International Journal of Mathematical Archive-6(11), 2015, 179-184
IJMA Available online through www.ijma.info ISSN 2229 – 5046

ON INFRA TOPOLOGICAL SPACES

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(Received On: 27-10-15; Revised & Accepted On: 19-11-15)

ABSRACT

In this paper we introduced and investigate infra-topological spaces which deduced from topological spaces and studies the properties of subsets of infra topological spaces such as infra topological space, infra derived set, infra-interior set , infra-closure set, infra-exterior set and infra-boundary set.

Keywords: *infra - topological space, infra -derived set, infra- interior point, infra- closure, infra-exterior point and infra-boundary set.*

1. INTRODUCTION

In 1983, A.S.Mashhour *et al.* [1] introduced the supra topological space and studied $\boldsymbol{s}$-ontinous functions and $\boldsymbol{s}^*$-continuous functions.In this paper we introduced Infra -Topological Space (ITS) and analogue concepts associated with infra- topological space. Such as, infra- derived set (resp.infra-closure, infra-interior, infra-exterior and infra-boundary) of subset $\boldsymbol{A}$ of infra-space $\boldsymbol{X}$. we will be denoted by $\boldsymbol{ids}(\boldsymbol{A})$ (resp. $\boldsymbol{icp}(\boldsymbol{A}), \boldsymbol{iip}(\boldsymbol{A}), \boldsymbol{iep}(\boldsymbol{A})$ and $\boldsymbol{ibp}(\boldsymbol{A})$). Many results of topologic space remain valid in infra- topological space, Whereas some become invalid in infra-topological space.

2. INFRA -TOPOLOGICAL SPACES

Definition 2.1: Let X be any arbitrary set. An *Infra-topological space* on X is a collection τ_{iX} of subsets of X such that the following axioms are satisfying:
Ax-1 $\emptyset, X \in \tau_{iX}$,.
Ax-2 The intersection of the elements of any subcollection of τ_{iX} in X.
i.e, $If\ O_i \in \tau_{iX},\ 1 \le i \le n \rightarrow \bigcap O_i \in \tau_{iX}$.

Terminology, the ordered pair (X, τ_{iX}) is called ***I****nfra-Topological Space.* we simply say X is a*Infra -space*.

Definition 2.2: Let (X, τ_{iX}) be an (ITS) and $A \subset X$. A is called *infra -open set* (IOS) if $A \in \tau_{iX}$.

Definition 2.3: Let X be any arbitrary set and $\tau = \{\emptyset, X\}$, then (X, τ_{iX}) is called *indiscrete infra-topology space* or is called *trivial infra- topological space*

Definition 2.4: Let X be any countable arbitrary set and $\tau = P(X)$ the set of all subsets of X, then (X, τ) is called *discrete infra-topology space* or is called *maximal infra- topological space*.

Theorem 2.1: Let (X, τ) be a *topological -space* (TS), then (X, τ) is an *infra-topological space* (IITS)**.**

Proof: Suppose that (X, τ) is a topological space, then by axioms it is clear that (X, τ) is infra topological space. The converse of above theorem is not true.

Example 2.1: If $X = \{a, b, c\}$ *and* $\tau_{iX} = \{\emptyset, X, \{a\}, \{b\}\}$, then (X, τ_{iX}) *is infra- topological space, but not topological space.*

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Theorem 2.2: Let (X, τ_{iX}) be *infra-topological space*. Then:

1. $\emptyset, X$ are *infra -open set.*
2. Any arbitrary intersections of infra- open sets are infra- open sets.
3. Finite union of infra-*open* sets may not be infra-*open* sets.

Proof

1. It is clear that $\emptyset, X$ are *infra open set* by Ax-1 and definition 2.1.
2. Let $C_{i\in J} \in \tau_{iX}$, by Ax-2 and definition 2.1.$\bigcap O_i \in \tau_{iX}$are *infra open set*.
3. By counter example 2.1:$\{a\},\{b\} \in \tau_{iX}$, but $\{a\} \cup \{b\} = \{a, b\} \notin \tau_{iX}$.

