ECCENTRICITY PROPERTIES OF $BG_4(G)$

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(Received On: 13-11-15; Revised & Accepted On: 30-11-15)

ABSTRACT

Let $G$ be a simple $(p, q)$ graph with vertex set $V(G)$ and edge set $E(G)$. $BG_{G,NINC,K_4}(G)$ is a graph with vertex set $V(G) \cup E(G)$ and two vertices are adjacent if and only if they correspond to two non adjacent vertices of $G$, a vertex and an edge not incident to it in $G$. For simplicity, denote this graph by $BG_4(G)$, Boolean graph fourth kind of $G$. In this paper, eccentricity properties of $BG_4(G)$ and its complement $\overline{BG_4(G)}$ are studied.

Keywords: Eccentricity, Boolean graph $BG_4(G)$.

2010 Mathematics Subject Classification: 05C69, 05C12.

1. INTRODUCTION

Let $G$ be a finite, simple, undirected $(p, q)$ graph with vertex set $V(G)$ and edge set $E(G)$. For a graph theoretic terminology refer to Harary [4], Buckley and Harary [3].

Let $G$ be a connected graph and $u$ be a vertex of $G$. The eccentricity $e(v)$ of $v$ is the distance to a vertex farthest from $v$. Thus, $e(v) = \max \{d(u, v) : u \in V\}$. The radius $r(G)$ is the minimum eccentricity of the vertices, whereas the diameter $diam(G)$ is the maximum eccentricity. For any connected graph $G$, $r(G) \leq diam(G) \leq 2r(G)$. $v$ is a central vertex if $e(v) = r(G)$. The center $C(G)$ is the set of all central vertices. The central subgraph $< C(G) >$ of a graph $G$ is the subgraph induced by the center. $v$ is a peripheral vertex if $e(v) = diam(G)$. The periphery $P(G)$ is the set of all such vertices. For a vertex $v$, each vertex at distance $e(v)$ from $v$ is an eccentric node of $v$.

A subgraph of $G$ is a graph having all of its vertices and edges in $G$. It is a spanning subgraph if it contains all the vertices of $G$. If $H$ is a subgraph of $G$, then $G$ is a super graph of $H$. For any set $S$ of vertices in $G$, the induced subgraph $< S >$ is the maximal subgraph with vertex set $S$.

A graph $G$ is complete if every pair of its vertices is adjacent. $K_n$ denotes the complete graph on $n$ vertices.

The complement $\overline{G}$ of a graph $G$ is the graph with vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$. A self-complementary graph is isomorphic to its complement.

A graph $G$ is connected if there is a path joining each pair of vertices. A component of a graph is a maximal connected subgraph. If a graph has only one component, then it is connected. Otherwise it is disconnected. The diameter $diam(G)$ of a connected graph $G$ is the length of any, longest geodesic (diametral path).

The Line graph $L(G)$ of a graph $G$ is the graph whose vertices correspond to the edges of $G$ and two vertices of $L(G)$ are adjacent if and only if the corresponding edges of $G$ are adjacent.

The Total graph $T(G)$ of a graph $G$ is the graph whose vertices correspond to the set of vertices and edges of $G$ and two vertices of $T(G)$ are adjacent if and only if the corresponding elements of $G$ are adjacent or incident.

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Let $G$ be a simple $(p, q)$ graph with vertex set $V(G)$ and edge set $E(G)$. $B_{G,NINC}(G)$ is a graph with vertex set $V(G) \cup E(G)$ and two vertices are adjacent if and only if they correspond to two non-adjacent vertices of $G$, a vertex and an edge not incident to it in $G$. For simplicity, denote this graph by $BG_4(G)$, Boolean graph fourth kind of $G$. In [1] properties of $BG_4(G)$ is studied. The vertices of $BG_4(G)$, which are in $V(G)$ are called point vertices and vertices in $E(G)$ are called line vertices. $V(BG_4(G)) = V(G) \cup E(G)$, $E(BG_4(G)) = E(T(G)) - E(L(G))$, where $T(G)$ is the total graph of $G$ and $L(G)$ is the line graph of $G$.

In [2, 5, 6, 7] Janakiraman, Bhanumathi and Muthammai have defined and studied the properties of Boolean graphs. Motivated by this, here we study the eccentric properties of Boolean graph $BG_4(G)$.

2.1 Eccentricity Properties of $BG_4(G)$

In this section, radius and diameter of $BG_4(G)$ are found out. Throughout this section, if $v \in V(G)$ and $e \in E(G)$, the corresponding vertices of $BG_4(G)$ are denoted by $v'$ and $e'$.

Observation 2.1.1: If $G$ is totally disconnected then the eccentricity of every vertex in $BG_4(G)$ is one and $BG_4(G)$ is $K_p$.

Proposition 2.1.1: If $BG_4(G)$ is connected, eccentricity of a point vertex is 1, 2 or 3.

