A, there exists i-pre open set U such that

x ∈ U⊆ i-pCl(U) ⊆ A. This implies that x ∈ i-pCl(U) ⊆ A *and* A∈ $\tau_j(X)$.Then A ∈ $(\mathrm{i},\mathrm{j})\Psi\mathrm{O}$(X).

Properties 3.17: For any topological space the following statements are true:

(1) Let(Y,τ_Y) be a subspace of a space (X, τ), if F ∈ PC(X) and F ⊆Y then F ∈ PC(Y).

(2) Let(Y,τ_Y) be a subspace of a space (X, τ), if F ∈ PC(Y) and Y ∈ PC(X) then F ∈ PC(X).

(3) Let(X τ) be a topological space, if Y is open subset of a space X and F ∈ PC(X) , then F∩Y∈PC(X).

Theorem 3.18: Let Y be a subspace of a bitopological space (X, τ_1, τ_2), if A ∈ $(\mathrm{i},\mathrm{j})\Psi\mathrm{O}$(X) and A ⊆Y then A ∈ $(\mathrm{i},\mathrm{j})\Psi\mathrm{O}$(Y).

Proof: Let A ∈ $(\mathrm{i},\mathrm{j})\Psi\mathrm{O}$(X) then A∈ $\tau_j(X)$. For x ∈ A, there exists i-pre closed set F in X such that x ∈ F⊆ A. Since A∈ $\tau_j(X)$ and A ⊆ Y then A∈ $\tau_j(Y)$.

Since F ∈ i-PC(X) and F ⊆ Y then F ∈ i-PC(Y). Hence A ∈ $(\mathrm{i},\mathrm{j})\Psi\mathrm{O}$(Y).

Corollary 3.19: Let X be a bitopological space, A and Y be two subsets of X such that A ⊆Y⊆X , Y∈ $RO(X, \tau_j)$, Y∈ $RO(X, \tau_i)$, then A ∈ $(\mathrm{i},\mathrm{j})\Psi\mathrm{O}$(Y) if and only if A ∈ $(\mathrm{i},\mathrm{j})\Psi\mathrm{O}$(X).

Proposition 3.20 Let Y be a subspace of a bitopological space (X, τ_1, τ_2), if A ∈ $(\mathrm{i},\mathrm{j})\Psi\mathrm{O}$(Y) and Y∈ i-PC(X) then for each x ∈ A, there exists i-pre closed set F in X such that x ∈ F⊆ A.

Proof: Let A ∈ $(\mathrm{i},\mathrm{j})\Psi\mathrm{O}$(Y). Then A ∈ $\tau_j(Y)$ and for each x ∈ A, there exists i-pre closed set F in Y such that x ∈ F⊆ A and since Y∈ i-PC(X) so F∈ i-PC(X).

Proposition 3.21: Let A and Y be any subsets of a bitopological space X, if A ∈ $(\mathrm{i},\mathrm{j})\Psi\mathrm{O}$(X)and Y∈ $RO(X, \tau_j)$, Y∈ $RO(X, \tau_i)$, then A∩Y ∈ $(\mathrm{i},\mathrm{j})\Psi\mathrm{O}$(X).

Proof: Let A ∈ $(\mathrm{i},\mathrm{j})\Psi\mathrm{O}$(X). Then A ∈ $\tau_j(X)$ and A=∪$F_\propto$, where $F_\propto$ ∈ i-PC(X) for each α. Then A∩Y = ∪$F_\propto$∩Y =∪($F_\propto$∩Y). Since Y∈ $RO(X, \tau_j)$ then Y is j-open. Therefore A∩Y ∈ $\tau_j(X)$. Since Y∈ $RO(X, \tau_i)$, then Y∈ i-PC(X).

Hence $F_\propto$∩Y ∈ i − PC(X) for each α. Therefore A∩Y ∈ $(\mathrm{i},\mathrm{j})\Psi\mathrm{O}$(X).