Theorem 2.3: let (X, τ_{iX}) and (X, τ_{iX}^*) be two *infra topological Spaces* on set X. Then the intersection τ_{iX} and τ_{iX}^* is an*infra topological space*.

Proof: let (X, τ_{iX}) and (X, τ_{iX}^*) be two infra topological Spaces on set X.

By *Ax*-1$\emptyset, X \in \tau_{iX}$ and $\emptyset, X \in \tau_{iX}^*$, so $\emptyset, X \in \tau_{iX} \cap \tau_{iX}^*$.Suppose that $O_i \in \tau_{iX} \cap \tau_{iX}^*$.

$1 \le i \le n$, implies that $O_i \in \tau_{iX}$ and $O_i \in \tau_{iX}^*$ Consequently, $\bigcap_{i=1}^n O_i \in \tau_{iX}$ and $\bigcap_{i=1}^n O_i \in \tau_{iX}^*$ and hence $\bigcap_{i=1}^n O_i \in \tau_{iX} \cap \tau_{iX}^*$.

Theorem 2.4: Let (X, τ_{iX}) and (X, τ_{iX}^*) be two *infra topological Spaces* on set X. then the union τ_{iX} and τ_{iX}^* is an *infra topological space.*

Proof: Let (X, τ_{iX}) and (X, τ_{iX}^*) be tow *infra topological infra spaces* on set X. By *Ax*-1,$\emptyset, X \in \tau_{iX}$ and $\emptyset, X \in \tau_{iX}^*$, so $\emptyset, X \in \tau_{iX} \cup \tau_{iX}^*$.

Suppose that $O_i \in \tau_{iX} \cup \tau_{iX}^*$; $1 \le i \le n$, implies that $O_i \in \tau_{iX}$ or $O_i \in \tau_{iX}^*$ for some $i, 1 \le i \le n$. So $\bigcap_{i=1}^n O_i \in \tau_{iX}$ or $\bigcap_{i=1}^n O_i \in \tau_{iX}^*$ and hence $\bigcap_{i=1}^n O_i \in \tau_{iX} \cup \tau_{iX}^*$.

Remark: The union of infra- topological spaces may not be infra-topological space, in general, by the following example.

Example 2.2: Let X be a set $X = \{a, b, c, d\}$, $\tau_{iX} = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}\}$ and $\tau_{iX}{}^* = \{\emptyset, X, \{c\}, \{d\}, \{b, c\}, \{c, d\}\}$. Now $\tau_{iX} \cup \tau_{iX}{}^* = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{c, d\}\}$. is not infra-topological space.

3. PROPERTIES OF SUBSETS ON INFRA TOPOLOGICAL SPACES

Definition 3.1: Let (X, τ_{iX}) be an (ITS) and $A \subset X$. A point $x \in X$is called *Infra-Cluster Point* (ICP) of A**,** if for all *Infra-open set* Ocontaining$x$**,** then $A \cap (O \backslash \{x\}) \neq \emptyset$**.**

Definition 3.2: Let (X, τ_{iX}) be an (ITS) and $A \subset X$. The set of all *Infra-Cluster Point* (ICP) of A is called the *Infra-Derived Set* (IDS) of A and is denoted by $ids(A)$**.**

Definition 3.3: Let (X, τ_{iX}) be *infra-topological space*. A subset $C \subset X$ is called *infra-closed set* in X if $X - C$ is *infra-open set* in X. That is C is *infra-closed set* (ICS) iff $X - C \in \tau_{iX}$.

Theorem 3.1: Let (X, τ_{iX}) be *infra-topological space*. Then:

i. $\emptyset, X \in \tau_{iX}$ are *infra-closed set.*
ii. Any arbitrary finite intersections of infra-closed sets is an infra-closed sets.

