International Journal of Mathematical Archive-6(11), 2015, 20-24
Available online through www.ijma.info ISSN 2229 – 5046

MORE ON THE COMPLEX REFLECTION ARRANGEMENT $\mathcal{A}(G_{25})$

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(Received On: 14-10-15; Revised & Accepted On: 07-11-15)

ABSTRACT

In this paper we study the complex reflection arrangement $\mathcal{A}(G_{25})$ and we found the bases, circuits, broken circuits, no broken circuits base, the triple arrangements, poincaré polynomial, and the characteristic polynomial of $\mathcal{A}(G_{25})$.

Key word: *Arrangement, bases, circuit, broken circuit.*

1. INTRODUCTION

An arrangement of hyperplanes is a finite collection of codimension one subspace in a finite dimensional vector space over R(ℂ). We fix a linear order on $\mathcal{A}$ and we write $\mathcal{A}$ = {H_1, … ,H_n}. Let $L = L(\mathcal{A})$ be the intersection poset of $\mathcal{A}$. L is the set of non-empty intersections of hyperplanes in $\mathcal{A}$ ordered by reverse inclusion. By convention L includes V as its unique minimal element. Then L is a ranked poset, with r(X) = codim(X), and all maximal elements have the same rank. The rank of $\mathcal{A}$ is then the rank of any maximal element of L, denoted by r. The arrangement $\mathcal{A}$ is called centered with center T($\mathcal{A}$) if T($\mathcal{A}$) = $\bigcap_{H \in \mathcal{A}}$ H ≠ ∅. If $\mathcal{A}$ is centered, then the coordinate may be chosen so that each hyperplane contains the origin and hence $\mathcal{A}$ is central. We call $\mathcal{A}$ centerless if the intersection of the finite family $\mathcal{A}$ is empty, i.e. $\bigcap_{H \in \mathcal{A}}$ H = ∅.

2. PRELIMINARY DEFINITIONS

***Definition* (2.1) [5][6]:** A finite poset L is said to be geometric lattice if it satisfies the following conditions:

1. (1) L is a lattice (hence has a minimum $\hat{0}$ and a maximam$\hat{1}$).
2. (2) Every X ∈ L is a join of atoms.
3. (3) If X, Y ∈ L cover X ∧ Y then X ∨ Y covers both X and Y.

Definition* (2.2) [1]:** Let L be a geometric lattice of rank r and let L_1 be the set of atoms. A set B = {b_1, b_2, …,b_n}⊆ L_1 is said to be a ***base of L if n = r and, ⋁B = $\hat{1}$.

***Note**:* ⋁B the intersection of all elements of B

i.e. (⋁B = $b_1 \cap b_2 \cap \ldots \cap b_n$).

Definition* (2.3) [10]:** Call B ⊆ L ***independent set, if r(⋁B) = |B| and ***dependent*** if r(⋁B) < |B|, where |B| is the cardinal number of B.

Definition* (2.4) [2][8]:** A ***circuit C ⊆ $\mathcal{A}$ is a minimal dependent sub arrangement (with respect to the inclusion) and for a given total order ⊴ of the hyperplanes of $\mathcal{A}$, $\bar{C}$ = C\{H} is said to be a ***broken circuit of*** $\mathcal{A}$, where H is the smallest hyperplane in C with respect to the ordering ⊴. The sub-arrangement B ⊆ $\mathcal{A}$ which contains no broken circuit is denoted by ***NBC*** base of $\mathcal{A}$ and if r(B) = k we denoted it by ***k_NBC*** base . Let X ∈ L, we call the NBC base of $\mathcal{A}$ a NBC base of X if $\bigcap_{H \in B} = $ X.