Definition 3.22: A subset A of a space(X, τ) is called regular preopen, if A= pInt(pCl(A)).

Proposition 3.23: Let A and Y be any subsets of a bitopological space X, if A $\in$ (i, j)ΨO(X) and Y is regular pre open in τ_i *and* τ_j then A$\cap Y$ $\in$ (i, j)ΨO(Y).

Proof: Let A $\in$ (i, j)ΨO(X). Then A $\in \tau_j(X)$ and A=$\cup F_\propto$, where $F_\propto \in$ i-PC(X) for each α. Then A$\cap Y$ = $\cup$ $F_\propto \cap Y$ =$\cup$ ($F_\propto \cap Y$).

Since Y$\in RPO(X, \tau_j)$ then Y $\in$j-PO(X). Therefore A$\cap Y$ $\in \tau_j(Y)$. Since Y$\in RPO(X, \tau_i)$, then Y$\in$ i-PC(X). Hence $F_\propto \cap Y \in$ i-PC(X) for each α. Then $F_\propto \cap Y \in$ i-PC(Y). Therefore A$\cap$ Y $\in$ (i, j)ΨO(Y).

Proposition 3.24: If Y is an i-open and j-open subspace of a bitopological space X and A $\in$ (i, j)ΨO(X) then A$\cap Y$ $\in$ (i, j)ΨO(Y).

Proof: Let A $\in$ (i, j)ΨO(X). Then A $\in \tau_j(X)$ and A=$\cup$ $F_\propto$, where $F_\propto \in$ i-PC(X) for each α. Then A$\cap Y$ = $\cup F_\propto \cap Y$ =$\cup$ ($F_\propto \cap Y$). Since Y $is\ j-open\ sub\ space$ of X, then Y $\in$j-PO(X). Therefore A$\cap Y$ $\in \tau_j(Y)$. Since Y is i-open subspace of X, then $F_\propto \cap Y \in$ i-PC(X) for each α. Therefore A$\cap Y \in$ (i, j)ΨO(Y).

Corollary 3.25: If either Y$\in RPO(X, \tau_j)$ and Y$\in RPO(X, \tau_i)$ or Y is an i-open and j-open subspace of a bitopological space X and A $\in$ (i, j)ΨO(X) then A$\cap Y$ $\in$ (i, j)ΨO(Y).

4. PAIRWISE-Ψ-OPERATORS:

Definition 4.1: A subset N of a bitopological space (X, τ_1, τ_2) is called (i,j)-Ψ neighbourhood of a subset A of X if there exists an (i,j)- Ψ open set U such that A ⊆ U⊆ N. When A = {x}, we say that N is (i,j)-Ψ neighbourhood of x.

Definition 4.2: A point x $\in$ X is said to be an (i, j)-Ψ interior point of A, if there exists an (i,j)- Ψ-open set U containing x such that U⊂ A. The set of all (i,j)-Ψ interior points of A is said to be (i,j)-Ψ interior of A and it is denoted by (i,j)-Ψ Int(A).

Definition 4.3: Let X be a bitopological space and A⊆ X, x $\in$ X, then x is (i,j)-Ψ interior of A if and if A is an (i,j)-Ψ neighbourhood of x.

Definition 4.4: A subset G of a bitopological space X is (i,j)-Ψ-open if and only if it is an (i,j)-Ψ neighbourhood of each of its points.

Proposition 4.5: Let A any subset of bitopological space X. If a point x in the (i,j)-Ψ-Int (A) , then there exists a i-pre closed set F of X containing x and F ⊆ A.

Proof: Suppose that x $\in$ (i,j)-Ψ-Int (A).Then there exists in (i,j)-Ψ-open set U of X containing x ,such that x $\in$ U ⊆ A. Since U is an (i,j)-Ψ-open set, there exists an i-pre closed set F, such that x $\in$ F ⊆ U ⊆ A.Hence x $\in$ F ⊆ A.