Proof:

i. Since $X - \emptyset = X \in \tau_{iX}$ *and* $X - X = \emptyset \in \tau_{iX}$ are infra-closed sets.
ii. Let $\{C_i : i \in I\}$be an arbitrary family of infra closed sets such that $C_i \in \tau_{iX}$ for all $i \in I$.Now, $X - C_i \in \tau_{iX}$ is infra-open set for all $i \in I$.
But $X - C_i = C_i{}^c \in \tau_{IX}$, then $\cap C_i{}^c = \cap (X - C_i) = X - \cap C_i \in \tau_{iX}, \forall i \in I$. Hence $\cap C_i \in \tau_{iX}, \forall i \in I$ is infra-closed set.

Remark: Finite union of infra-closed sets may not be infra-closed sets, in general.

Definition 3.4: Let (X, τ_{iX}) be an (ITS) and $A \subset X$. *The Infra Closure Point (*ICP) of A is a set denoted by $icp\ (A)$ and given by: $icp\ (A) = \cap \{C_i : A \subset C_i,\ X - C_i \in \tau_{iX}\}$.That is, $icp\ (A)$ is the intersection of all infra closed set contained the set A**.**

Remark: Since $icp(A)$ is the intersection of all infra closed sets containing in A, then $A \subset icp(A)$ and $icp(A)$ is the smallest infra closed sets .

Definition 3.5: Let (X, τ_{iX}) be an (ITS) and $A \subset X$. *The Infra-Interior Points (*IIP) of A is a set denoted by $iip\ (A)$ and given by: $iip\ (A) = \cup \{O_i : O_i \subset A,\ O_i \in \tau_{iX}\}$.That is, $iip(A)$ is the union of all infra-open set contained in the set A.

Remark: Since $iip\ (A)$ is the union of all infra-open sets contained in A, then $iip(A) \subset A$ and $icp\ (A)$ is the smallest infra-open sets. Also if O is infra-open setcontained in A**,** then $O \subset iip\ (A)$.

Definition 3.6 Let (X, τ_{iX}) be an (ITS) and $A \subset X$. *The Infra-Exterior Points (*IEP) of A is a set denoted by $iep\ (A)$ and given by: $iep\ (A) = iip(A^c)$.That is, Set of all infra-interior point of complement of A .

Definition 3.7: Let (X, τ_{iX}) be an (ITS) and $A \subset X$. *The Infra-Boundary Points (*IBP) of A is a set denoted by $ibp\ (A)$ and given by:$ibp\ (A) = X \backslash iip(A) \cup iep(A)$

Theorem 3.2: Let (X, τ_{iX}) be an (ITS) and $A, B \subset X$.The *Infra-Derived Set Axioms* (IDSA) satisfies the followings:

(IDSA $)_1$: $ids(\emptyset) = \emptyset$.
(IDSA $)_2$: If $A \subset B$, then $ids(A) \subset ids(B)$.
(IDSA $)_3$: if $x \in ids(A)$, then $x \in ids(A \backslash \{x\})$.
(IDSA $)_4$: $ids(A \cap B) \subset ids(A) \cap ids(B)$.
(IDSA $)_5$ $ids(A \cup B) = ids(A) \cup ids(B)$.

Proof:
(IDSA $)_1$: Suppose that $ids(\emptyset) \neq \emptyset \to \exists\ x \in ids(A) \ni \emptyset \cap (O \backslash \{x\}) \neq \emptyset$
$\to\ x \in \emptyset\ and\ x \notin \emptyset$.That is contradiction.
$\to\ ids(\emptyset) = \emptyset$.

(IDSA$)_2$: Suppose that $A \subset B$. Let $x \in ids(A) \to \forall\ O \ni x, A \cap (O \backslash \{x\}) \neq \emptyset$.
$\to \forall\ O \ni x, B \cap (O \backslash \{x\}) \neq \emptyset$.
$\to x \in ids(B)$
$\to ids(A) \subset ids(B)$.

(IDSA$)_3$: Assume that $x \in ids(A) \to \forall\ O \ni x, A \cap (O \backslash \{x\}) \neq \emptyset$.
$\to \forall\ O \ni x, A \cap (O \cap \{x\}^c) \neq \emptyset$.
$\to \forall\ O \ni x, A \cap (O \cap \{x\}^c \cap \{x\}^c) \neq \emptyset$.
$\to \forall\ O \ni x, A \cap (\{x\}^c \cap O \cap \{x\}^c) \neq \emptyset$.
$\to \forall\ O \ni x, (A \cap \{x\}^c) \cap (O \cap \{x\}^c) \neq \emptyset$.
$\to \forall\ O \ni x, (A \backslash \{x\}) \cap (O \backslash \{x\}) \neq \emptyset$.
$\to x \in ids(A \backslash \{x\})$.

