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FIXED POINT THEOREMS FOR T_k -CONTRACTIONS IN K-METRIC SPACES

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ABSTRACT

A basic difference between ordinary metric and k-metric is in the triangle inequality. In this paper we have shown that instead of this difference, by restricting the domain of effectively involved constant, some fixed point theorems for T_k -contractions can be obtained as in cone metric spaces.

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1. INTRODUCTION

H. Pajoohesh introduced the concept of k-metrics in 2012 which are valued in lattice ordered groups and that allows to talk about distance in non-abelian lattice ordered groups. He also characterized intrinsic metrics on lattice ordered rings and established that if a lattice ordered ring is representable then every intrinsic metric therein is a k-metric. Being motivated by these facts we study some fixed point theorems for T -contractive mappings in this setting. In this paper we actually traslated the results of [1] in the language of k -metric spaces. As in [2] a k-metric, where k is a real number ≥ 1, on a nonempty set X is a mapping $d: X\times X \to R$ such that

(i) $d(x,y)\geq 0 \;\; \forall \;\; x,y\in X$,
(ii) $d(x,y)=0 \Leftrightarrow x=y$,
(iii) $d(x,y)=d(y,x) \;\; \forall \;\; x,y\in X$,
(iv) $d(x,y)\leq k(d(x,z)+d(z,y)) \;\; \forall \;\; x,y,z\in X$.

The ordered pair (X,d) is called a k -metric space.

Let us consider the mapping $d: R\times R\to R$ defined by $d(x,y)=(x-y)^2 \quad \forall x,y\in R.$ The fact $(a+b)^2\leq 2(a^2+b^2) \quad \forall a,b\in R$ ensures that the mapping d enjoys all the properties of being a k -metric for $k=2$.

From the definition and the example, just given above, it is clear that every metric is a k -metric $(k=1)$, but a k -metric may not be a metric and every k -metric is an l -metric, where $l\geq k$.

Open balls, closed balls, diameter of non empty sets, open sets (A subset O of a k-metric space (X,d) is said to be open in (X,d) if $\forall \;\; x\in O \;\; \exists \;\; \varepsilon>0$ such that the open ball $B_d(x,\varepsilon)\subset O$.), closed sets, closure and interior of a set, convergence of a sequence, Cauchy sequence, completeness of k-metric spaces are defined as in case of metric spaces. It is also seen that every k -metric space is first countable and T_4.

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2. T_k – CONTRACTIONS AND FIXED POINT THEOREMS

We begin with the following definitions which bear the character of the mappings involved in the theorems of this section.

Definition 1: *Let* (X, d) *be a k-metric space and T:* $X \to X$ *. T is said to be* **sequentially convergent** *in* X *if for every sequence {* y_n *} in* X *, {T* y_n *} is convergent implies that {* y_n *} is also convergent.*

Definition 2: *Let* (X, d) *be a k-metric space and T:* $X \to X$ *. T is said to be* **sub-sequentially convergent** *in* X *if for every sequence {* y_n *}, {T* y_n *} is convergent implies that {* y_n *} has a convergent subsequence.*

Definition 3: *Let* (X, d) *be a k-metric space and* $T, S : X \to X$ *are functions. The mapping S is said to be* T_k **-contraction** *if there is a constant* $\alpha \in \left[0, \frac{1}{k}\right)$ *such that*

$$d(TSx, TSy) \le \frac{\alpha}{k} d(Tx, Ty), \ \forall x, y \in X$$

Theorem 1: *Let* (X, d) *be a complete k-metric space and T:* $X \to X$ *be a one to one and continuous function. In addition let S:* $X \to X$ *be a* T_k *-contraction continuous function. Then*

(i) for every $x_0 \in X, \lim_{n\to\infty} d(TS^n x_0, TS^{n+1} x_0) = 0.$

(ii) there is $y_0 \in X$ such that $\lim_{n\to\infty} TS^n x_0 = y_0$

(iii) if T is sub-sequentially convergent, then there is a unique $z_0 \in X$ such that $Sz_0 = z_0$

(iv) if T is sequentially convergent, then for each $x_0 \in X$, the sequence { $S^n x_0$ } converges to z_0 .

Proof: Let $x_1, x_2 \in X$, then

$$\begin{aligned} d(Tx_1, Tx_2) &\le k[d(Tx_1, TSx_1) + d(TSx_1, Tx_2)] \\ &\le kd(Tx_1, TSx_1) + k^2[d(TSx_1, TSx_2) + d(TSx_2, Tx_2)] \\ &\le kd(Tx_1, TSx_1) + k^2\left[\frac{\alpha}{k} d(Tx_1, Tx_2) + d(TSx_2, Tx_2)\right] \end{aligned}$$

$$d(Tx_1, Tx_2) \le \frac{k}{1 - k\alpha}[d(Tx_1, TSx_1) + kd(TSx_2, Tx_2)] ------> (A)$$

(i) Let $x_0 \in X$ and consider the sequence { x_n } given by $x_n = Sx_{n-1} = S^2 x_{n-2} = ... = S^n x_0, \forall n \in \mathbb{N}$

Now,

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &= d(TS^n x_0, TS^{n+1} x_0) \\ &= d(TS(S^{n-1} x_0), TS(S^n x_0)) \\ &\le \frac{\alpha}{k} d(T(S^{n-1} x_0), T(S^n x_0)) \\ &= \frac{\alpha}{k} d(TS(S^{n-2} x_0), TS(S^{n-1} x_0)) \\ &\le \frac{\alpha^2}{k^2} d(T(S^{n-2} x_0), T(S^{n-1} x_0)) \end{aligned}$$

continuing this process, we get

$$d(Tx_n, Tx_{n+1}) \le \frac{\alpha^n}{k^n} d(Tx_0, TSx_0)$$

So $\lim_{n\to\infty} d(TS^n x_0, TS^{n+1}x_0) = 0$, since $0 \le \frac{\alpha}{k} < 1$.