Proof: Consider a point $v$ in $V(G)$. If $G$ has an isolated vertex then the eccentricity of that vertex in $BG_4(G)$ is one. Assume $G$ has no isolated vertex.

To find $d(u', v')$ in $BG_4(G)$ where $u, v \in V(G)$. $G$ has no isolated vertex. Take $u, v \in V(G)$. If $u$ and $v$ are non adjacent in $G$ then $d(u', v') = 1$ in $BG_4(G)$. Suppose $u$ and $v$ are adjacent in $G$. If $u$ and $v$ have a common non incident edge or non adjacent vertex then we have a shortest path $u'w'v'$ or $u'e'v'$. Hence $d(u', v') = 2$.

Suppose $u$ and $v$ does not have a common non incident edge or vertex in $G$, but $u$ has a non adjacent vertex $w$ and $v$ has a non incident edge $e$ and vice versa. Then we have a shortest path $u'w'e'v'$. since $w$ is non incident with $e$ in $G$. Hence, $d(u', v') = 3$.

Suppose a vertex is adjacent to other vertices and incident with all the edges of $G$, then that vertex is isolated in $BG_4(G)$.

To find $d(v', e')$ in $BG_4(G)$ where $e \in E(G)$ and $v \in V(G)$.

If $e$ is not incident with $v$ in $G$ then $d(v', e') = 1$ in $BG_4(G)$.

Suppose $e$ is incident with $v$ in $G$. Let $e = vv_1 \in E(G)$. In $BG_4(G)$, $e'$ is not incident to $v'$. If there exists another vertex $v_2$, which is not adjacent to $v$ in $G$, then $v'v_2'e'$ is a shortest path in $BG_4(G)$ and hence $d(v', e') = 2$ in $BG_4(G)$.

If there exists no such vertex, then $deg_Gv = p-1$ and if there exist non incident edge $e_1$ in $G$, then $\forall e_1 \neq v', e'$ (where $v_2$ is not incident to $e_1$ and $e$ in $G$) is a shortest path and hence $d(v', e') = 3$ in $BG_4(G)$.

Suppose this is also not possible. That is, if $deg_Gv = p-1$ and if there does not exist non incident edge $e_1$ in $G$, then that vertex is isolated vertex in $BG_4(G)$ that means $BG_4(G)$ is disconnected.

Hence, if $BG_4(G)$ is connected, then eccentricity of point vertices is 1, 2 or 3.

Proposition 2.1.2: If $BG_4(G)$ is connected, then the eccentricity of a line vertex is 2, 3 or 4.

Proof: From the previous theorem $d(e', v') = 1, 2$ or 3 in $BG_4(G)$.

Now to find $d(e_1', e_2')$ for $e_1, e_2 \in E(G)$.

Case-(i): $e_1$ and $e_2$ are non adjacent

If there exist a vertex $v$ which is not incident with both $e_1$ and $e_2$ then $e_1'v'e_2'$ is a shortest path and hence $d(e_1', e_2') = 2$ in $BG_4(G)$. If there exists no vertex $v$, not incident with both $e_1$ and $e_2$ and there is no edge adjacent to both $e_1$ and $e_2$.

Consider the edges $e_1 = u_1v_1$ and $e_2 = u_2v_2$ in $G$, $e_1'v_2'u_1'e_2'$ is a shortest path in $BG_4(G)$ then $d(e_1', e_2') = 3$ in $BG_4(G)$.
Suppose $u_1$ and $v_1$ are adjacent to all other vertices in $G$ and there are only four vertices in $G$, then $e_1'w_1'e_2'w_2'e_3'$ is a shortest path in $BG_4(G)$. Hence, $d(e_1', e_2') = 4$ in $BG_4(G)$.

Case-(ii): $e_1$ and $e_2$ are adjacent

Let $e_1 = u_1v_1$ and $e_2 = u_2v_2$ in $G$. $e_1'$ is adjacent to $v_2'$ and $e_2'$ is adjacent to $u_1'$ and also $u_1'$ and $v_2'$ also adjacent. Hence, $e_1'v_2'u_1'e_2'$ is a shortest path in $BG_4(G)$. Hence, $d(e_1', e_2') = 3$.

Hence, distance from $e'$ to any other vertex is 1, 2, 3 or 4 in $BG_4(G)$. This implies that the eccentricity of a line vertex is 2, 3 or 4.

Theorem 2.1.1: Let $G$ be a $(p, q)$ graph. Eccentricity of a point vertex is one if and only if $G$ has an isolated vertex.

Proof: Let $u$ be an isolated vertex in $G$. Then the vertex $u'$ is adjacent to all other line vertices and point vertices of $BG_4(G)$. Hence, $\deg u' = p+q-1$ and eccentricity of a vertex $u'$ is one in $BG_4(G)$.

On the other hand, assume that eccentricity of a point vertex $u$ is one in $BG_4(G)$, This implies, $\deg u = p+q-1$. Therefore, $u$ is not adjacent to any vertex in $G$. Hence, $u$ is an isolated vertex in $G$.