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3. THE REFLECTION ARRANGEMENT $\mathcal{A}(G_{25})$

In the classification of Shephard and Todd **[3]** G_{25} is the complex reflection group of order 648 generated by reflections of order "3" and its reflection arrangement is given by:

$$Q\ (\mathcal{A}(G_{25})) = xyz \quad \prod_{0\le i,j\le 2}(x + \omega^i y + \omega^j), \text{ where } \omega = \frac{2\pi i}{e^3}$$

Let $H_i = \ker \alpha_{H_i}$, $1 \le i \le 12$ such that:

$H_1 : x = 0$	$H_5 : x + y + \omega z = 0$	$H_9 : x + \omega y + \omega^2 z = 0$
$H_2 : y = 0$	$H_6 : x + y + \omega^2 z = 0$	$H_{10} : x + \omega^2 y + z = 0$
$H_3 : z = 0$	$H_7 : x + \omega y + z = 0$	$H_{11} : x + \omega^2 y + z = 0$
$H_4 : x + y + z = 0$	$H_8 : x + \omega y + \omega z = 0$	$H_{12} : x + \omega^2 y + \omega^2 z = 0$

Thus $\mathcal{A}(G_{25})$ contains "12" hyperplanes which are:
$\{H_1,H_2,H_3,H_4,H_5,H_6,H_7,H_8,H_9,H_{10},H_{11},H_{12}\}$.

3.1. Bases Of $L(\mathcal{A}(G_{25}))$

The set of atoms of $L(\mathcal{A}(G_{25}))$ is $L_1 = \{H_1,H_2,H_3,H_4,H_5,H_6,H_7,H_8,H_9,H_{10},H_{11},H_{12}\}$ and the sets $B_i \subseteq L_1$, i = 1,…,185 are bases since $|B_i| = 3$ for all i = 1,…,185 and r = 3, $\vee B_i = \hat{I}$, i = 1,…,185.

The bases of $L(\mathcal{A}(G_{25}))$ are:

$\{H_1,H_2,H_3\} = B_1$	$\{H_2,H_3,H_5\} = B_{48}$	$\{H_3,H_6,H_9\} = B_{96}$	$\{H_5,H_7,H_{10}\} = B_{142}$
$\{H_1,H_2,H_4\} = B_2$	$\{H_2,H_3,H_6\} = B_{49}$	$\{H_3,H_6,H_{10}\} = B_{97}$	$\{H_5,H_7,H_{11}\} = B_{143}$
$\{H_1,H_2,H_5\} = B_3$	$\{H_2,H_3,H_7\} = B_{50}$	$\{H_3,H_6,H_{11}\} = B_{98}$	$\{H_5,H_7,H_{12}\} = B_{144}$
$\{H_1,H_2,H_6\} = B_4$	$\{H_2,H_3,H_8\} = B_{51}$	$\{H_3,H_6,H_{12}\} = B_{99}$	$\{H_5,H_8,H_9\} = B_{145}$
$\{H_1,H_2,H_7\} = B_5$	$\{H_2,H_3,H_9\} = B_{52}$	$\{H_3,H_7,H_{10}\} = B_{100}$	$\{H_5,H_8,H_{10}\} = B_{146}$
$\{H_1,H_2,H_8\} = B_6$	$\{H_2,H_3,H_{10}\} = B_{53}$	$\{H_3,H_7,H_{11}\} = B_{101}$	$\{H_5,H_8,H_{12}\} = B_{147}$
$\{H_1,H_2,H_9\} = B_7$	$\{H_2,H_3,H_{11}\} = B_{54}$	$\{H_3,H_7,H_{12}\} = B_{102}$	$\{H_5,H_9,H_{11}\} = B_{148}$
$\{H_1,H_2,H_{10}\} = B_8$	$\{H_2,H_3,H_{12}\} = B_{55}$	$\{H_3,H_8,H_{10}\} = B_{103}$	$\{H_5,H_9,H_{12}\} = B_{149}$
$\{H_1,H_2,H_{11}\}= B_9$	$\{H_2,H_4,H_5\} = B_{56}$	$\{H_3,H_8,H_{11}\} = B_{104}$	$\{H_5,H_{10},H_{12}\}= B_{150}$
$\{H_1,H_2,H_{12}\}= B_{10}$	$\{H_2,H_4,H_6\} = B_{57}$	$\{H_3,H_8,H_{12}\} = B_{105}$	$\{H_5,H_{11},H_{12}\}= B_{151}$
$\{H_1,H_3,H_4\} = B_{11}$	$\{H_2,H_4,H_8\} = B_{58}$	$\{H_3,H_9,H_{10}\} = B_{106}$	$\{H_6,H_7,H_8\} = B_{152}$
$\{H_1,H_3,H_5\} = B_{12}$	$\{H_2,H_4,H_9\} = B_{59}$	$\{H_3,H_9,H_{11}\} = B_{107}$	$\{H_6,H_7,H_9\} = B_{153}$
$\{H_1,H_3,H_6\} = B_{13}$	$\{H_2,H_4,H_{11}\} = B_{60}$	$\{H_3,H_9,H_{12}\} = B_{108}$	$\{H_6,H_7,H_{10}\} = B_{154}$
$\{H1,H_3,H_7\} = B_{14}$	$\{H_2,H_4,H_{12}\} = B_{61}$	$\{H_4,H_5,H_7\} = B_{109}$	$\{H_6,H_7,H_{12}\} = B_{155}$
$\{H_1,H_3,H_8\} = B_{15}$	$\{H_2,H_5,H_6\} = B_{62}$	$\{H_4,H_5,H_8\} = B_{110}$	$\{H_6,H_8,H_9\} = B_{156}$
$\{H_1,H_3,H_9\} = B_{16}$	$\{H_2,H_5,H_7\} = B_{63}$	$\{H_4,H_5,H_9\} = B_{111}$	$\{H_6,H_8,H_{10}\} = B_{157}$
$\{H_1,H_3,H_{10}\} = B_{17}$	$\{H_2,H_5,H_9\} = B_{64}$	$\{H_4,H_5,H_{10}\} = B_{112}$	$\{H_6,H_8,H_{11}\} = B_{158}$
$\{H_1,H_3,H_{11}\} = B_{18}$	$\{H_2,H_5,H_{10}\} = B_{65}$	$\{H_4,H_5,H_{11}\} = B_{113}$	$\{H_6,H_8,H_{12}\} = B_{159}$
$\{H_1,H_3,H_{12}\} = B_{19}$	$\{H_2,H_5,H_{12}\} = B_{66}$	$\{H_4,H_5,H_{12}\} = B_{114}$	$\{H_6,H_9,H_{10}\} = B_{160}$
$\{H_1,H_4,H_5\} = B_{20}$	$\{H_2,H_6,H_7\} = B_{67}$	$\{H_4,H_6,H_7\} = B_{115}$	$\{H_6,H_9,H_{11}\} = B_{161}$
$\{H_1,H_4,H_6\} = B_{21}$	$\{H_2,H_6,H_8\} = B_{68}$	$\{H_4,H_6,H_8\} = B_{116}$	$\{H_6,H_{10},H_{11}\}= B_{162}$
$\{H1,H_4,H_7\} = B_{22}$	$\{H_2,H_6,H_{10}\} = B_{69}$	$\{H_4,H_6,H_9\} = B_{117}$	$\{H_6,H_{10},H_{12}\}= B_{163}$
$\{H_1,H_4,H_9\} = B_{23}$	$\{H_2,H_6,H_{11}\} = B_{71}$	$\{H_4,H_6,H_{10}\} = B_{118}$	$\{H_6,H_{11},H_{12}\}= B_{164}$
$\{H_1,H_4,H_{10}\} = B_{24}$	$\{H_2,H_7,H_8\} = B_{72}$	$\{H_4,H_6,H_{11}\} = B_{119}$	$\{H_7,H_8,H_{10}\} = B_{165}$
$\{H_1,H_4,H_{11}\} = B_{25}$	$\{H_2,H_7,H_9\} = B_{73}$	$\{H_4,H_6,H_{12}\} = B_{120}$	$\{H_7,H_8,H_{11}\} = B_{166}$
$\{H_1,H_5,H_6\} = B_{26}$	$\{H_2,H_7,H_{11}\} = B_{74}$	$\{H_4,H_7,H_8\} = B_{121}$	$\{H_7,H_8,H_{12}\} = B_{167}$
$\{H_1,H_5,H_7\} = B_{27}$	$\{H_2,H_7,H_{12}\} = B_{75}$	$\{H_4,H_7,H_9\} = B_{122}$	$\{H_7,H_9,H_{10}\} = B_{168}$
$\{H_1,H_5,H_8\} = B_{28}$	$\{H_2,H_8,H_9\} = B_{76}$	$\{H_4,H_7,H_{11}\} = B_{123}$	$\{H_7,H_9,H_{11}\} = B_{169}$
$\{H_1,H_5,H_{11}\} = B_{29}$	$\{H_2,H_8,H_{10}\} = B_{77}$	$\{H_4,H_7,H_{12}\} = B_{124}$	$\{H_7,H_9,H_{12}\} = B_{170}$
$\{H_1,H_5,H_{12}\} = B_{30}$	$\{H_2,H_8,H_{12}\} = B_{78}$	$\{H_4,H_8,H_9\} = B_{125}$	$\{H_7,H_{10},H_{11}\}= B_{171}$
$\{H_1,H_6,H_8\} = B_{31}$	$\{H_2,H_9,H_{10}\} = B_{79}$	$\{H_4,H_8,H_{10}\} = B_{126}$	$\{H_7,H_{10},H_{12}\}= B_{172}$
$\{H_1,H_6,H_9\} = B_{32}$	$\{H_2,H_9,H_{11}\} = B_{80}$	$\{H_4,H_8,H_{11}\} = B_{127}$	$\{H_7,H_{11},H_{12}\}= B_{173}$
$\{H_1,H_6,H_{10}\} = B_{33}$	$\{H_2,H_{11},H_{12}\} = B_{81}$	$\{H_4,H_8,H_{12}\} = B_{128}$	$\{H_8,H_9,H_{10}\} = B_{174}$
$\{H_1,H_6,H_{12}\} = B_{34}$	$\{H_3,H_4,H_7\} = B_{82}$	$\{H_4,H_9,H_{10}\} = B_{129}$	$\{H_8,H_9,H_{11}\} = B_{175}$
$\{H_1,H_7,H_8\} = B_{35}$	$\{H_3,H_4,H_8\} = B_{83}$	$\{H_4,H_9,H_{11}\} = B_{130}$	$\{H_8,H_9,H_{12}\} = B_{176}$
$\{H_1,H_7,H_9\} = B_{36}$	$\{H_3,H_4,H_9\} = B_{84}$	$\{H_4,H_9,H_{12}\} = B_{130}$	$\{H_8,H_{10},H_{11}\}= B_{177}$
$\{H_1,H_7,H_{10}\} = B_{37}$	$\{H_3,H_4,H_{10}\} = B_{85}$	$\{H_4,H_{10},H_{11}\}= B_{131}$	$\{H_8,H_{10},H_{12}\} =B_{178}$
$\{H_1,H_7,H_{12}\} = B_{38}$	$\{H_3,H_4,H_{11}\} = B_{86}$	$\{H_4,H_{10},H_{12} = B_{132}$	$\{H_8,H_{11},H_{12}\} =B_{179}$