(IDSA $)_4$: Since $A \cap B \subset A \ \wedge\ A \cap B \subset B$
$\to ids(A \cap B) \subset ids(A) \wedge ids(A \cap B) \subset ids(B))$.
$\to ids(A \cap B) \subset ids(A) \cap ids(B))$.

It can be easily shown by example the equality is not hold like topological space.
(IDSA $)_5$: Since $A \subset A \cup B$ and $B \subset A \cup B$, then $ids(A) \subset ids(A \cup B)$ and $ids(B) \subset ids(A \cup B)$, hence $ids(A) \cup ids(B) \subset ids(A \cup B)$. Conversely,

Suppose that $x \in ids(A \cup B) \to \forall\ O \ni x, (A \cup B) \cap (O \backslash \{x\}) \neq \emptyset$.
$\to \forall\ O \ni x, A \cap (O \backslash \{x\}) \neq \emptyset \cup B \cap (O \backslash \{x\}) \neq \emptyset$.

$\to x \in ids(A) \cup ids(B)$.Hence,
$ids(A \cup B) = ids(A) \cup ids(B)$.

Theorem 3.3: Let (X, τ_{iX}) be an (ITS) and $A, B \subset X$.The *Infra Closure Point Axioms* **(**ICPA) satisfying the following conditions:

(ICPA$)_1$: A is infra-closed iff$A = icp(A)$.
(ICPA$)_2$: $icp(\emptyset) = \emptyset\ and\ icp(X) = X$.
(ICPA$)_3$: $icp\big(icp(A)\big) = icp(A)$.
(ICPA$)_4$: If $A \subset B$, then $icp(A) \subset icp(B)$.
(ICPA$)_5$: $icp(A \cap B) \subset icp(A) \cap icp(B)$.

Proof:
$(\text{ICPA})_1$: Suppose that A is infra-closed set. Since $A \subset A$ and $A \cap A = A \to icp(A) \subset A$, Also $A \subset icp(A) \to A = icp(A)$. Conversely, Let $A = icp(A)$, obviously,
$icp(A)$is the smallest infra-closed set. Hence A is infra-closed set.

$(\text{ICPA})_2$: Since $X\ and\ \emptyset$ are infra- closed sets, so by$(\text{ICPA})_1 icp(\emptyset) = \emptyset\ and\ icp(X) = X$. $(\text{ICPA})_3$: Since$icp(A)$ is the intersection of all infra-closed sets are closed sets, then $icp\big(icp(A)\big) = icp(A)$.

$(\text{ICPA})_4$: Consider $A \subset B$. Since $A \subset icp(A)$ and $B \subset icp(B)$, so $icp(A) \subset icp(B)$.
$(\text{ICPA})_5$: Since $(A \cap B \subset A \wedge A \cap B \subset B)$, then $icp(A \cap B) \subset icp(A)$and
$icp(A \cap B) \subset icp(B) \to icp(A \cap B) \subset icp(A) \cap icp(B)$.

Remark: In $(\text{ICPA})_5$ the equality does not work like topological space.

Theorem 3.4: Let (X, τ_{iX}) be an (ITS) and $A, B \subset X$.The *Infra-Interior Points Axioms* (IIPA) *given by:*

$(\text{IIPA})_1$: A is infra-open set iff$A = iip(A)$.
$(\text{IIPA})_2$: $iip(X) = X$and$iip(\emptyset) = \emptyset$
$(\text{IIPA})_3$: $iip\big(iip(A)\big) = iip(A)$.
$(\text{IIPA})_4$: If $A \subset B$, then $iip(A) \subset iip(B)$.
$(\text{IIPA})_5$: $iip(A \cap B) = iip(A) \cap iip(B)$.