Note 2.1: If $G$ has an isolated vertex then $BG_4(G)$ is connected and radius of $BG_4(G)$ is one.

Theorem 2.1.2: Let $G$ and $BG_4(G)$ be connected. Eccentricity of a vertex is one in $G$ and has at least one non incident edge in $G$ if and only if eccentricity of that vertex in $BG_4(G)$ is three.

Proof: Let $v \in V(G)$ and $e \in E(G)$ which is non incident with $v$ in $G$.

Assume $e(v) = 1$ in $G$. Then the vertex $v$ is adjacent to all vertices of $G$. Hence $v'$ is not adjacent to any point vertex in $BG_4(G)$ and $v'$ is adjacent to $e'$ in $BG_4(G)$.

By proposition 2.1.1 we have $d(v', u') = 3$ where $u$ is adjacent to $v$ and incident with $e$ in $G$. Hence, $e(v') = 3$.

On the other hand, assume that eccentricity of a point vertex in $BG_4(G)$ is three. By proposition 2.1.1, we have $\deg_{BG_4(G)} v = p-1$, and eccentricity of $v$ is one in $G$. Suppose all the edges of $G$ are incident with $v$, then $v$ will be isolated in $BG_4(G)$.

So $G$ has at least one edge that is not incident with $v$. Hence, the theorem is proved.

Theorem 2.1.3: Let $G$ be a graph. If radius of $G$ is greater than one then eccentricity of a point vertex in $BG_4(G)$ is two.

Proof: Let $G$ be a connected graph. Assume $u \in V(G)$, $v$ is not adjacent to $u$ in $G$. This implies that, in $BG_4(G)$, $u'$ and $v'$ are adjacent.

Suppose $u$ is adjacent to $v$ in $G$ and $w \in G$ is not adjacent to both $u$ and $v$ then $v'w'u'$ is a shortest path in $BG_4(G)$.

Hence, $d(u', v') \leq 2$ and $d(v', e') \leq 2$ since $e(v) \neq 1$ in $G$. Hence, eccentricity of a point vertex is two in $BG_4(G)$.

Theorem 2.1.4: Eccentricity of a line vertex is 2 in $BG_4(G)$ if and only if $G$ has more than four vertices and $r(G) \neq 1$.

Proof: Let $G$ be a graph with at least five vertices and $r(G) \neq 1$. In $G$, any two edges $e_1$ and $e_2$ have a common non incident vertex. Hence $e_1'v'e_2'$ is a shortest path from $e_1'$ to $e_2'$.

Hence $d(e_1', e_2') = 2$ in $BG_4(G)$.

Every line vertex $e'$ corresponding to $e = uv$ is adjacent to $p-2$ point vertices in $BG_4(G)$ and since $r(G) \neq 1$, every vertex $u$ or $v$ in $G$ has at least one non adjacent vertex $w$ in $G$. Then there exist shortest path $e'w'u'$ or $e'w'v'$ in $BG_4(G)$.

Therefore, $d(e', v') = 2$. Hence, eccentricity of a line vertex is 2.

On the other hand, assume that in $BG_4(G)$ the eccentricity of a line vertex is 2. Let $e = uv \in E(G)$, In $BG_4(G)$, $e(e') = 2$.

This implies that distance between $e'$ and other line vertices are exactly 2. Hence, for any two edges $e_1$ and $e_2$ in $G$, there is a common non incident vertex. This implies $p \geq 5$. 

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Also, \( e(e') = 2 \) implies \( d(u', e') = 2 \) and \( d(v', e') = 2 \) in \( B_{4}(G) \), which implies there exist \( w \in V(G) \) such that \( w \) is not adjacent to both \( u \) and \( v \) in \( G \). (That is, \( dw'e', v'w'v' \) are paths in \( B_{4}(G) \)). Thus \( r(G) > 1 \). Hence, the theorem is proved.

**Theorem 2.1.5:** \( G = K_{4} \) if and only if eccentricity of each line vertex is four in \( B_{4}(G) \).

**Proof:** Assume that \( G = K_{4} \). Let \( V(G) = \{v_{1}, v_{2}, v_{3}, v_{4}\} \) and \( E(G) = \{e_{1}, e_{2}, e_{3}, e_{4}, e_{5}, e_{6}\} \). Let \( e_{1} = v_{1}v_{2}, e_{2} = v_{2}v_{3} \). A line vertex \( e'_{1} \) is adjacent to \( v'_{3} \) and \( v'_{4} \) and we have a shortest path \( e'_{1}v'_{4}e'_{2} \) in \( B_{4}(G) \). In \( B_{4}(G) \), \( d(e'_{1}, e'_{2}) = 2 \) for adjacent edges in \( G \).

For non adjacent edges we have a shortest path of distance 4. Hence, eccentricity of a line vertex is 4.