$\{H_1,H_8,H_9\} = B_{39}$	$\{H_3,H_4,H_{12}\} = B_{87}$	$\{H_4,H_{11},H_{12}\}= B_{133}$	$\{H_9,H_{10},H_{11}\} =B_{180}$
$\{H_1,H_8,H_{10}\} = B_{40}$	$\{H_3,H_5,H_7\} = B_{88}$	$\{H_5,H_6,H_7\} = B_{134}$	$\{H_9,H_{10},H_{12}\} =B_{181}$
$\{H_1,H_8,H_{11}\} = B_{41}$	$\{H_3,H_5,H_8\} = B_{89}$	$\{H_5,H_6,H_8\} = B_{135}$	$\{H_9,H_{11},H_{12}\} =B_{182}$
$\{H_1,H_9,H_{11}\} = B_{42}$	$\{H_3,H_5,H_9\} = B_{90}$	$\{H_5,H_6,H_9\} = B_{136}$	$\{H_2,H_{10},H_{12}\} =B_{183}$
$\{H_1,H_9,H_{12}\} = B_{43}$	$\{H_3,H_5,H_{10}\} = B_{91}$	$\{H_5,H_6,H_{10}\} = B_{137}$	$\{H_2,H_{11},H_{12}\} =B_{184}$
$\{H_1,H_{10},H_{11}\} = B_{44}$	$\{H_3,H_5,H_{11}\} = B_{92}$	$\{H_5,H_6,H_{11}\} = B_{138}$	$\{H_4,H_{10},H_{12}\} =B_{185}$
$\{H_1,H_{10},H_{12}\} = B_{45}$	$\{H_3,H_5,H_{12}\} = B_{93}$	$\{H_5,H_6,H_{12}\} = B_{139}$	
$\{H_1,H_{11},H_{12}\} = B_{46}$	$\{H_3,H_6,H_7\} = B_{94}$	$\{H_5,H_7,H_8\} = B_{140}$	
$\{H_2,H_3,H_4\} = B_{47}$	$\{H_3,H_6,H_8\} = B_{95}$	$\{H_5,H_7,H_9\} = B_{141}$	