(ii) For, $m, n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned} d(Tx_n, Tx_m) &= d(TS^n x_0, TS^m x_0) \\ &\le \frac{k}{1-k\alpha}[d(TS^n x_0, TS^{n+1}x_0) + kd(TS^{m+1}x_0, TS^m x_0)](by \quad A) \\ &\le \frac{k}{1-k\alpha}\left[\frac{\alpha^n}{k^n} d(Tx_0, TSx_0) + k\frac{\alpha^m}{k^m} d(Tx_0, TSx_0)\right] \\ &\le \frac{k}{1-k\alpha} \left[\frac{\alpha^n}{k^n} + k\frac{\alpha^m}{k^m}\right] d(Tx_0, TSx_0) \end{aligned}$$

which implies,

$$\lim_{n,m\to\infty} d(Tx_n, Tx_m) = 0$$
$$\lim_{n,m\to\infty} d(TS^n x_0, TS^m x_0) = 0$$

Therefore, { $TS^n x_0$ } is a Cauchy sequence in X. By completeness of X, $\exists$ $y_0 \in X$ such that

$$\lim_{n\to\infty} TS^n x_0 = y_0$$

(iii) Let T be sub-sequentially convergent, then since { $TS^n x_0$ } is convergent the sequence { $S^n x_0$ } has a subsequence { $S^{n_i} x_0$ } such that

$$\lim_{n_i\to\infty} S^{n_i} x_0 = z_0 \in X \tag{1}$$

$$\Rightarrow \lim_{n_i\to\infty} TS^{n_i} x_0 = Tz_0, \quad [Since, \quad T \quad is \quad continuous]$$

$$\Rightarrow Tz_0 = y_0$$

By (1),

$$\lim_{n_i\to\infty} S^{n_i+1} x_0 = Sz_0 [\text{sinceS}'\text{iscontinuous}]$$

$$\Rightarrow \lim_{n_i\to\infty} TS^{n_i+1} x_0 = TSz_0 \Rightarrow y_0 = TSz_0$$

Then, we have

$$TSz_0 = Tz_0$$
$$\Rightarrow Sz_0 = z_0 \quad [\text{sinceT}'\text{isoneone}]$$

Therefore z_0 is a fixed point of S.

We now prove the uniqueness of fixed point of S.

Let $z_0, z_1 \in X$ so that $Sz_0 = z_0$ and $Sz_1 = z_1$. Then

$$0 \le d(Tz_0, Tz_1) = d(TSz_0, TSz_1) \le \frac{\alpha}{k} d(Tz_0, Tz_1)$$
$$\Rightarrow d(Tz_0, Tz_1) = 0 \Rightarrow Tz_0 = Tz_1 \Rightarrow z_0 = z_1 (\text{sinceTisoneone}).$$

(iv) This is a special case of (iii).

Note 1: *In the above* ***Theorem*** *1 if we take T as identity map and $k=1$ then we obtain the classical Banach fixed point theorem.*

Theorem 2: *Let (X,d) be a complete k-metric space and $T:X\to X$ be an injective and continuous function. Suppose that $S:X\to X$ is a mapping such that S^n is a T_k-contraction for some $n\in N$. Then S has a unique fixed point in X*

Proof: As in general case.

Theorem 3: *Let, (X,d) be a complete k-metric space and $T:X\to X$ be an injective and continuous mapping. For $c>0$ and $x_0\in X$, set $B(Tx_0,c)=\{y\in X: d(Tx_0,y)<c\}$. Suppose $S:X\to X$ be a T_k-contraction continuous mapping for all $x,y\in B(Tx_0,c)$ satisfying $d(TSx_0,Tx_0)<(1-\alpha)\frac{c}{k}$.*

Then S has a unique fixed point in $\overline{B}(Tx_0,c)$.

Proof: Set $x_n = Sx_{n-1}$ $\forall n\in N$. By the given condition

$$d(Tx_1,Tx_0)=d(TSx_0,Tx_0)\le(1-\alpha)c/k<c \Rightarrow Tx_1\in B(Tx_0,c)$$

Also whenever $Tx\in B(Tx_0,c)$, $TSx\in B(Tx_0,c)$.

For, let $Tx\in B(Tx_0,c)$, then $d(Tx_0,Tx)<c$ and

$$\begin{aligned} d(Tx_0,TSx) &\le k[d(TSx_0,Tx_0)+d(TSx_0,TSx)] \\ &\le k(1-\alpha)\frac{c}{k}+k\frac{\alpha}{k}d(Tx_0,Tx) \\ &< (1-\alpha)c+\alpha c = c \end{aligned}$$

$$\Rightarrow TSx\in B(Tx_0,c)$$

So, $TSx_1 = Tx_2\in B(Tx_0,c)$ and consequently $\{Tx_n\}$ is a sequence in $B(Tx_0,c)$. Now a proof of this theorem follows from the **Theorem** 1 and completeness of $\overline{B(Tx_0,c)}$.

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