Conversely, assume eccentricity of every line vertex is four. Then \( d(e'_{1}, u'_{1}) \leq 4 \) in \( B_{4}(G) \) and \( d(e'_{1}, e'_{2}) \leq 4 \) in \( B_{4}(G) \) for \( u \in V(G) \) and \( e_{1}, e_{2} \in E(G), |V(G)| \leq 4 \) since eccentricity of a line vertex is four (by theorem 2.1.4). \( \overline{G} \) has anyone edge, then eccentricity of a line vertex is three, which is a contradiction to the assumption. Hence, \( G = K_{2}, K_{3} \) or \( K_{4} \).

If \( G = K_{2} \) or \( K_{3} \) then \( B_{4}(G) \) is disconnected. Hence, \( G = K_{4} \).

**Note 2.2:** If \( G = K_{4} \), eccentricity of every point vertex is 3 and eccentricity of every line vertex is 4 in \( B_{4}(G) \). Therefore, \( B_{4}(K_{4}) \) is bi-eccentric with diameter 4.

**Theorem 2.1.6:** \( G \neq K_{4}, p = 4 \) and \( G \) has at least two edges if and only if eccentricity of each line vertex is three in \( B_{4}(G) \).

**Proof:** Assume that \( G \) has four vertices and \( G \neq K_{4} \). Therefore \( G \) has at least one edge. Any edge in \( G \) is non incident with other two points in \( G \). Then the line vertex \( e'_{i} \) is adjacent to non incident vertex \( u'_{i} \) in \( B_{4}(G) \).

If \( e_{1} \) and \( e_{2} \) are non adjacent, by proposition 2.2.2, \( d(e'_{1}, e'_{2}) = 3 \) in \( B_{4}(G) \) since \( G \neq K_{4} \) and \( p = 4 \).

If \( e_{1} \) and \( e_{2} \) are adjacent then \( d(e'_{1}, e'_{2}) = 3 \) in \( B_{4}(G) \).

If \( e_{1} \) is non incident with \( v_{1} \) then \( d(e'_{1}, v'_{1}) = 1 \) and \( e_{1} \) is incident with \( v_{2} \) then \( d(e'_{1}, v'_{2}) = 3 \). Hence, eccentricity of a line vertex is 3.

On the other hand, assume that eccentricity of each line vertex is three in \( B_{4}(G) \). Hence \( d(e'_{1}, u'_{1}) \leq 3 \) and \( d(e'_{1}, e'_{2}) \leq 3 \) in \( B_{4}(G) \), for \( u \in V(G) \) and \( e_{1}, e_{2} \in E(G) \). Hence, by theorem 2.1.4, \( p \leq 4 \) and \( G \neq K_{4} \) by theorem 2.1.5. Suppose \( p \leq 3 \), \( B_{4}(G) \) is disconnected. Hence, \( G \neq K_{4} \) and \( p = 4 \).

**Theorem 2.1.7:** If \( G = K_{p}, p \geq 5 \), then \( B_{4}(G) \) is a bi eccentric graph with radius 2.

**Proof:** Let \( G = K_{p}, p \geq 5 \).

**Case-(i): To find \( d(v'_{1}, v'_{2}) \) in \( B_{4}(G) \) where \( v_{1}, v_{2} \in V(G) \).**

Any two vertices of \( G \) are adjacent in \( G \). So the point vertices of \( B_{4}(G) \) are non adjacent. Every pair of vertices have the common non incident edge \( e \) in \( G \). Then there exist a shortest path \( v_{1}'e'v_{2}' \) in \( B_{4}(G) \). Hence \( d(v'_{1}, v'_{2}) = 2 \) in \( B_{4}(G) \).

**Case-(ii): To find \( d(v', e') \) in \( B_{4}(G) \) where \( v \in V(G) \) and \( e \in E(G) \).**

Suppose a vertex \( v \) is incident to an edge \( e \) in \( G \). Let \( e = uv, e_{1} = uv_{1}, e_{2} = v_{2}v_{3} \) be the edges of \( G \). There exist a shortest path \( v'v_{2}'e'v_{1}' \) in \( B_{4}(G) \). Hence \( d(v', e') = 3 \) in \( B_{4}(G) \).

**Case-(iii): To find \( d(e'_{1}, e'_{2}) \) in \( B_{4}(G) \) where \( e_{1}, e_{2} \in E(G) \).**

\( K_{4} \) is an induced subgraph of \( B_{4}(G) \). Thus it follows that \( d(e'_{1}, e'_{2}) \neq 1 \). Every pair of edges has a common non incident vertex \( v \) in \( G \) since \( G \) has more than four vertices. Therefore, there exist a shortest path \( e_{1}'v'e_{2}' \) in \( B_{4}(G) \). Hence, \( d(e'_{1}, e'_{2}) = 2 \).

Hence, radius of \( B_{4}(G) \) is two and diameter is three.
Note 2.3:
(i) If $G = K_3$, $BG_d(G)$ is a disconnected graph.
(ii) If $G = K_4$, the diameter of $BG_d(G)$ is four by theorem 2.1.5.

Theorem 2.1.8: If $G$ is a graph with radius 1 and diameter 2, then $BG_d(G)$ has isolated vertex or $\text{diam}(BG_d(G)) = 3$.