Some properties of (i,j)-Ψ-interior operators on a set are given in the following:

Properties 4.6: For any subsets A and B of a bitopological space X

The (i, j)-Ψ-interior of A is the union of all (i, j)-Ψ-open sets Contained in A:

1. (i, j)-Ψ-Int (A) is an (i,j)-Ψ-open set in X contained in A.
2. (i, j)-Ψ-Int (A) is largest (i, j)-Ψ-open set in X contained in A.
3. A is an (i, j)-Ψ-open set if and only if A= (i,j)-Ψ-Int(A).
4. (i, j)-Ψ-Int(ϕ) = ϕ.
5. (i, j)-Ψ-Int(X) = X.
6. A ⊆(i, j)-Ψ-Int(A).
7. If A⊆ B, then (i,j)-Ψ-Int(A) ⊆ (i,j)-Ψ-Int(B).
8. (i, j)-Ψ-Int(A) $\cap$ (i,j)-Ψ-Int(B) = (i,j)-Ψ-Int (A $\cap$ B).
9. (i,j)- Ψ-Int(A) $\cup$(i,j)-Ψ-Int(B) ⊆ (i,j)-Ψ-Int(A $\cup$ B).

In general (i, j)-Ψ-Int (A) $\cup$(i,j)-Ψ-Int (B) ≠ (i,j)- Ψ-Int (A $\cup$ B) as it shown in the following example:

Example 4.7: Let X = {a,b,c},τ_1 = {ϕ,{a},{b,c},X},τ_2 = { ϕ,{b,c},X}.

Then (i,j)-ΨO(X) = { ϕ,{b,c},X}.If we take A = {a,b} and B = {b,c}

then (i,j)-Ψ-Int (A) = ϕ,(i,j)-Ψ-Int (B) = {b,c},(i,j)-Ψ-Int (A $\cup$ B) = (i,j)-Ψ-Int (X) = X.

In general (i,j)-Ψ-Int (A) ⊆ j-Int (A), but (i,j)-Ψ-Int (A) ≠j-Int(A) which is shown in the following example;

Example 4.8: Let X = {a,b,c}, τ_1 = {ϕ,{a},{a,b},{a,c},X}, τ_2 = { ϕ,{a},{b,c},X}.

Then (i,j)- ΨO(X) = { ϕ,{b,c},X}.If we take A = {a} then (i,j)- Ψ-Int (A) = ϕ, but j-Int(A) = A.

Hence (i,j)- Ψ-Int (A) ≠j-Int(A).

Definition 4.9: The intersection of all (i,j)- Ψ- closed set containing F is called (i,j)-Ψ-closure of F and we denoted it by (i,j)-Ψ-cl(F).

Remark 4.10: Let F be any subset of a space X. A point x ∈X is in the (i,j)- Ψ-closed of F, if and only if F∩U ≠ ϕ for every (i,j)- Ψ-open set U containing x.

Proposition 4.11: Let A be any subset of a bitopological space X. If a point x in the (i,j)- Ψ-closure of A, then F∩A ≠ ϕ for every i-pre closed set F of X containing x.

Proof: Suppose that x ∈ (i,j)-Ψ-cl(A)Then A∩U ≠ ϕ for every (i,j)-Ψ-open set U containing x. Since U is an (i,j)- Ψ-open set, there exists an i-pre closed set F containing x, such that F⊆U. Hence F∩A ≠ ϕ.

Some properties of (i,j)-Ψ-closure operators on a set are given.

Properties 4.12: For any subsets A and B of A bitopological space X,

1. The (i,j)-Ψ-closure of A is the intersection of all (i,j)-Ψ-closed sets containing A.
2. (i,j)-Ψ-cl(A) is an (i,j)-Ψ-closed set in X containing A.
3. (i,j)-Ψ-cl(A) is smallest (i,j)-Ψ-closed set in X containing A.
4. A is an (i,j)-Ψ-closed set if and only if A= (i,j)-Ψ-cl(A).
5. (i,j)-Ψ-cl(ϕ) = ϕ.
6. (i,j)-Ψ-cl(X) = X.
7. A ⊆(i,j)-Ψ-cl(A).
8. If A⊆ B, then (i,j)-Ψ-cl(A) ⊆ (i,j)-Ψ-cl(B).
9. (i,j)-Ψ-cl(A ∩ B) ⊆ (i,j)-Ψ-cl(A) ∩ (i,j)-Ψ-cl(B) .
10. (i,j)-Ψ-cl(A) $\cup$(i,j)-Ψ-cl(B) = (i,j)-Ψ-cl(A $\cup$ B).