Proof:
$(\text{IIPA})_1$ **:** Suppose that A is infra-open set. Since $A \subset A$, then A is infra-open set containing itself, so $A \subset iip\ (A)$ and $iip\ (A) \subset A$, that implies $A = iip(A)$. Conversely, Let $A = iip(A)$, suppose that $A = iip(A)$. Since $iip(A)$ is infra-open set, then A is infra- open set.

$(\text{IIPA})_2$**:** Since $X, \emptyset$ are infra-open sets, by $(\text{IIPA})_1$, we have $iip(X) = X$ and $iip(\emptyset) = \emptyset$.

$(\text{IIPA})_3$: Since $iip(A)$ is infra-open set. so by $(\text{IIPA})_1 iip\big(iip(A)\big) = iip(A)$.

$(\text{IIPA})_4$: Suppose that If $A \subset B$. Let$O_i \in iip(A) \to O_i \subset A \to O_i \subset B \to O_i \in iip(B)$.

Therefore $iip(A) \subset iip(B)$.

$(\text{IIPA})_5$: Let $O_i \in iip(A) \cap iip(B) \leftrightarrow O_i \in iip(A) \wedge O_i \in iip(B)$.
$\leftrightarrow \cup\, O_i, O_i \subset A, \forall\ i\ \wedge \cup\, O_i, O_i \subset B, \forall i$.
$\leftrightarrow \cup\, O_i, O_i \subset A \cap B, \forall i$.
$\leftrightarrow O_i \in iip\ (A \cap B), \forall i$.

Theorem 3.5: Let $(X\tau_{iX})$ be an (ITS) and $A, B \subset X$.The *Infra -Exterior points Axioms* (IEPA) *given by:*

$(\text{IEPA})_1$: $iep(X) = \emptyset$and$iep(\emptyset) = X$.
$(\text{IEPA})_2$: $iep(A) \subset A^C$.
$(\text{IEPA})_3$: $iep(A \cup B) = iep(A) \cap iep(B)$.
$(\text{IEPA})_4$: If $A \subset B$, then $iep(B) \subset iep(A)$.
$(\text{IEPA})_5$ $iep(A \cap B) \subset iep(A) \cup iep(B)$.

Proof:
$(\text{IEPA})_1$: $iep(X) = iip(X^c) = iip(\emptyset) = \emptyset$ and $iep(\emptyset) = iip(\emptyset^c) = iip(X) = X$.

$(\text{IEPA})_2$: $iep(A) = iip(A^c) \subset A^C$.

$(\text{IEPA})_3$: $iep(A \cup B) = iip(A \cup B)^c = iip(A^c \cap B^c) = iip(A^c) \cap iip(B^c)$
$= iep(A) \cap iep(B)$.

$(\text{IEPA})_4$: let $A \subset B \to B^c \subset A^c \longrightarrow iip(B^c) \subset iip(A^c) \to iep(B) \subset iep(A)$.

$(\text{IEPA})_5$: $iep(A \cap B) = iip(A \cap B)^c = iip(A^c \cup B^c) \subset iip(A^c) \cup iip(B^c) = iep(A) \cup iep(B)$.

Theorem 3.6: Let (X, τ_{iX}) be an (ITS) and $A \subset X$.The *Infra-Boundary Points Axioms* (IBPA) *given by:*

$$(\text{IBPA})_1 \ : \quad ibp(X) = ibp(\emptyset) = \emptyset.$$
$$(\text{IBPA})_2 \ : \quad ibp(A \cap B) = ibp(A) \cup ibp(B).$$

Proof:
$(\text{IBPA})_1$: $ibp(X) = X \backslash iip(X) \cup iep(X) = X \backslash X \cup \emptyset = X \backslash X = \emptyset.$

$ibp(\emptyset) = X \backslash iip(\emptyset) \cup iep(\emptyset) = X \backslash \emptyset \cup X = X \backslash X = \emptyset$

$$\begin{aligned}(\text{IBPA})_2: ibp(A \cap B) &= X \backslash iip(A \cap B) \cup iep(A \cap B) \\ &= X \backslash iip(A) \cap iip(B) \cup iep(A \cap B) \\ &= X \backslash iip(A) \cup X \backslash iip(B) \cup iep(A \cap B) \\ &= X \backslash iip(A) \cup X \backslash iip(B) \cup iep(A) \cup iep(B)) \\ &= ibp(A) \cup ibp(B).\end{aligned}$$

The following theorem illustrate the relations between $ids(A), icp(A), iip(A), iep(A)$and $ibp(A)$.