3.2. The Circuits and Broken Circuits of $\mathcal{A}(G_{25})$

We will give all the minimal dependent subsets of L_1 (set of atoms) with $\trianglelefteq$ in the following table:

$\{H_1,H_4,H_8\} = C_1$	$\{H_2,H_5,H_{11}\} = C_{13}$	$\{H_3,H_{10},H_{11}\} = C_{25}$
$\{H_1,H_4,H_{12}\} = C_2$	$\{H_2,H_6,H_9\} = C_{14}$	$\{H_3,H_{10},H_{12}\} = C_{26}$
$\{H_1,H_5,H_9\} = C_3$	$\{H_2,H_6,H_{12}\} = C_{15}$	$\{H_3,H_{11},H_{12}\} = C_{27}$
$\{H_1,H_5,H_{10}\} = C_4$	$\{H_2,H_7,H_{10}\} = C_{16}$	$\{H_4,H_5,H_6\} = C_{28}$
$\{H_1,H_6,H_7\} = C_5$	$\{H_2,H_8,H_{11}\} = C_{17}$	$\{H_4,H_7,H_{10}\} = C_{29}$
$\{H_1,H_6,H_{11}\} = C_6$	$\{H_2,H_9,H_{12}\} = C_{18}$	$\{H_5,H_8,H_{11}\} = C_{30}$
$\{H_1,H_7,H_{11}\} = C_7$	$\{H_3,H_4,H_5\} = C_{19}$	$\{H_5,H_9,H_{10}\} = C_{31}$
$\{H_1,H_8,H_{12}\} = C_8$	$\{H_3,H_4,H_6\} = C_{20}$	$\{H_6,H_7,H_{11}\} = C_{32}$
$\{H_1,H_9,H_{10}\} = C_9$	$\{H_3,H_5,H_6\} = C_{21}$	$\{H_6,H_9,H_{12}\} = C_{33}$
$\{H_2,H_4,H_7\} = C_{10}$	$\{H_3,H_7,H_8\} = C_{22}$	$\{H_7,H_8,H_9\} = C_{34}$
$\{H_2,H_4,H_{10}\} = C_{11}$	$\{H_3,H_7,H_9\} = C_{23}$	$\{H_{10},H_{11},H_{12}\} =C_{35}$
$\{H_2,H_5,H_8\} = C_{12}$	$\{H_3,H_8,H_9\} = C_{24}$	

All these subsets which we listed above are called circuits. Since $r(\vee B) < |B|$.