Proof: Let $G$ be a graph with radius 1 and diameter 2. Assume $e(v) = 1$ in $G$. Suppose $v$ is incident with all the edges of $G$, then $v'$ is isolated in $BG_d(G)$. If not, by proposition 2.1.1, $e(v') = 3$ in $BG_d(G)$ and $e(e') \leq 3$ for $G \neq K_n$. Hence, $\text{diam}(BG_d(G)) = 3$.

Theorem 2.1.9: If $G$ is a 2-self centered graph with $p \geq 5$, then $BG_d(G)$ is 2-self centered.

Proof: Let $G$ be a 2-self centered graph with $p \geq 5$. By theorem 2.1.4, in $BG_d(G)$ eccentricity of a point vertex is 2 since $r(G) \neq 1$. By theorem 2.1.4, in $BG_d(G)$ eccentricity of a line vertex is 2 since $G$ has more than four vertices and $r(G) \neq 1$. Therefore, $BG_d(G)$ is a 2-self centered graph.

Cor 2.1.1: If $G$ is a 2-self centered graph with four vertices, then $BG_d(G)$ is a bi-eccentric graph with radius 2.

Proof: Let $G$ be a 2-self centered graph with four vertices. By theorem 2.1.2, eccentricity of a point vertex is 2 in $BG_d(G)$ and by theorem 2.1.6, eccentricity of a line vertex is 3 in $BG_d(G)$. Hence, $BG_d(G)$ is a bi-eccentric graph with radius 2.

Theorem 2.1.10: If $G$ is a graph with radius 2 and diameter 3 then $BG_d(G)$ is a graph with radius 2 and diameter 3.

Proof: Let $G$ be a graph with radius 2 and diameter 3. Consider the vertices $u, v \in V(G)$.

Case-(i): To find $d(u', v')$ in $BG_d(G)$ where $u, v \in V(G)$.

If $u$ and $v$ are non adjacent in $G$, then they are adjacent in $BG_d(G)$.

If $u$ and $v$ are adjacent in $G$ and $e(u) = 3$ or $e(v) = 3$, then $u$ and $v$ have the common non incident vertex in $G$. Therefore, $d(u', v') = 2$ in $BG_d(G)$. If $e(u) = e(v) = 3$, then $u$ and $v$ have the common non incident vertex in $G$. Hence $d(u', v') = 2$ in $BG_d(G)$. If $e(u) = e(v) = 2$ and $u$ and $v$ do not have a common non incident edge and not have a common non adjacent vertex, then $d(u', v') = 3$ in $BG_d(G)$. If there exist a common non incident edge or common non adjacent vertex then $d(u', v') = 2$ in $BG_d(G)$.

Case-(ii): To find $d(u', e')$ in $BG_d(G)$ where $u \in V(G)$ and $e \in E(G)$.

If a vertex $u$ is incident with an edge $e = uv$ in $G$, then $d(u', e') = 2$ in $BG_d(G)$ since $e(u) = 2$ or $e(v) = 3$ (that is the eccentric vertex of $u$ in $G$ is adjacent to $u$ and $e$ in $BG_d(G)$).

If $u$ is not incident with $e$ in $G$, then $d(u', e') = 1$ in $BG_d(G)$.

Case-(iii): To find $d(e_1', e_2')$ in $BG_d(G)$ where $e_1, e_2 \in E(G)$.

By proposition 2.1.2, $d(e_1', e_2') = 2$ or 3.

Hence, $BG_d(G)$ is a graph with radius 2 and diameter 3.

Theorem 2.1.11: If $G$ is a graph with radius 2 and diameter 4, then $BG_d(G)$ is a 2-self centered graph.

Proof: Let $G$ be a graph with radius 2 and diameter 4.

Case-(i): To find $d(u', v')$ in $BG_d(G)$ where $u, v \in V(G)$.

Let $uv = e \in E(G)$. The non adjacent vertices of $G$ are adjacent in $BG_d(G)$. If the eccentricity of a vertex $v \in V(G)$ is 3 or 4, then $u$ and $v$ have the common non adjacent vertex $w$ in $G$. Therefore, there exists a shortest path $u'w'v'$ in $BG_d(G)$. Hence $d(u', v') = 2$ in $BG_d(G)$.

If $u$ and $v$ have eccentricity 2, then $d(u', v') = 2$ or 3 in $BG_d(G)$. (The proof is similar as in theorem 2.1.10 case (i)). By theorem 2.1.9, $d(u', v') = 2$ in $BG_d(G)$.
Case-(ii): To find $d(u', e')$ in $BG_4(G)$, where $u \in V(G)$, $e \in E(G)$.

If $u$ is not incident with $e$ in $G$, then $d(u', e') = 1$ in $BG_4(G)$. If $u$ is incident with $e$ in $G$ then $d(u', e') = 2$. (the proof is similar to theorem 2.1.10 case(ii)).