Theorem 3.7: Let (X, τ_{iX}) be an (ITS) and $A \subset X$.then:

1. $A \subset icp(A) \to ids(A) \subset ids(icp(A))$.
2. $iip(A) \subset A \to ids(iip(A)) \subset ids(A)$.
3. If A is infra-closed , then $ids(A) \subset A$.
4. $icp(A) = A \cup ids(A)$.
5. $ibp(A) = icp(A) \backslash iip(A)$.
6. $icp(A) = ibp(A) \cup iip(A)$.
7. $ibp(A) \subset icp(A)$.
8. $iip(A) \cap ibp(A) = \emptyset$.

Proof:

(1) Let $A \subset icp(A)$. By(IDSA $)_2 ids(A) \subset ids(icp(A))$.

(2) Let $iip(A) \subset A$. By(IDSA $)_2 ids(iip(A)) \subset ids(A)$.

(3) Let$A$ be a infra closed set and $x \in ids(A)$, then $\forall\, O \ni x, A \cap (O - (x)) \neq \emptyset$ Hence $x \in A$ and $ids(A) \subset A$.

(4) Since $A \subset icp(A)$ and $ids(A) \subset ids(icp(A)) \subset icp(A)$. we have $A \cup ids(A) \subset icp(A)$. Another direction, To show that $icp(A) \subset A \cup ids(A)$.

Let $x \in icp(A)$, but $A \subset icp(A)$, then $x \in A$ or $x \notin A$.

Probability 1) If$x \in A$, then $x \in A \cup ids(A)$.

Probability 2) if $x \notin A$,Let $x \notin ids(A) \to \exists O \ni x, A \cap (O \backslash \{x\} = \emptyset$, but $x \notin A$, that is contradiction, therefore $x \in ids(A)$ and $x \in A \cup ids(A)$.

So $icp(A) = A \cup ids(A)$.

(5) By def^n:$ibp(A) = X \backslash iip(A) \cup iep(A)$
$= X \backslash iip(A) \cap X \backslash iep(A)$
$= X \backslash iip(A) \cap icp(A)$
Since $iip(A) \subset icp(A) \subset X \to icp(A) \cap (X \backslash iip(A) = icp(A) \backslash iip(A)$. Then we have
$ibp(A) = icp(A) \backslash iip(A)$.

(6) By (1) $ibp(A) = icp(A) \backslash iip(A)$
$\to ibp(A) \cup iip(A) = icp(A) \backslash iip(A) \cup iip(A) = icp(A)$.

(7) By (2) it is clear that $ibp(A) \subset icp(A)$.

(8) $iip(A) \cap ibp(A) = iip(A) \cap icp(A) \backslash iip(A) = \emptyset$.

Theorem 3.8: Let X be any finite set and $A \subset X$ and order ($o(A) = 1$). The collection $\tau_{iX} = \{\emptyset, X\} \cup \{A \subset X \text{ such that } o\, A\text{=1}$ is *infra topological space*.

Proof: Since $\emptyset, X \in \{\emptyset, X\}$, so $\emptyset, X \in \tau_{iX}$ *and* Ax-1 is hold. Now, Assume that $A_i \in \tau_{iX}$, $1 \leq i \leq n$. And $o(A) = 1$) Then $A_i \cap \emptyset = \emptyset, \forall i$ *and* $A_i \cap X = A_i, \forall i$, so that Ax-2 is hold .

The pair (X, τ_{iX}) is called the Particular singleton set of infra-topological space on X.

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Source of support: Nil, Conflict of interest: None Declared