Thus the Broken circuits of $\mathcal{A}(G_{25})$ are:

$\{H_4,H_8\}= \bar{C}_1$	$\{H_5,H_{11}\} = \bar{C}_{13}$	$\{H_{10},H_{11}\} = \bar{C}_{25}$
$\{H_4,H_{12}\} = \bar{C}_2$	$\{H_6,H_9\} = \bar{C}_{14}$	$\{H_{10},H_{12}\} = \bar{C}_{26}$
$\{H_5,H_9\} = \bar{C}_3$	$\{H_6,H_{12}\} = \bar{C}_{15}$	$\{H_{11},H_{12}\} = \bar{C}_{27}$
$\{H_5,H_{10}\} = \bar{C}_4$	$\{H_7,H_{10}\} = \bar{C}_{16}$	$\{H_5,H_6\} = \bar{C}_{28}$
$\{H_6,H_7\}= \bar{C}_5$	$\{H_8,H_{11}\} = \bar{C}_{17}$	$\{H_7,H_{10}\} = \bar{C}_{29}$
$\{H_6,H_{11}\} = \bar{C}_6$	$\{H_9,H_{12}\} = \bar{C}_{18}$	$\{H_8,H_{11}\} = \bar{C}_{30}$
$\{H_7,H_{11}\} = \bar{C}_7$	$\{H_4,H_5\} = \bar{C}_{19}$	$\{H_9,H_{10}\} = \bar{C}_{31}$
$\{H_8,H_{12}\}= \bar{C}_8$	$\{H_4,H_6\} = \bar{C}_{20}$	$\{H_7,H_{11}\} = \bar{C}_{32}$
$\{H_9,H_{10}\} = \bar{C}_9$	$\{H_5,H_6\} = \bar{C}_{21}$	$\{H_9,H_{12}\} = \bar{C}_{33}$
$\{H_4,H_7\} = \bar{C}_{10}$	$\{H_7,H_8\} = \bar{C}_{22}$	$\{H_8,H_9\} = \bar{C}_{34}$
$\{H_4,H_{10}\} = \bar{C}_{11}$	$\{H_7,H_9\} = \bar{C}_{23}$	$\{H_{11},H_{12}\} =\bar{C}_{35}$
$\{H_5,H_8\} = \bar{C}_{12}$	$\{H_8,H_9\} = \bar{C}_{24}$	

3.3. No Broken Circuits Base Of $L(\mathcal{A}(G_{25}))$

We will give the *NBC* base of elements of $L(\mathcal{A}(G_{25}))$ of rank two

The elements of $L(\mathcal{A}(G_{25}))$ of rank two are:

1) $H_1 \cap H_4 \cap H_8 \cap H_{12} = a_1$	8) $H_3 \cap H_7 \cap H_8 \cap H_9 = a_8$	15) $H_5 \cap H_7 = a_{15}$
2) $H_1 \cap H_5 \cap H_9 \cap H_{10} = a_2$	9) $H_3 \cap H_{10} \cap H_{11} \cap H_1 = a_9$	16) $H_5 \cap H_{12} = a_{16}$
3) $H_1 \cap H_6 \cap H_7 \cap H_{11} = a_3$	10) $H_1 \cap H_2 = a_{10}$	17) $H_6 \cap H_8 = a_{17}$
4) $H_2 \cap H_4 \cap H_7 \cap H_{10} = a_4$	11) $H_1 \cap H_3 = a_{11}$	18) $H_6 \cap H_{10} = a_{18}$
5) $H_2 \cap H_5 \cap H_8 \cap H_{11} = a_5$	12) $H_2 \cap H_3 = a_{12}$	19) $H_7 \cap H_{12} = a_{19}$
6) $H_2 \cap H_6 \cap H_9 \cap H_{12} = a_6$	13) $H_4 \cap H_9 = a_{13}$	20) $H_8 \cap H_{10} = a_{20}$
7) $H_3 \cap H_4 \cap H_5 \cap H_6 = a_7$	14) $H_4 \cap H_{11} = a_{14}$	21) $H_9 \cap H_{11} = a_{21}$