Properties 4.13: For any subset A of a bitopological space X,

1. X\((i,j)-Ψ-cl(A)) = (i,j)-Ψ-Int(X\A)
2. X\((i,j)-Ψ-Int(X\A)) = (i,j)-Ψ-cl(X\A)
3. (i,j)-Ψ-Int(A) = X\((i,j)-Ψ-cl(X\A)).

It is clear that j-cl(F) ⊆ (i,j)- Ψ-cl(F), the converse may be false as shown in the following example:

Example 4.15: Let X = {a,b,c}, τ_1 = {ϕ,{a},{a,b},{a,c},X},τ_2 = { ϕ,{a}, {b,c},X}.

Then (i, j)-ΨO(X) = { ϕ,{b,c},X}.If we take F = {b,c} then j-cl(F) = {b,c} and (i,j)-Ψ-cl(F) = X.

This shows that (i,j)-Ψ-cl(F) is not a subset of j-cl(F).

Remark 4.16: If A is any subset of a bitopological space X, then (i,j)-Ψ-Int(A) ⊆j-Int (A) ⊆ A⊆ A ⊆ j-cl (A) ⊆ (i,j)-Ψ-cl(A).

Definition 4.17: Let A be a subset of A bitopological space X.A point x ∈x is said to be (i,j)-Ψ-limit point of A , if for each (i,j)-Ψ-open set U containing x, U ⋂(A\{x}) ≠ϕ. The set of all (i,j)-Ψ-limit point of A is called (i,j)-Ψ-derived set of A and is denoted by (i,j)-Ψ-D(A).

In general it is clear that (i,j)-Ψ-D(A) ⊆ j-D(A), but the converse may not be true as shown in the following example:

Example 4.18: Let X = {a, b, c},τ_1 = {ϕ,{a},{a, b},{a, c},X},τ_2 = { ϕ,{a}, {b, c},X}.

Then (i, j)-ΨO(X) = {ϕ, {b, c}, X}.If we take A={a, c}, then (i, j)-Ψ-D(A) ={b, c} and j-D(A) = {b}.

Hence (i,j)-Ψ-D(A) is not a subset of j-D(A).

Theorem 4.19: Let X be a bitopological space and A be a subset of X, then A ⋃ (i,j)-Ψ-D(A) is (i,j)-Ψ-closed.

Proof: Let x ∉ (i,j)-Ψ-D(A).This implies that x ∉ A and x ∉ (i,j)-Ψ-D(A).

Since x ∉ (i,j)-Ψ-D(A), then there exists an (i,j)-Ψ-open U of X, Which contains no point of A other than x, but x ∉A.So U contains no point of A. Then U⊆X\A.Again U is an (i,j)-Ψ-open set for each of its points. But as U does not contain any point of A, no point of U can be (i,j)-Ψ-limit point of A. Therefore no point U can belong to (i,j)-Ψ-D(A).Then U⊆X\ (i,j)-Ψ-D(A).Hence it follows that x∈X\A⋂(X\(i,j)-Ψ-D(A) = X\(A⋃(i,j)-Ψ-D(A)).Therefore A⋃(i,j)-Ψ-D(A) is an (i,j)-Ψ-closed. Hence (i,j)-Ψ-cl(A) ⊆ A⋃(i,j)-Ψ-D(A).

Remark 4.20: If a subset A of a bitopological space X is (i,j)-Ψ-closed, then A contains the set of all of its (i,j)-Ψ-limit points.

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