Case-(iii): To find $d(e_1', e_2')$ in $BG_4(G)$, where $e_1, e_2 \in E(G)$.

The graph $G$ has more than four vertices since diameter of $G$ is four. By the theorem 2.1.4, eccentricity of a line vertex is 2.

Thus, the eccentricity of every point vertex and line vertex is two in $BG_4(G)$. Hence, $BG_4(G)$ is 2-self centered.

Theorem 2.1.12: If $G$ is a graph with $r(G) \geq 3$, then $BG_4(G)$ is a 2-self centered graph.

Proof: Let $G$ be a graph with $r(G) \geq 3$.

Case-(i): To find $d(u', v')$ in $BG_4(G)$, where $u, v \in V(G)$.

If $u$ and $v$ are non adjacent in $G$, then $d(u', v') = 1$ in $BG_4(G)$.

If $u$ and $v$ are adjacent in $G$, then $u$ and $v$ have a common non adjacent vertex in $G$.

By the definition, $d(u', v') = 2$ in $BG_4(G)$.

Case-(ii): To find $d(u', e')$ in $BG_4(G)$, where $u \in V(G)$, $e \in E(G)$.

If $u$ is incident with an edge $e$ in $G$, then $u$ and $e$ have the common non incident (adjacent) vertex in $G$. By proposition 2.1.2, we have $d(u', e') = 2$ in $BG_4(G)$.

If $u$ is non incident with an edge $e$ in $G$, then $e'$ and $v'$ are adjacent in $BG_4(G)$.

Case-(iii): To find $d(e_1', e_2')$ in $BG_4(G)$, where $e_1 = u_1v_1$, $e_2 = u_2v_2 \in E(G)$.

$G$ has more than five vertices since $r(G) = 3$. By theorem 2.1.4, $d(e_1', e_2') = 2$.

Therefore, $e(u') = 2$ and $e(e') = 2$ in $BG_4(G)$. Hence, $BG_4(G)$ is a 2-self centered graph.

2.2 Eccentricity properties of $BG_4(G)$

In this section, radius and diameter of $BG_4(G)$ are found out. Throughout this section, if $v \in V(G)$ and $e \in E(G)$, the corresponding vertices of $BG_4(G)$ are denoted by $v'$ and $e'$.

Proposition 2.2.1: If $BG_4(G)$ is connected, eccentricity of a point vertex is 1, 2 or 3.

Proof: Consider a point vertex $v$ in $V(G)$.

To find $d(u', v')$ in $BG_4(G)$ where $u, v \in V(G)$.

Case-(i): If a vertex $u$ has degree $p-1$ and incident with all the edges of $G$, then in $BG_4(G)$, $u'$ is adjacent to all the line vertices and point vertices. Hence, eccentricity of point vertex $v$ is one.

Case-(ii): If a vertex $u$ is adjacent to $v$ in $G$ then $d(u', v') = 1$ in $BG_4(G)$. If a vertex $u$ is non adjacent to $v$ in $G$, then $u'e_1'v'e_2'$ is a shortest path and $d(u', v') = 3$ in $BG_4(G)$, where edge $e_1$ is incident with $u$ and edge $e_2$ is incident with $v$ in $G$. 

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To find \( d(u', e') \) in \( \overrightarrow{BG_4}(G) \) where \( e \in E(G) \).

**Case-(i):** If a vertex \( u \) is incident with an edge \( e \) in \( G \), then \( d(u', e') = 1 \) in \( \overrightarrow{BG_4}(G) \).

**Case-(ii):** If a vertex \( u \) is not incident with an edge \( e \) in \( G \), then there exist a shortest path \( u'e_{1}'e' \) and \( d(u', e') = 2 \) in \( \overrightarrow{BG_4}(G) \), where \( e_1 \) is incident with \( u \) in \( G \).

Hence, distance from \( u \) to any other vertex is 1, 2 or 3 in \( \overrightarrow{BG_4}(G) \). Hence, the eccentricity of a point vertex is 1, 2 or 3.

**Proposition 2.2.2:** If \( \overrightarrow{BG_4}(G) \) is connected, then the eccentricity of a line vertex is 1 or 2.

**Proof:** From the proposition 2.2.1 \( d(e_1', v') = 1 \) or 2.

Now to find \( d(e_1', e_2') \) for \( e_1, e_2 \in E(G) \).

\( K_q \) is an induced sub graph of \( \overrightarrow{BG_4}(G) \). Hence, \( d(e_1', e_2') = 1 \). Hence the eccentricity of a line vertex is 1 or 2.

**Observation 2.2.1:** If a graph \( G \) has a vertex of degree \( p - 1 \) and incident with all the edges of \( G \), that is \( G = K_{1, p-1} \), then radius of \( \overrightarrow{BG_4}(G) \) is one.