Now we will give the *NBC* base of elements of $L(\mathcal{A}(G_{25}))$ of rank two as follows:

Elements of $L(\mathcal{A}(G_{25}))$ of rank two	The ***NBC*** bases	The cardinal number of ***NBC*** bases
a_1	$\{H_1,H_4\},\{H_1,H_8\},\{H_1,H_{12}\}$	3
a_2	$\{H_1,H_5\},\{H_1,H_9\},\{H_1,H_{10}\}$	3
a_3	$\{H_1,H_6\},\{H_1,H_7\},\{H_1,H_{11}\}$	3
a_4	$\{H_2,H_4\},\{H_2,H_7\},\{H_2,H_{10}\}$	3
a_5	$\{H_2,H_5\},\{H_2,H_8\},\{H_2,H_{11}\}$	3
a_6	$\{H_2,H_6\},\{H_2,H_9\},\{H_2,H_{12}\}$	3
a_7	$\{H_3,H_4\},\{H_3,H_5\},\{H_3,H_6\}$	3
a_8	$\{H_3,H_7\},\{H_3,H_8\},\{H_3,H_9\}$	3
a_9	$\{H_3,H_{10}\},\{H_3,H_{11}\},\{H_3,H_{12}\}$	3
a_{10}	$\{H_1,H_2\}$	1
a_{11}	$\{H_1,H_3\}$	1
a_{12}	$\{H_2,H_3\}$	1
a_{13}	$\{H_4,H_9\}$	1
a_{14}	$\{H_4,H_{11}\}$	1
a_{15}	$\{H_5,H_7\}$	1
a_{16}	$\{H_5,H_{12}\}$	1
a_{17}	$\{H_6,H_8\}$	1
a_{18}	$\{H_6,H_{10}\}$	1
a_{19}	$\{H_7,H_{12}\}$	1
a_{20}	$\{H_8,H_{10}\}$	1
a_{21}	$\{H_9,H_{11}\}$	1

***Definition* (3.1) [11]:** Let $\mathcal{A}$ be an arrangement.

1. A subset $\mathcal{B}$ of $\mathcal{A}$, is called ***sub-arrangement*** of $\mathcal{A}$.
2. For any subspace X $\in L_{\mathcal{A}}$ there are two smaller arrangements associated to it.
   (i) $\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\}$, is a sub-arrangement of $\mathcal{A}$, called ***localization arrangement.***
   (ii) $\mathcal{A}^X = \{X \cap H \mid H \in \mathcal{A} - \mathcal{A}_X \text{ and } X \cap H \neq \phi\}$, is the arrangement within the vector space X called ***restriction arrangement***.

Definition* (3.2) [2]:** Let $\mathcal{A}$ be an arrangement and let H $\in \mathcal{A}$, let $\mathcal{A}^{`} = \mathcal{A} - \{H\}$ and $\mathcal{A}^{H}$. We call $(\mathcal{A}, \mathcal{A}^{`}, \mathcal{A}^{``})$ ***a triple of arrangements and H the ***distinguished hyperplane***.

3.4. The Triple Arrangements of $\mathcal{A}(G_{25})$

Given the reflection arrangement

$$\mathcal{A}(G_{25}) = \{H_1,H_2,H_3,H_4,H_5,H_6,H_7,H_8,H_9,H_{10},H_{11},H_{12}\}.$$

Let H_1 be the distinguished hyperplane then

$$\mathcal{A}^{`}(G_{25}) = \{H_2,H_3,H_4,H_5,H_6,H_7,H_8,H_9,H_{10},H_{11},H_{12}\}.$$

By reveres inclusion as subspace of V we get $\mathcal{A}_{H1} = \{H_1\}$ and

$$\mathcal{A}^{H1} = \mathcal{A}^{``}(G_{25}) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$$

where:

$\alpha_1 = H_1 \cap H_2$**:** y = x = 0
$\alpha_2 = H_1 \cap H_3$**:** x = z = 0
$\alpha_3 = H_1 \cap H_4$**:** x = 0, y = -z
$\alpha_4 = H_1 \cap H_5$**:** x = 0, y = $-\omega z$
$\alpha_5 = H_1 \cap H_6$**:** x = 0, y = $-\omega^2 z$

Thus the triple arrangements of $\mathcal{A}(G_{25})$ are $(\mathcal{A}(G_{25}), \mathcal{A}^{`}(G_{25}), \mathcal{A}^{``}(G_{25}))$.

Definition* (3.3) [7]:** Let $\mathcal{A}$ be an arrangement and let $L = L(\mathcal{A})$. Define the ***Möbius function $\mu_{\mathcal{A}} = \mu : L \times L \longrightarrow Z$ as follows:

$\mu(X, X) = 1$ if X $\in L$,
$\sum_{X \leq Z \leq Y} \mu(X, Z) = 0$ if X, Y, Z $\in L$ and X<Y,
$\mu(X,Y) = 0$ otherwise.

Definition* (3.4) [12]:** Let $\mathcal{A}$ be an arrangement with lattice L and Möbius function. Let t be an indeterminate. Define the ***poincaré polynomial of $\mathcal{A}$ by

$$\boldsymbol{\pi(\mathcal{A}, t) = \sum_{X \in L} \mu(X)(-t)^{r(X)}}\ .$$

To compute the poincaré polynomial of $\mathcal{A}(G_{25})$, note that if $X \in L_{\mathcal{A}(G25)}$ s.t rk(X) = 1, then $\mu(X) = -1$, if rk(X) = 2, and X is in two planes then $\mu(X) = 1$, and if rk(X) = 2, s.t X in four planes then $\mu(X) = 3$. This allows calculation of $\mu(\{0\})$ which is equal to "-28".

Thus $\pi(\mathcal{A}(G25)) = 1 + 12t + 39t^2 + 28t^3$.

The Poincaré polynomial of $\mathcal{A}^{`}(G_{25})$ can be computed by the same way after removing the distinguished hyperplane H_1.

Thus $\pi(\mathcal{A}^{`}(G25)) = 1 + 11t + 34t^2 + 12t^3$.

Now since $\pi(\mathcal{A}, t) = \pi(\mathcal{A}^{`}, t) + t\pi(\mathcal{A}^{``}, t)$ by [4]

Then $\pi(\mathcal{A}^{``}(G25)) = 1 + 5t + 4t^2$.

***Definition* (3.5) [9]:** Define the characteristic polynomial of $\mathcal{A}$ by

$$\mathcal{X}(\mathcal{A}, t) = t^{\ell}\pi(\mathcal{A}, -t^{-1}) = \sum_{X \in L} \mu(X) t^{\dim(x)}.$$

By applying the above equation on $\mathcal{A}(G_{25})$ we get:

$$\chi(\mathcal{A}(G_{25}), t) = t^3 - 12t^2 + 39t - 28.$$

And the characteristic polynomial of $\mathcal{A}^{`}(G_{25})$ can be computed using the same equation by removing distinguished hyperplane H_1.

Thus

$$\chi(\mathcal{A}^{`}(G_{25})) = t^3 - 11t^2 + 34t - 24.$$

So by **[4]** we get:

$$\mathcal{X}(\mathcal{A}^{``}(G_{25}), t) = t^2 - 5t + 4.$$

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Source of support: Nil, Conflict of interest: None Declared