**Theorem 2.2.1:** If \( G = K_p \), then \( \overrightarrow{BG_4}(G) \) is a two self centered graph.

**Proof:** Let \( G = K_p \).

**Case-(i):** To find \( d(v_1', v_2') \) in \( \overrightarrow{BG_4}(G) \), where \( v_1, v_2 \in V(G) \).

All the vertices of \( K_p \) are adjacent to each other and \( G \) is an induced sub graph of \( \overrightarrow{BG_4}(G) \). Hence, \( d(v_1', v_2') = 1 \) in \( \overrightarrow{BG_4}(G) \).

**Case-(ii):** To find \( d(v', e') \) in \( \overrightarrow{BG_4}(G) \), where \( v \in V(G) \) and \( e \in E(G) \).

If a vertex \( v \) is incident with an edge \( e \), then \( d(v', e') = 1 \) in \( \overrightarrow{BG_4}(G) \).

Suppose a vertex \( v \) is not incident with an edge \( e \), then there exists a shortest path \( v'e_{1}'e' \) in \( \overrightarrow{BG_4}(G) \), where \( e_1 \) is incident with \( v \) in \( G \). Therefore, \( d(v', e') = 2 \) in \( \overrightarrow{BG_4}(G) \).

**Case-(iii):** To find \( d(e_1', e_2') \) in \( \overrightarrow{BG_4}(G) \) where \( e_1, e_2 \in E(G) \).

Since \( K_q \) is an induced sub graph of \( \overrightarrow{BG_4}(G) \). \( d(e_1', e_2') = 1 \) in \( \overrightarrow{BG_4}(G) \).

Therefore, radius of \( \overrightarrow{BG_4}(G) \) is a 2-self centered graph.

**Theorem 2.2.2:** If \( G \neq K_{1, n} \) is a graph with radius 1 and diameter 2, then \( \overrightarrow{BG_4}(G) \) is a 2-self centered graph.

**Proof:** Let \( G \) be a graph with radius 1 and diameter 2. Consider \( v_1, v_2 \in V(G) \), \( d(v_1', v_2') = 1 \) in \( \overrightarrow{BG_4}(G) \) if \( v_1 \) and \( v_2 \) are adjacent in \( G \). \( d(v_1', v_2') = 2 \) in \( \overrightarrow{BG_4}(G) \) if \( v_1 \) and \( v_2 \) are non adjacent in \( G \) since diameter of \( G \) is 2. \( d(v_1', e') = 1 \) in \( \overrightarrow{BG_4}(G) \) if \( v_1 \) is non incident with an edge \( e \) in \( G \) since there exist a shortest path \( v_1'e_{1}'e' \) in \( \overrightarrow{BG_4}(G) \), where \( e_1 = v_1v_2 \) in \( G \).
d(e_1', e_2') = 1 in $BG_4(G)$ since $K_q$ is an induced sub graph of $G$.

Therefore, $BG_4(G)$ is a 2-self centered graph.

\textbf{Theorem 2.2.3:} If $G$ is a 2-self centered graph then $BG_4(G)$ is a 2-self centered graph.

\textbf{Proof:} Assume $G$ is a 2-self centered graph.

\textbf{Case-(i): To find} $d(v_1', v_2')$ in $BG_4(G)$ where $v_1, v_2 \in V(G)$.

Suppose $v_1$ and $v_2$ are adjacent in $G$, then $d(v_1', v_2') = 1$ in $BG_4(G)$. Suppose $v_1$ and $v_2$ are non adjacent in $G$, $d(v_1', v_2') = 2$ in $BG_4(G)$, since $G$ is an induced sub graph of $BG_4(G)$.

\textbf{Case-(ii): To find} $d(v', e')$ in $BG_4(G)$, where $v \in V(G)$ and $e \in E(G)$.

If a vertex $v$ is incident with $e$ in $G$, then $d(v', e') = 1$ in $BG_4(G)$.

If a vertex $v$ is not incident with $e$ in $G$, then there exists a shortest path $v'e_1'e'$ in $BG_4(G)$ where $e_1$ is incident with $v$ in $G$. Therefore, $d(v', e') = 2$ in $BG_4(G)$.

\textbf{Case-(iii): To find} $d(e_1', e_2')$ in $BG_4(G)$, where $e_1, e_2 \in E(G)$.

In $BG_4(G)$, $K_q$ is an induced sub graph. Hence, $d(e_1', e_2') = 1$ in $BG_4(G)$.

Therefore, $BG_4(G)$ is a 2-self centered graph.

\textbf{Theorem 2.2.4:} If $G$ is a graph with radius 2 and diameter 3, then $BG_4(G)$ is a graph with radius 2 and diameter 3.

\textbf{Proof:} Assume a graph $G$ with radius 2 and diameter 3.

\textbf{Case-(i): To find} $d(v_1', v_2')$ in $BG_4(G)$ where $v_1, v_2 \in V(G)$.

The adjacent vertices of $G$ are adjacent in $BG_4(G)$. Consider a vertex $v$ such that $e(v) = 2$ and $w$ is an eccentric vertex of $v$. Then, $d(v', w') = 2$ in $BG_4(G)$, otherwise $d(u', v') = 3$ in $BG_4(G)$ since $u$ and $v$ are at distance 3 in $G$.

\textbf{Case-(ii): To find} $d(v', e')$ in $BG_4(G)$ where $v \in V(G)$ and $e \in E(G)$.

By proposition 2.2.1, $d(v', e') = 1$ or 2 in $BG_4(G)$.

\textbf{Case-(iii): To find} $d(e_1', e_2')$ in $BG_4(G)$ where $e_1, e_2 \in E(G)$.

In $BG_4(G)$ $d(e_1', e_2') = 1$ since $K_q$ is an induced sub graph of $BG_4(G)$.

Hence, $BG_4(G)$ is a graph with radius 2 and diameter 3.

\textbf{Theorem 2.2.5:} If $G$ is a graph with radius 2 and diameter 4, then $BG_4(G)$ is a bi-eccentric graph with radius two.

\textbf{Proof:} Let $G$ be a graph with radius 2 and diameter 4.
Case-(i): To find \( d(v_1', v_2') \) in \( \overline{BG_4(G)} \) where \( v_1, v_2 \in V(G) \).

If \( v_1 \) and \( v_2 \) are adjacent in \( G \), then \( d(v_1', v_2') = 1 \) in \( \overline{BG_4(G)} \).

If \( v_1 \) and \( v_2 \) are non-adjacent in \( G \). Assume \( e(v_1') = 2 \) in \( G \), then \( e(v_1') = 2 \) in \( \overline{BG_4(G)} \). If \( e(v_1') = 3 \) in \( G \), then \( e(v_1') = 3 \) in \( \overline{BG_4(G)} \) by the proposition 2.2.1.

Case-(ii): To find \( d(v', e') \) in \( \overline{BG_4(G)} \) where \( v' \in V(G) \) and \( e \in E(G) \).

By proposition 2.2.1, \( d(v', e') = 1 \) or \( 2 \) in \( \overline{BG_4(G)} \).

Case-(iii): To find \( d(e_1', e_2') \) in \( \overline{BG_4(G)} \) where \( e_1, e_2 \in E(G) \).

\( K_4 \) is an induced sub graph of \( \overline{BG_4(G)} \). Thus \( e(e') = 1 \) in \( \overline{BG_4(G)} \).

Therefore, \( \overline{BG_4(G)} \) is a bi-eccentric graph with radius two.

**Theorem: 2.2.6** If \( G \) is a connected graph \( p \geq 3 \) with radius greater than 2, then \( \overline{BG_4(G)} \) is a bi-eccentric graph with radius 2.

**Proof:** Let \( G \) be a graph with \( r(G) \geq 3 \).

Case-(i): To find \( d(v_1', v_2') \) in \( \overline{BG_4(G)} \), where \( v_1, v_2 \in V(G) \).

Consider \( v_1, v_2 \in V(G) \) and \( e_1, e_2 \in E(G) \). If \( v_1 \) and \( v_2 \) are adjacent, then \( d(v_1', v_2') = 1 \) in \( \overline{BG_4(G)} \).

If \( v_1 \) is an eccentric vertex of \( v_2 \), then \( d(v_1', v_2') = 3 \) in \( \overline{BG_4(G)} \) since there exist a shortest path \( v_1'e_1'e_2'v_2' \) in \( \overline{BG_4(G)} \), where \( v_1 = v_1w_1 \) and \( v_2 = v_2w_2 \) in \( G \). Hence \( d(v_1', v_2') = 3 \) in \( \overline{BG_4(G)} \).

Case-(ii): To find \( d(v', e') \) in \( \overline{BG_4(G)} \), where \( v' \in V(G) \) and \( e \in E(G) \).

If a vertex \( v \) is incident with an edge \( e \) in \( G \), then \( d(v', e') = 1 \) in \( \overline{BG_4(G)} \).

If a vertex \( v \) is non-incident with an edge \( e \) in \( G \), then there exist a shortest path \( v'e_1'e' \) in \( \overline{BG_4(G)} \), where \( e_1 = vv_1 \) in \( G \). Hence, \( d(v', e') = 2 \) in \( \overline{BG_4(G)} \).

Case-(iii): To find \( d(e_1', e_2') \) in \( \overline{BG_4(G)} \), where \( e_1 = uv, e_2 = u_1v_1 \in E(G) \).

\( d(e_1', e_2') = 1 \) in \( \overline{BG_4(G)} \) since \( K_4 \) is an induced sub graph of \( \overline{BG_4(G)} \). Hence, eccentricity of a line vertex is 2.

Therefore, \( \overline{BG_4(G)} \) is a bi-eccentric graph with radius 2.

**CONCLUSION**

It is proved that either \( BG_4(G) \) is disconnected or it is a graph of diameter at most 4. Also, we have characterized graphs \( G \) for which \( BG_4(G), \overline{BG_4(G)} \) are 2-self centered, bi-eccentric, etc.
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Source of support: Nil, Conflict of interest: None Declared

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